On the Number of Vertices with Specified Eccentricity

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Abstract

The *eccentricity* of a vertex v in a graph is the maximum of the distances from v to all other vertices. The *diameter* of a graph is the maximum of the eccentricities of its vertices. Fix the parameters n, d, c. Over all graphs with order n and diameter d, we determine the maximum (within 1) and the minimum of the number of vertices with eccentricity c.

We solve an extremal problem involving distances in connected graphs. Let G be a connected graph. The *distance* $d(u,v)$ between two vertices u and v in G is the length of a shortest u,v-path. The *eccentricity* of a vertex v in G is $\epsilon(v) = \max_{u \in V(G)} d(u,v)$. The *radius* and *diameter* of G are the minimum and maximum eccentricities, respectively. A *diametric path* is a shortest u,v-path when $d(u,v)$ is the diameter of G.

Let $\mathcal{G}_{n,d}$ be the family of connected graphs with order n and diameter d. Over $\mathcal{G}_{n,d}$, let $h_{n,d}(c)$ and $f_{n,d}(c)$, respectively, be the minimum and maximum number of vertices with eccentricity c. In this paper, we determine these functions almost completely. Since diameter is maximum eccentricity, these values are 0 if $c > d$. Also, no vertex w has eccentricity less than half $\mathrm{diam} G$, since $d(u,v) \le d(u,w) + d(w,v)$ for all u, v, w. Furthermore, the existence of a diametric path requires at least $d+1$ vertices. We therefore may assume that $d/2 \le c \le d < n$. We also assume that $d > 1$, since otherwise G is a clique and all vertices have eccentricity d.

We consider only simple graphs. A *nontrivial* graph is a graph with at least one edge. Let $N_G(v)$ or simply $N(v)$ denote the set of neighbors of vertex v in graph G. A *vertex expansion* in a graph G is a replacement of a vertex $v \in V(G)$ by a clique Q of new vertices, such that the neighborhood outside Q of each vertex of Q is $N_G(v)$. Let C_n and P_n denote the cycle and the path of order n.

1 Preliminary Lemmas

We begin with elementary remarks useful for both the minimization and the maximization problems. A simple class of graphs provides extremal examples in many cases.

Lemma 1 *For $k > 0$, let $G(k,0) = C_{2k}$. For $k \ge l > 0$, let $G(k,l)$ be the graph obtained by extending a path of length $l-1$ from one vertex of C_{2k} and adding one pendant edge at the diametrically opposite vertex of C_{2k}. The graph $G(k,l)$ has order $2k+l$, diameter $k+l$, and no vertices of eccentricity less than k. The count of vertices in $G(k,l)$ with various eccentricities is*

case	eccentricity	multiplicity
$l > 1$	k	$2k - 2l + 2$
	$k < \epsilon(v) < k + l - 1$	3
	$k + l - 1 \le \epsilon(v) \le k + l$	2
$l = 1$	k	$2k - 1$
	$k + 1$	2
$l = 0$	k	$2k$

Proof: Vertices on the cycle have eccentricity at least k; vertices on the pendant paths have eccentricity greater than k. From the peripheral vertices, eccentricity decreases by one with each step toward the center until eccentricity k is reached (see Fig. 1). □

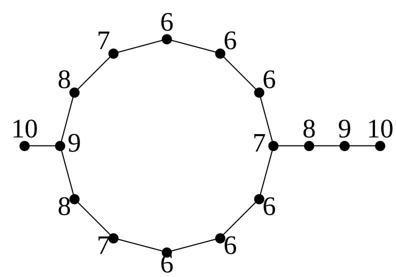

Fig. 1. $G(6,4)$, labeled by eccentricity

Lemma 2 *A vertex expansion in a nontrivial graph does not change the eccentricity of any vertex, and the eccentricity of the new vertices equals the eccentricity of the vertex they replaced. In particular,* $h_{n+1,d}(c) \le h_{n,d}(c)$ *and* $f_{n+1,d}(c) \ge f_{n,d}(c) + 1$.

Proof: Suppose that G' is obtained from G by a vertex expansion at v. No chordless path of length more than one contains two vertices of the new clique Q, since those vertices have the same neighborhoods outside Q. Thus every shortest x,y-path in G' is a copy of an x,y-path in G (or of a path ending at v if $y \in Q$). Thus distances and eccentricities do not change. The remark about h follows by expanding a vertex of eccentricity other than c in an extremal graph for the parameters n, d, c. For the remark about f, expand a vertex of eccentricity c. □

Lemma 3 *Suppose that* $G \in \mathcal{G}_{n,d}$. *If* $n - d \le c \le d$ *and* $c \ge d/2$, *then the vertices* v_c *and* v_{d-c} *on a diametric path* P *with vertices* $v_0, \ldots, v_d$ *have eccentricity* c. *If* $n < c + d$, *then* $\epsilon(v_i) < c$ *for* $d - c < i < c$.

