Degree-associated Reconstruction Number of Graphs

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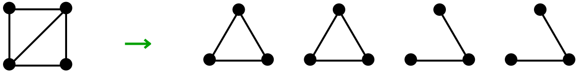
Joint work with
Michael Barrus

The Classical Problem

Def. A card of a graph G is an induced subgraph $G - v$. The deck of a graph is the multiset of its cards.



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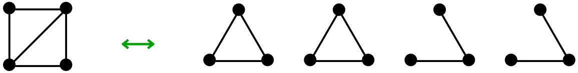

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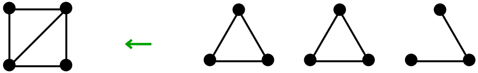

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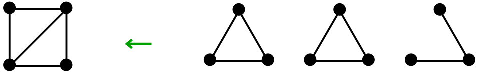

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Def. (Harary-Plantholt [1985]) The reconstruction number $\mathrm{rn}(G)$ is the minimum number of cards needed to determine G.

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How much can be saved?

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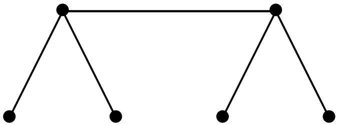
- **Trees:** sufficient conditions for $\mathrm{drn}(G) = 2$.
Main Result: If G is a caterpillar, then $\mathrm{drn}(G) = 2$ unless G is a star or the graph below:



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Cor. Characterization of when $\mathrm{drn}(G) = 1$.

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Let H share $k+2$ dacards with G, including (C, k). Note that $C = H - u$ for some u. If $H \not\cong G$, then u has a neighbor in H with degree k in C, so $\Delta(H) = k+1$.

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Equality holds for $tK_{m,m}$ with $t > 1$;
that is, $\text{drn}(tK_{m,m}) = m+2$ (Ramachandran [2006]).

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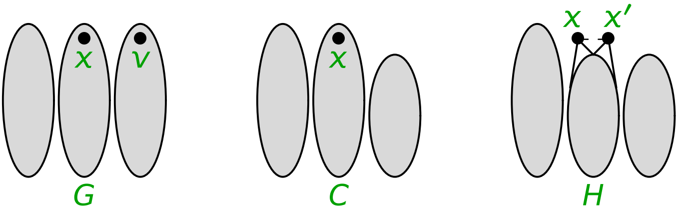

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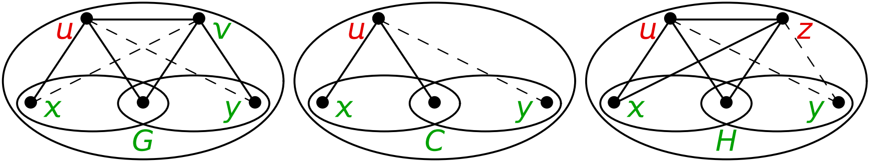
Otherwise, pick $u \in N(v)$ with $N[u] \neq N[v]$. Form H by adding to C a clone z of u. Now $H - u = H - z = C$ and $d_H(u) = d_H(z) = d$, but $d_H(x) > d_C(x) > d_c(y) = d_H(y)$. ■



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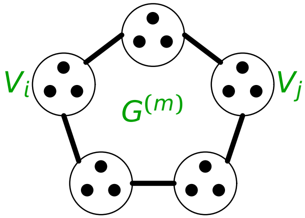

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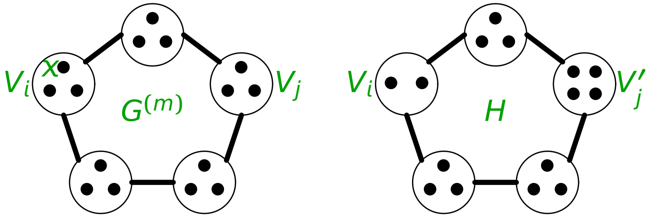

