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NOTE

TWO EASY DUALITY THEOREMS FOR PRODUCT PARTIAL ORDERS

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Two duality theorems are proved about the direct product of two partial orders. First, the size of the largest unichain (a chain fixed in one coordinate) equals the smallest number of semiantichains (collections of elements in which no pair are comparable if they agree in either coordinate) needed to cover the elements of the product order. With analogous definitions, the size of a largest uniantichain equals the size of a smallest semichain covering.

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In this note we prove two easy duality theorems yielding integer max-min relations for integer programming problems arising from products of partial orders. Our interest in these results stems from their relation to two additional integer programming problems for which equality has not been established. The results given here and the open 'companion' questions generalize well-known results for ordinary partial orders (henceforth 'posets').

A *chain* is a totally ordered subset of a poset; an *antichain* is a totally unordered subset. The classical theorem of Dilworth [1] states that the largest antichain in a poset has the same size as the smallest covering of its elements by chains. Since any antichain intersects any chain at most once, Dilworth's result may be interpreted as the following integral max-min relation, where C is the incidence matrix of (maximal) chains versus elements for the poset and $\mathbf{1}$ represents a vector of ones of appropriate length:

$$\max\{\mathbf{1}\cdot x\colon Cx\leq \mathbf{1},\, x\geq \mathbf{0},\, x \text{ integer}\}=\min\{\mathbf{1}\cdot y\colon yC\geq \mathbf{1},\, y\geq \mathbf{0},\, y \text{ integer}\}.$$

Dilworth's Theorem thus provides a prototype for integral max-min relations that arise in Fulkerson's theory of antiblocking pairs of polyhedra (see [6]). This theory suggest asking whether such a relation still holds when the roles played by chains and antichains are reversed. In the poset literature, these are referred to as the 'con-

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jugate' pair of integer programs. As Fulkerson points out, the answer is affirmative: the largest chain and smallest antichain covering have the same size. Furthermore, this result is easily proved, and a minimum covering obtained, by successively stripping off antichains consisting of all currently maximal elements. Fulkerson also notes that, although this latter result appears entirely symmetric to Dilworth's Theorem, the result of Dilworth is evidently deeper. Loosely speaking, an 'augmenting path' procedure is apparently needed to give an algorithmic proof of Dilworth's theorem (see [5]), in contrast to the simple 'greedy' procedure outlined above. Even the non-algorithmic proofs of Dilworth's Theorem (see [1, 10]) are more difficult than that of the conjugate result.

Thus, one encounters an apparent increase in complexity when passing from the 'maximum chain' problem to the 'minimum chain cover' problem. This phenomenon occurs frequently in combinatorial optimization. For example, a 'greedy' procedure suffices to find independent set of maximum weight in a matroid [4], but [2] uses an augmentation-type algorithm to find a minimum collection of independent sets to cover a matroid's elements. Similarly, determining the largest degree of a vertex (largest set of edges forming a 'star') in a bipartite graph is trivial, but covering the edges by vertices (actually, by the corresponding 'stars') apparently requires an augmenting path procedure (see, e.g., [5]). For both of these problems, as for the poset problems above, integral max-min relations hold. The largest vertex degree equals the edge-coloring number (smallest cover by matchings, i.e., by sets of non-incident edges), by a theorem of König, and the smallest vertex cover has the same size as the maximum matching, by the König–Hall–Egervary Theorem (see [6]).

As a final example, consider the maximum matching problem in a general graph. Edmonds's algorithm [3] yields a solution; it and more recent refinements are of the augmenting-path type. Again, passing to the conjugate problem of determining the edge-coloring produces an evident jump in complexity; Holyer [9] has recently shown the latter problem to be NP-complete. No integral max-min relation holds; a triangle has no pair of independent edges, but two vertices are needed to cover the edges. Similarly, the edge-coloring number of the triangle exceeds its maximum degree by one.