Proof: By symmetry, we need only consider v_i, where $d/2 \le i \le c$. Define c' by

$$c' = \begin{cases} c & \text{if } i = c \\ c - 1 & \text{if } i < c. \end{cases}$$

Every vertex of P is within distance c' of v_i, and when $i = c$ we have $d(v_i, v_0) = c'$. If v_i does not have eccentricity at most c', then there is a vertex $w \notin V(P)$ such that $d(v_c, w) = c' + 1$. Let Q be a shortest w, v_i-path, and let v_q be the vertex at which Q first enters P. For $q > c'$, let R be a shortest w, v_0-path; when $q \le c'$, let R be a shortest w, v_d-path. Let v_r be the vertex where R first enters P. If R contains $v_{c'}$, then the portion of R before $v_{c'}$ is a $w, v_{c'}$-path, and hence the length of R is at least $c' + 1 + c'$ or at least $(c' + 1) + (d - c')$. Since each of these exceeds d, we conclude that v_r and v_q are on opposite sides of $v_{c'}$ along P (although q may equal c'). Fig. 2 shows one such arrangement.

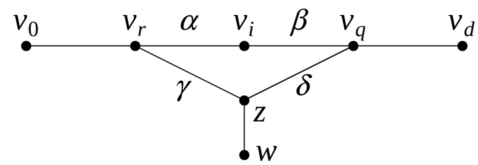

Fig. 2. Forcing eccentricity c

Let z be the last vertex of $Q \cap R$. Because each is a shortest path from w, we may assume that Q and R coincide before z. We consider four distances along P, Q, R. Let $\alpha = d(v_i, v_r)$, $\beta = d(v_i, v_q)$, $\gamma = d(v_r, z)$, $\delta = d(v_q, z)$. Because P is a shortest v_0, v_d-path, we have $\gamma + \delta \ge \alpha + \beta$. Because Q is a shortest v_i, w-path, we have $\alpha + \gamma \ge \beta + \delta$. Summing the two inequalities yields $\gamma \ge \beta$. Thus

$$n \ge |V(P \cup Q \cup R)| = (d + 1) + (d(v_i, w) - \beta) + (\gamma - 1) \ge d + 1 + c'.$$

When $i = c = c'$, this contradicts the hypothesis $n \le c + d$. When $i < c = c' + 1$, this contradicts the hypothesis $n < c + d$. $\square$

These tools, which will be useful when considering $f_{n,d}$, enable us to compute $h_{n,d}$.

Theorem 4 *For $n > d$ and $d/2 \le c \le d$,*

$$h_{n,d}(c) = \begin{cases} 2 & \text{if } c = d \\ 0 & \text{if } n > c + d \text{ and } d/2 \le c < d \\ 2 & \text{if } n \le c + d \text{ and } d/2 < c < d \\ 1 & \text{if } n \le c + d \text{ and } c = d/2. \end{cases}$$

Proof: The endpoints of a diametric path have eccentricity d, and P_{d+1} with Lemma 2 shows that this is the minimum when $c = d$. For $d/2 \le c < d$, the graph $G(c + 1, d - c - 1)$ of Lemma 1 has order $c + d + 1$, diameter d, and no vertices of eccentricity c. With Lemma 2, this completes the case $n > c + d$. The lower bound for the case $n \le c + d$ is given by Lemma 3. For the upper bound when $n \le c + d$ we use P_{d+1} and Lemma 2. $\square$

2 The Maximization Problem

We now study the maximum number of vertices with eccentricity c. We begin with constructions for the lower bound.

Lemma 5 *For $n > d$ and $d/2 \le c \le d$,*

$$f_{n,d}(c) \ge \begin{cases} n - 2(d-c) & \text{if } n \ge c+d \text{ and } c \ge d-1 \\ n - 3(d-c) + 2 & \text{if } n \ge c+d \text{ and } \frac{2d-1}{3} \le c < d-1 \\ n - d + 1 & \text{if } n < c+d \text{ or } \frac{d}{2} < c \le \frac{2d-1}{3} \\ n - d & \text{if } c = \frac{d}{2}. \end{cases}$$

Proof: Since these formulas increase by one with each augmentation of n, by Lemma 2 it suffices to consider the smallest value of n in each case. For Case 1 and Case 2, let $n = c + d$ and let $G = G(c, d-c)$. By Lemma 1, the number of vertices in G with eccentricity c is $2c - (d-c) = (c+d) - 2(d-c)$ when $d - c \le 1$ and is $2c - 2(d-c) + 2 = (c+d) - 3(d-c) + 2$ when $d - c > 1$. For Case 3 and Case 4, let $n = d + 1$ and let $G = P_{d+1}$. Here the number of vertices with each eccentricity is 2, except that there is only one central vertex when d is even. □

When $n \ge c + d$ and $c < (2d-1)/3$, the constructions of Cases 2 and 3 in Lemma 5 both apply, but the construction in Case 3 provides more vertices of eccentricity c.