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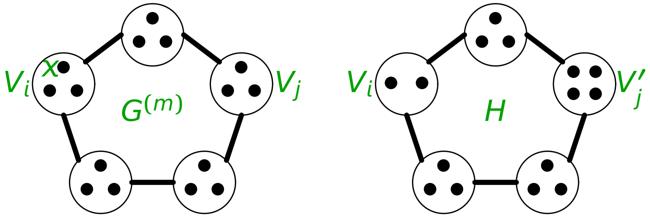

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$\therefore$ $m + 1$ dacards are shared, but $H \not\cong G^{(m)}$. ■

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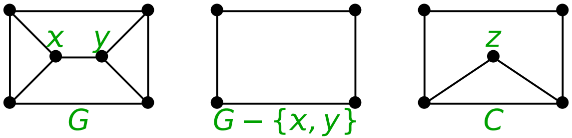

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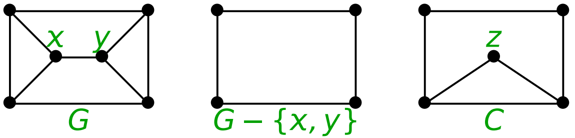

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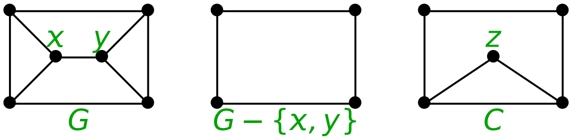

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Ex. The Petersen graph, hypercube (dimension ≥ 3), and both $K_n \square K_2$ and $C_n \square K_2$ (for $n \ge 4$) are all coherent, by ad hoc arguments. Hence they satisfy $\mathrm{drn}(G) = 3$.

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The Plan: First prove it for caterpillars with special structure. In general, show that the dacards obtained by deleting the first spine vertex and a leaf neighbor of it determine T unless T has special structure.

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Lem. If $n \ge 4$, then $\mathrm{drn}(P_n) = 2$.

Pf. Forced by two dacards [only] using a centroid vertex and a non-centroid neighbor of a centroid vertex. ■

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Def. Deleting all leaves of a tree yields its skeleton.
A caterpillar is a tree whose skeleton is a path, its spine.
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Thm. If $T = C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$ and $T \neq P_{s+2}$, let $r = \min\{i\colon a_i > 0 \text{ and } i > 1\}$. The two dacards formed by deleting the leaf neighbor of v_1 and a leaf neighbor of v_r determine T unless
(1) $T = C(1, 0^{(p)}, 1, 0^{(q)}, 1)$ with $p, q \geq 1$, or
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Thm. If T is a caterpillar with special structure listed above, then $\text{drn}(T) = 2$.

The Final Steps

More Special Structures:

If $T = C(k+1, k^{(m)}, k+1)$ with $k, m \geq 1$, then $\text{drn}(T) = 2$.
If $T = C(2, 0^{(s-2)}, 2)$ with $s \geq 3$, then $\text{drn}(T) = 2$.
If $T = C(k+2, (0,k)^{(m)}, 0, k+2)$ with $k \geq 0$ and $m \geq 1$, then $\text{drn}(T) = 2$.

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Thm. If $T = C(a_1, \ldots, a_s)$, then the dacards for the first endpoint of the spine and one of its leaf neighbors determine T unless $(a_1, \ldots, a_s)$ is one of four exceptional types.

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Thm. If $T = C(a_1, \ldots, a_s)$, then the dacards for the first endpoint of the spine and one of its leaf neighbors determine T unless $(a_1, \ldots, a_s)$ is one of four exceptional types.

Thm. If both $(a_1, \ldots, a_s)$ and $(a_s, \ldots, a_1)$ are of the exceptional types, then $C(a_1, \ldots, a_s)$ is one of the caterpillars with special structure previously shown to be determined by two dacards.

Open Problems

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Conj. Only finitely many trees T satisfy $\mathrm{drn}(T) > 2$.

- So far we know only H_1 and H_2. For a proof like that outlined above, the special structures would include all caterpillars (except H_1). One would hope to choose two dacards in a consistent way to determine all other trees (except H_2), perhaps with more special structures.