In summary, there seem to be three levels of complexity for packing and covering problems: those solvable by greedy algorithms, those solvable by augmenting-path algorithms, and those that are NP-complete. A problem and its conjugate often lie at neighboring levels. When they lie at the two lower levels, integral max-min relations usually follow. When they lie at the two upper levels, such relations generally fail. (Although the examples cited above are of the 'antiblocking' type, we note that examples of the 'blocking' type are also well known – see [6].)

Interest in the first integral max-min relation obtained below arises from the open question of whether the similar relation holds for the conjugate pair of problems. The result here is established by a greedy procedure, which suggests that the conjugate problems may also satisfy an integral max-min relation and be solvable by augmentation methods. The second relation seems to require an augmenting path procedure. Thus it would be of considerable interest to know the extent to which the conjugate problems obtained from those studied below follow the pattern outlined above.

These problems concern certain combinatorial structures in direct products of finite posets, structures that generalize the antichains and chains of general finite posets. The *direct product* $P\times Q$ of two posets P and Q is an order defined on the product of their underlying sets by $(p, q)\le(p', q')$ if and only if $p\le p'$ and $q\le q'$. A *unichain* in $P\times Q$ is a chain in which one of the coordinates is fixed. A *semiantichain* in $P\times Q$ is a collection of elements in which pairs of elements are not comparable if they agree in either coordinate. We show that the size of the largest unichain in a direct product equals the size of the smallest covering of the elements by semiantichains. The related open question is whether the size of the largest semiantichain always equals the size of the smallest covering by unichains [11, 12, 13]. If so, Dilworth's Theorem follows by letting one of the component orders be a single point. As discussed in [13], the Greene–Kleitman [8] generalization of Dilworth's Theorem also follows, by letting one of the orders be a chain of k elements.

Theorem 1. *The largest unichain in a direct product of posets has the same size as the smallest covering by semiantichains.*

Proof. The size of the largest unichain in $P\times Q$ is the size of the longest chain in P or Q. Let k be the size of the largest chain in P, l the size of the largest chain in Q, and $m=\max\{k, l\}$. We need to obtain a semiantichain covering of size m.

Let A and B be partitions of P and Q into antichains, obtained by successively stripping off all the maximal elements, with A_i and B_i denoting the ith antichains from the top in the respective collections. Note that $|A|=k$ and $|B|=l$. Pairing A_i with B_j yields an antichain $A_i\times B_j=\{(a, b): a\in A_i, b\in B_j\}$ in $P\times Q$. Each of the elements in $P\times Q$ appears in exactly one antichain in the collection $A\times B=\{A_i\times B_j: A_i\in A, B_j\in B\}$.

We need only piece together these kl antichains into m semiantichains $S_1,\dots,S_m$. For distinct indices $i_1,\dots,i_r$ and distinct indices $j_1,\dots,j_r$, the collection of elements $A_{i_1}\times B_{j_1}\cup\cdots\cup A_{i_r}\times B_{j_r}$ is a semiantichain. Partitioning the index pairs $\{1,\dots,k\}\times\{1,\dots,l\}$ into m such sets is equivalent to providing a $k\times l$ Latin rectangle with m labels. A circulant labeling suffices, i.e., $A_i\times B_j\subset S_{i+j-1\ (\mathrm{mod}\ m)}$. □

Reversing the roles played by chains and antichains in passing to the direct product leads to additional natural problems. Define a *uniantichain* to be an antichain in $P\times Q$ in which one of the coordinates is fixed, and define a *semichain* to be a collection of elements of $P\times Q$ in which pairs of elements must be comparable if they agree in either coordinate. Here again the open question is whether the largest semichain and smallest covering by uni-antichains must have the same size. Analagously to the preceding situation, letting one of the component orders be a single point yields the trivial "max chain size = min antichain cover size", and letting one of the component orders be an antichain of k elements yields Greene's generalization [7] of that.

However, for the largest uniantichain and smallest semichain covering we can mimic the preceding proof to obtain the theorem below. This time we must begin with a minimum chain decomposition of the component orders, so we place the algorithm in the 'augmenting path' class.

Theorem 2. *The largest uniantichain in a direct product of posets has the same size as the smallest covering by semichains.*

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