To obtain upper bounds on $f_{n,d}(c)$, the formulas in Lemma 5 suggest that we obtain lower bounds on the number of vertices that must have eccentricity other than c, since the desired bound is independent of n in each case. Let $g_{n,d}(c) = n - f_{n,d}(c)$. Our approach is to start with a diametric path $P = v_0, v_1, \ldots, v_d$ in $G \in \mathcal{G}_{n,d}$ and to deduce the existence of many vertices with eccentricity not c. The vertices of P before v_{d-c} and after v_c have eccentricity greater than c; thus $g_{n,d}(c) \ge 2(d-c)$. This proves optimality of the constructions in Lemma 5 when $c \ge d-1$ and when $c = d/2$. For $d/2 < c < d-2$, we must improve the lower bound on $g_{n,d}(c)$.

Although we will try to show that the bounds of Lemma 5 are optimal, in fact we can only prove that $g_{n,d}(c) \ge 3(d-c) - 3$ when $\frac{2d-2}{3} \le c < d-1$. The cases in which we cannot show that $g_{n,d}(c) \ge 3(d-c) - 2$ lead in fact to surprising examples where $g_{n,d}(c) = 3(d-c) - 3$.

Theorem 6 *For $n > d$ and $d/2 < c < d-1$,*

$$g_{n,d}(c) \ge \begin{cases} 3(d-c) - 2 & \text{if } n \ge c+d \text{ and } \frac{2d-2}{3} \le c < d-1 \text{ and } d-c \text{ is even} \\ 3(d-c) - 3 & \text{if } n \ge c+d \text{ and } \frac{2d-2}{3} \le c < d-1 \text{ and } d-c \text{ is odd} \\ d - 1 & \text{if } n < c+d \text{ or } \frac{d}{2} < c \le \frac{2d-1}{3}. \end{cases}$$

Furthermore, if $g_{n,d}(c) = 3(d-c) - 3$, then a graph achieving this has a diametric pair at the end of paths of length $\frac{d-c+1}{2}$ that emerge from antipodal vertices on a cycle of length $2c-2$.

Proof: Suppose that $n > d$ and $d/2 < c < d-1$, and consider $G \in \mathcal{G}_{n,d}$. Let $P = v_0, \ldots, v_d$ be a diametric path in G. For $n < c + d$, Lemma 3 implies that $\epsilon(v_i) \ne c$ when $i \notin \{d-c, c\}$. Thus we may assume that $n \ge c + d$.

In this range, the desired lower bound is $\min\{3(d-c) - 2, d-1\}$, so it suffices to obtain either $3(d-c) - 2$ or $d - 1$ as a lower bound. If no vertex of P between v_{d-c} and v_c has eccentricity c, then at least $d-1$ vertices of G have eccentricity other than c. Thus we may assume that P has a vertex of eccentricity c between v_{d-c} and v_c.

Let v_i be the last vertex on P before v_c that has eccentricity c, and let z be a vertex at distance c from v_i. Since $d(v_i, u) < c$ for $u \in V(P)$, we have $z \notin V(P)$. Let Q be a shortest z, v_d-path, and let v_t be the first vertex of Q that is also on P. Since $d(z, v_d) \le d$ and $d(z, v_i) = c$, we have $t > i$.

We define several sets of vertices that will have eccentricity other than c.

$$A = \{v_j : \ i < j < c\}$$
$$B = \{u \in V(Q - P) : \ d(u, v_d) < d - c\}$$
$$C = \{u \in V(Q - P) : \ d(u, v_d) > c\}$$

For $u \in A$, the choice of i yields $\epsilon(v_j) \neq c$. By the triangle inequality, every vertex of G having distance less than $d - c$ from one end of P has distance more than c from the other end; thus $\epsilon(u) > c$ for $u \in B$. For $u \in C$, the definition of C yields $\epsilon(u) \geq d(u, v_d) > c$. Thus $\epsilon(u) \neq c$ for $u \in A \cup B \cup C$. Note also, since $c > d - c$, that A, B, C are pairwise disjoint.

Define $v_{d-i'}$, z', Q', $v_{d-t'}$, A', B', C' symmetrically from the other end of P. As above, $\epsilon(u) \neq c$ for $u \in A' \cup B' \cup C'$, and A', B', C' are pairwise disjoint. Fig. 3 shows a case with $t, t' > c$ and $(A' \cup B' \cup C') \cap (A \cup B \cup C) = \emptyset$. By construction, $i \geq d - i'$. Since $d(u, v_d) < d - i$ for $u \in A \cup B$ and $d(u, v_0) < d - i'$ for $u \in A' \cup B'$ a vertex common to these two sets would yield a v_0, v_d-path shorter than P. Thus we may assume that $(A \cup B) \cap (A' \cup B') = \emptyset$.

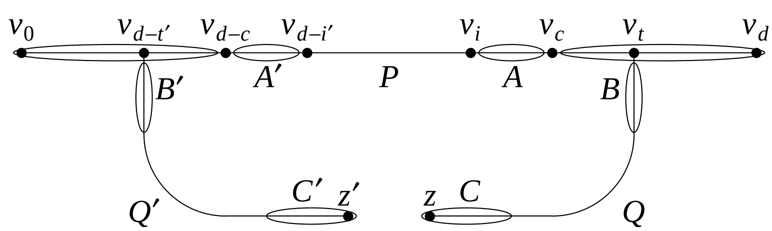

Fig. 3. Sets of vertices with eccentricity not c

To obtain the desired lower bound on $g_{n,d}(c)$, we will show that $\gamma = |A \cup B \cup C \cup A' \cup B' \cup C'|$ is large. In addition to the $2(d - c)$ outermost vertices of P, it suffices to show that $\gamma \geq d - c - 2$ or that $\gamma \geq 2c - d - 1$.

Let $k = d(z, v_d) - c$. Since Q is a shortest z, v_d-path, we have $d(v_t, z) = d(z, v_d) - (d - t)$. By the triangle inequality, $d(v_t, z) \geq d(v_i, z) - (t - i)$. Thus $(c + k) - (d - t) \geq c - (t - i)$, which yields $k \geq d - 2t + i$. If $t \leq c$, then $k \geq (d - c) + (i - c)$, and we obtain $|A \cup C| \geq c - i - 1 + k \geq d - c - 1$.

Thus we may assume that $t > c$, which yields $|B| = \min\{t - c - 1, c + k - (d - t)\}$. Similarly, we define $k' = d(z', v_0) - c$ and may assume that $t' > c$. Fig. 3 illustrates the case where $k, k' > 0$ and the six relevant sets are pairwise disjoint. For all cases, we have computed the inequalities $2t \geq d - k + i$ and $2t' \geq d - k' + i'$. Invoking these and then $i, i' \leq c - 1$ yields the computations below and the analogous lower bounds on $|A' \cup B' \cup C'|$ in terms of k'.

range	$\lvert A\rvert$	$\lvert B\rvert$	$\lvert C\rvert$	$\lvert A \cup B \cup C\rvert \geq$
$k \geq 0$	$c - i - 1$	$t - c - 1$	k	$\frac{d-c-3+k}{2}$
$0 > k \geq d - 2c - 1$	$c - i - 1$	$t - c - 1$	0	$\frac{d-c-3-k}{2}$
$d - 2c - 1 > k \geq 1 - c$	$c - i - 1$	$c + k - d + t$	0	$\frac{3c-d+k-1}{2}$

With $k \geq 1 - c$, the last bound becomes $(2c - d)/2$. Each of these bounds is at least half the needed amount, except when $k = 0$. We may sum the bounds for $|A \cup B \cup C|$ and $|A' \cup B' \cup C'|$ unless these two sets intersect. When $C \cap B' = C' \cap B = \emptyset$, summing the two lower bounds and subtracting $|C \cap C'|$ yields at least the desired amount unless $C = C'$, in which case it yields $\gamma \geq d - c - 3$. We postpone the analysis of that case.

Case 1: *$C \cap B'$ or $C' \cap B$ is nonempty.* By symmetry, we may assume $C \cap B' \neq \emptyset$. In following Q' from v_0, let y be the first vertex encountered in C. Let H be the cycle formed by the $v_t, v_{d-t'}$-subpath of P, the $v_{d-t'}, y$-subpath of Q', and y, v_t-subpath of Q. We define several vertex sets with respect to H (see Fig. 4). Let $D = \{u \in V(H) - V(P)\colon\ d(u, v_0) < d - c\}$. Since $d(u, v_d) \leq c$ for $u \in V(Q) - V(C)$, we have $D \subseteq B' \cup C$. Also $|D| = t' - c - 1$. Let $C_0 = C - V(H)$, and let $C_1 = C - D - C_0$.

The cycle H consists of $t - (d - t')$ edges on P, $t' - c - 1$ edges from $v_{d-t'}$ through D, $|C_1|$ edges to absorb the vertices of C_1, $c + 1 - (d - t)$ edges on Q from C to v_t. Thus H has length $2(t + t' - d) + |C_1|$. The distance from z to v_i is at most $|C_0|$ plus half the length of H; thus $c \leq |C_0| + |C_1|/2 + t + t' - d$. Now we compute

$$|B \cup B' \cup C| \geq |B \cup D \cup C_0| + |C_1|/2 \geq (t - c - 1) + (t' - c - 1) + (c - t - t' + d) = d - c - 2.$$

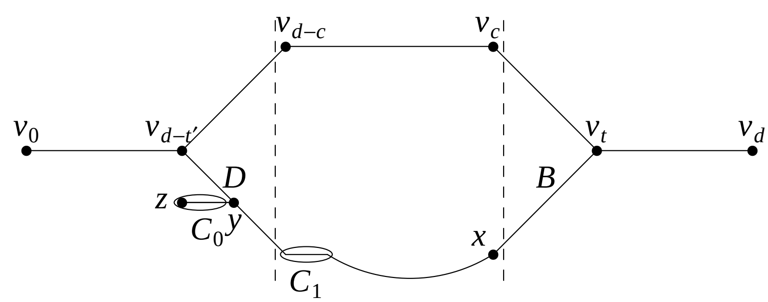

Fig. 4. Case 1: $C \cap B' \neq \emptyset$

Case 2: $B \cap C' = B' \cap C = \emptyset$ and $C = C'$. If $\gamma = d - c - 3$, then the inequalities used in obtaining $\gamma \geq d - c - 3$ hold with equality, and we need only one more vertex with eccentricity other than c. The equalities yield $i = i' = c - 1$, $k = k' \geq 0$, and $t = t' = (d - k + c - 1)/2$. Thus $t - i = d - t - k$, which implies that the v_i, z-path along P and Q is a shortest v_i, z-path (similarly for $v_{d-i'}$ and z'). Between B and C there are $1 + 2c - d$ vertices on Q.

Let x be the first vertex of Q after B (see Fig. 4); we have $d(x, v_d) = d - c$. By the triangle inequality, $d(x, v_0) \geq c$. We are finished if $\epsilon(x) > c$, so we may assume that $d(x, v_0) = c$. A shortest x, v_0-path and the x, v_d-portion of Q together form a v_0, v_d-path P^* of length d.

We show next that $d(x, u) < c$ for $u \in B \cup C \cup B'$ (A, A' are empty and $C' = C$). Using Q and P, we have $d(x, u) < c$ for $u \in B \cup C \cup \{v_c, \ldots, v_{d-1}\}$. Because $\epsilon(u) > c$ when $d(u, v_0) < d - c$, we may assume that P^* enters the set of vertices within distance $d - c - 1$ of v_0 at v_{d-c-1} or at the most distant vertex of B'. Since $d - t' - k' = (d - i') - (d - t')$, we can modify the last part of the x, v_0-portion of P^* to obtain $d(x, u) < c$ for $u \in B' \cup \{v_1, \ldots, v_{d-c}\}$.

We may apply the same analysis to P^* that we applied to P; we are finished unless this analysis also falls into Case 2. Let k^* and z^* be the value and vertex in the analysis of P^* that play the role of k and z in the analysis of P. If $k^* > 0$, then $\epsilon(z^*) > c$. Since $d(x, z^*) = c$, we have shown that z^* is not in $B \cup C \cup B'$ and not on P within distance $d - c - 1$ of either end. Hence z^* is a new vertex of large eccentricity and we are finished. Thus we may assume that $k^* = 0$.

Since we could have started with P^* as P, we may also assume that $k = 0$. Then when we generate the new P^*, again we are finished unless $k^* = 0$. The unresolved case is the case in which the theorem statement claims only $g_{n,d}(c) \geq 3(d-c) - 3$. □

The remaining unresolved case is when $n \geq c + d$, $\frac{2d-2}{3} \leq c < d - 1$, and $d - c$ is odd. In this situation we have proved that $3(d-c) - 3 \leq g_{n,d}(c) \leq 3(d-c) - 2$. Surprisingly, more subtle constructions achieve the lower bound for some such values of (d, c) when n is sufficiently large.

Theorem 7 *If $n \geq 3d + c - 3$, $\frac{3d}{4} \leq c < d - 1$, and $d - c$ is odd, then $g_{n,d}(c) = 3(d-c) - 3$.*

Proof: For such values we construct a graph G illustrated in Fig. 5. Each segment in the illustration represents a path of indicated length. Let $s = (d - c - 1)/2$; this is an integer since $d - c$ is odd. To form G, begin with a cycle C of length $2c - 2$ and two paths of length $s + 1$ extending from antipodal vertices on C to the unique vertices of eccentricity d. In the language we have been using, we may label the vertices of the diametric path around the top of C as the path P with vertices $v_0, \ldots, v_d$. Here v_{d-t} is the entrance of P to C and v_t is its departure. We have seen that $g_{n,d}(c) < 3(d-c) - 2$ requires $t = (d + c - 1)/2$. Let P' be the other v_0, v_d-path of length d, with vertices $v'_0, \ldots, v'_d$, of which $2s + 4$ are shared with P.

We next add two paths W, W' of length $2\lfloor c/2 \rfloor + 3$. Listed by vertices, these are $W = v_{d-c}, w_1, \ldots, w_{2\lfloor c/2\rfloor+2}, v'_c$ and $W' = v'_{d-c}, w'_1, \ldots, w'_{2\lfloor c/2\rfloor+2}, v_c$. Note that since $2s + 1 = d - c$, the endpoints of W and W' have distance s from v_{d-t} and v_t along P and P'. Note also that W and W' have no common vertex. We next add a path X from $w_{\lfloor c/2\rfloor-2s+1}$ to $w'_{\lfloor c/2\rfloor+2s+2}$ and a path X' from $w'_{\lfloor c/2\rfloor-2s+1}$ to $w_{\lfloor c/2\rfloor+2s+2}$. The length of X and X' is $2s - 1$ if c is even; $2s$ if c is odd. Finally, we add the four edges $\{v_{d-c+1}w_1, v'_{d-c+1}w'_1, v_{c-1}w'_{2\lfloor c/2\rfloor+2}, v'_{c-1}w_{2\lfloor c/2\rfloor+2}\}$.

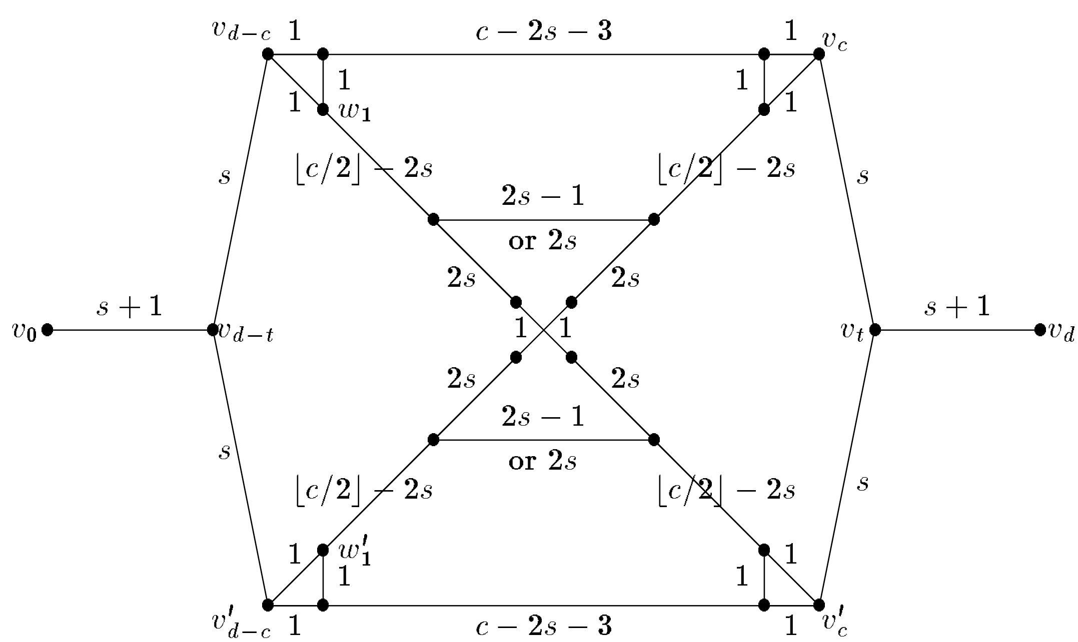

Fig. 5. A general construction for $g_{n,d}(c) = 3(d-c) - 3$

Altogether, we have created $3d+c-3$ vertices. By Lemma 2, the construction is valid for $n \geq 3d+c-3$. Vertices v_0 and v_d have eccentricity d. All of v_j and v'_j with $j < d-c$ or $j > c$ have eccentricity greater than c; there are $2(s+1)+2(2s-1) = 6s = 3(d-c)-3$ such vertices. It remains to be shown that all other vertices have eccentricity exactly c. We use the four-fold symmetry of the construction to shorten the analysis.

We first show that all distances involving the remaining vertices are at most c. A convenient computation to avoid most discussion by parity is that $2\lfloor c/2 \rfloor$ plus the length of X or X' equals $c+2s-1$. Also, $c \leq 3d/4$ yields $3s+2 \leq \lceil c/2 \rceil$, using $s = (d-c-1)/2$ and the opposite parity of d and c.

First consider $G' = G - \{v_0, \ldots, v_{d-t-1}\} - \{v_{t+1}, \ldots, v_d\}$. Within G', we define a collection of cycles such that every two vertices appear together in at least one of the cycles and each cycle has length at most $2c+1$. The cycle C has length $2c-2$. The path from v_{d-c} to v_c via W, X, W' has length $c-2s+1$, which exceeds the distance along C by 2. Modifying C by replacing the shorter with the longer path yields a cycle C' of length $2c$. Combining the two paths yields a cycle C'' of length $2c-4s$. The cycle A consisting of $X \cup X'$ plus the central portions of W and W' has length at most $12s+2$, which is less than $2c$. Let B be the cycle consisting of X, the central portion of W', the initial portions of W and W', and the portion of C between v_{d-c} and v'_{d-c} via v_{d-t}. The length of B is $c+4s+2$, which is less than $2c$. Finally, let D be the cycle formed by $W - v'_c$, the edge from $W - v'_c$ to v'_{c-1}, and the portions of P' and P meeting at v_{d-t}. The length of D is $2c$ when c is odd and $2c+1$ when c is even. Together with all the cycles isomorphic to these by symmetry, we have the desired list of cycles in G'.

Now consider vertices in $P \cap P'$. It suffices by symmetry to specify paths of length at most c from v_d to $W \cup X \cup (P - \{v_0, \ldots, v_{d-c-1}\})$. We reach those on P since the length of P is d. We reach w_1 within distance c by traveling along P until the last step. From v_d, the path along P to W', along W' to X, along X to W, and along W to w_2 has length exactly c. If instead we travel in the other direction on W for the last segment, we can reach s of those vertices within distance c because $3s+2 \leq \lceil c/2 \rceil$ yields $s \leq \lfloor c/2 \rfloor - 2s - 1$. We reach the remainder of W from v_d along P' and W; the length is at most $\lfloor c/2 \rfloor + 3s + 2$, which is at most c.

It now remains only to exhibit, for each $u \in V(G')$, a vertex $f(u) \in V(G)$ such that $d_G(u, f(u)) = c$. By symmetry, we need only consider (roughly) half the vertices in each of X, W, P'. Let $f(v_{d-c}) = f(w_1) = v_d$; these distances we have computed. For $2 \leq j \leq 1 + \lfloor c/2 \rfloor$, let $f(w_j) = v'_{c-j+1}$, except that when c is even and $j = 1 + c/2$, let $f(w_j) = v'_{c-j+2}$.

Observe that w_j and $f(w_j)$ lie on the cycle D. Except for the exception, the distance between them on D via v'_{d-c} is $j + 2s + (c-j+1) - (d-c) = c$. When $j = 1 + \lfloor c/2 \rfloor$, the path from w_j to v'_{c-j+1} through the junctions $w_{\lfloor c/2 \rfloor + 2 + 2s}$, $w'_{\lfloor c/2 \rfloor - 2s + 1}$, w'_1, and v'_{d-c+1} also has length c if c is odd, but it is a shortcut of length $c-1$ if c is even. In this case, setting $f(w_j)$ to v'_{c-j+2} brings the length of the "shortcut" back to c. This distance computed along the *other half* of D is now also c. By checking the possible alternative routes, it can be verified that no other shortcuts arise. For example, when $j \leq \lfloor c/2 \rfloor$, one may try the route along X, W', v_c, v'_c; the length is again c. One may try the route along $w_1, v_{d-c+1}, v_c, v'_c$; the length is $c - 3 + 2j$, which exceeds c for $j \geq 2$ and explains our choice of $f(w_1)$.

Since distance is symmetric, we may assign $f(f(w_j)) = w_j$. Since $c - \lfloor c/2 \rfloor + 1 \leq d/2$, we have thus also established eccentricity c for a vertex of each isomorphism class on P'. It remains only to consider X. Let x_j be the vertex of X whose distance from the junction $w_{\lfloor c/2 \rfloor - 2s + 1}$ is the same as that of w_j, where $j > \lfloor c/2 \rfloor - 2s$. It suffices to consider $\lfloor c/2 \rfloor - 2s < j \leq \lfloor c/2 \rfloor - s$, by symmetry. Let $f(x_j) = f(w_j) = v'_{c-j+1}$. Observe that v'_{c-j+1} is the vertex diametrically opposite x_j on the cycle C' of length $2c$. There are no shortcuts. □

We make no attempt to show that $n = 3d + c - 3$ is the minimum number of vertices that permits such a construction. When $d - c = 3$, the constraint $c \geq 3d/4$ requires $d \geq 12$. Since all cases with $c \leq 2d/3$ are settled, this leaves $(d, c) = (10, 7)$ and $(d, c) = (11, 8)$. In Figure 6 we illustrate special constructions for these cases. These constructions are not unique. Another small family of constructions with $f_{n,d}(c) = n - 3(d - c) + 3$ resolves the open cases for $d - c = 5$ and $16 \leq d \leq 19$ (also for $d - c = 3$ using a few more vertices than Figure 6). The family also has constructions for $(d, c) = (22, 15)$ and $(d, c) = (23, 16)$; in Figure 7 we show its largest member. Except for $(22, 15)$ and $(23, 16)$, the gap of 1 remains between the upper bound and construction when $2d/3 < c < 3d/4$ and $d - c$ is odd and $d - c \geq 7$.

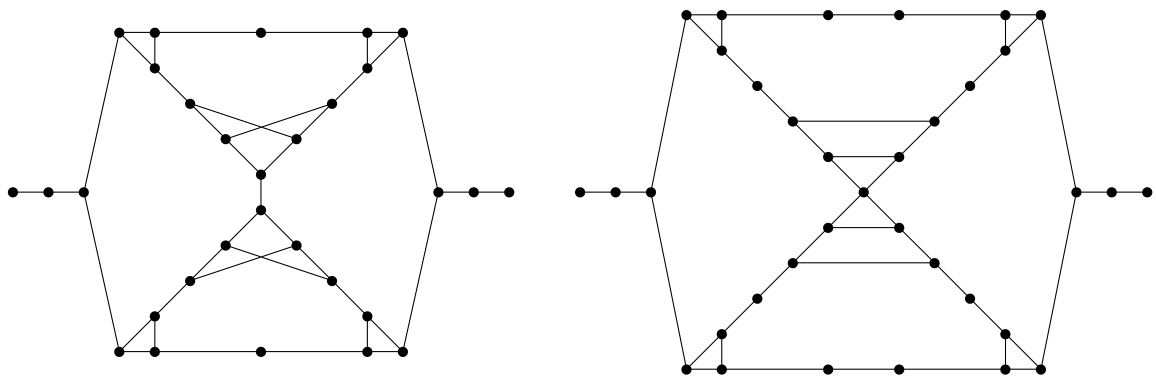

Fig. 6. $g_{n,10}(7) = 6$ for $n \geq 30$, and $g_{n,11}(8) = 6$ for $n \geq 35$.

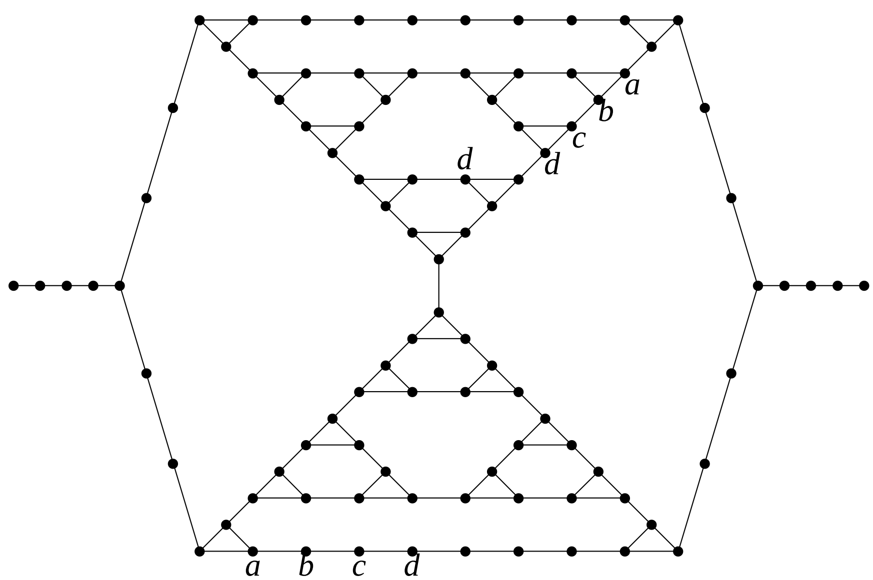

Fig. 7. $g_{n,23}(16) = 18$ for $n \geq 96$.