Degree-associated reconstruction number of graphs

Michael D. Barrus[*] Douglas B. West[†]

March 10, 2010

Abstract

A *card* of a graph G is a subgraph formed by deleting one vertex. The Reconstruction Conjecture states that each graph with at least three vertices is determined by its multiset of cards. A *dacard* specifies the degree of the deleted vertex along with the card. The *degree-associated reconstruction number* $\mathrm{drn}(G)$ is the minimum number of dacards that determine G. We show that $\mathrm{drn}(G) = 2$ for almost all graphs and determine when $\mathrm{drn}(G) = 1$. For k-regular n-vertex graphs, $\mathrm{drn}(G) \le \min\{k+2, n-k+1\}$. For vertex-transitive graphs (not complete or edgeless), we show that $\mathrm{drn}(G) \ge 3$, give a sufficient condition for equality, and construct examples with large drn. Our most difficult result is that $\mathrm{drn}(G) = 2$ for all caterpillars except stars and one 6-vertex example. We conjecture that $\mathrm{drn}(G) \le 2$ for all but finitely many trees.

1 Introduction

The well-known Graph Reconstruction Conjecture of Kelly [7, 8] and Ulam [22] has been open for more than 50 years. It asserts that every graph with at least three vertices can be (uniquely) reconstructed from its "deck" of vertex-deleted subgraphs. Here the *deck* of a graph G is the multiset of unlabeled induced subgraphs formed by deleting one vertex from G, and these subgraphs are *cards* in the deck. The conjecture has been proved for many special classes, and many properties of G may be deduced from the deck. Nevertheless, the full conjecture remains open. Surveys of results on reconstruction include [3, 4, 9, 10].

Usually, a graph is determined by less than its full deck. Introduced by Harary and Plantholt [6], the *reconstruction number* of a graph G, denoted $\mathrm{rn}(G)$, is the minimum number of cards from the deck of G that suffice to determine G, in the sense that no graph not isomorphic to G has this multiset in its deck (a graph may have many copies of a single card in its deck). The Reconstruction Conjecture is the statement that $\mathrm{rn}(G)$ is well defined for each graph G with at least three vertices (with $\mathrm{rn}(G) \le |V(G)|$). Reconstruction numbers are known for various classes of graphs; see [1, 6, 11, 12, 14, 15].

Motivated by reconstruction questions for directed graphs, Ramachandran [18] proposed a variation. A *degree-associated card* (or *dacard*) of a graph (or digraph) is a pair (C, d)

[*]Department of Mathematics, University of Illinois, Urbana, IL 61801, `mbarrus2@math.uiuc.edu`.

[†]Department of Mathematics, University of Illinois, Urbana, IL 61801, `west@math.uiuc.edu`. Research partially supported by the National Security Agency under Award No. H98230-09-1-0363.

consisting of a card C in the deck and the degree (or in/out-degree pair) d of the deleted vertex. The multiset of dacards is the *dadeck* (the *degree-associated deck*). Ramachandran [21] defined the *degree-associated reconstruction number* $\mathrm{drn}(G)$ of a graph G to be the minimum number of dacards that suffice to determine G. We abbreviate the term to *degree-reconstruction number*. Ramachandran studied it for complete graphs, edgeless graphs, cycles, complete bipartite graphs, and disjoint unions of identical graphs.

Each dacard provides more information than the corresponding card, so $\mathrm{drn}(G) \le \mathrm{rn}(G)$ for every graph G. Supplying the degree of the missing vertex is equivalent to supplying the total number of edges in the graph with the card. In contrast, a single card never determines $|E(G)|$. The usual counting argument for determining $|E(G)|$ from the deck uses all the cards, although Myrvold [16] showed that for an n-vertex graph the number of edges and the vertex degrees can be determined from $n-1$ cards (if $n \ge 7$). The point here is that since the full deck determines the total number of edges, the full deck provides the *same information* as the full dadeck, but a partial dadeck generally carries more information than the corresponding partial deck.

In this paper we continue the study of degree reconstruction numbers. Myrvold [13] and Bollobás [2] proved that $\mathrm{rn}(G) = 3$ for almost every graph. From this result, we conclude in Section 2 that $\mathrm{drn}(G) \le 2$ for almost every graph. We prove that $\mathrm{drn}(G) = 1$ if and only if G or its complement $\overline{G}$ has an isolated vertex or a vertex of degree 1 whose deletion leaves a vertex-transitive graph. We also prove that $\mathrm{drn}(G) \le \min\{k+2, n-k+1\}$ when G is a k-regular graph with n vertices.

In Section 3 we study vertex-transitive graphs. For a vertex-transitive graph G, we prove that $\mathrm{drn}(G) \ge 3$ when G is not complete or edgeless, and we give a sufficient condition for equality. We prove that this condition holds for the Petersen graph, the k-dimensional hypercube, and the cartesian product of a complete graph with an edge. The condition is sufficient but not necessary, since it fails for the n-vertex cycle C_n, even though $\mathrm{drn}(C_n) = 3$ for $n \ge 4$. Also, if G has nonadjacent vertices with distinct neighborhoods, and $G^{(m)}$ arises from G by expanding each vertex into a set of m independent vertices, then $\mathrm{drn}(G^{(m)}) = rm + 2$, where r is the maximum number of vertices in G having the same neighborhood. As a special case, $\mathrm{drn}(tK_{m,m}) = m + 2$ for $t > 1$ (Ramachandran [21]), where $K_{m,m}$ is the complete bipartite graph with parts of size m, and tG denotes the disjoint union of t copies of G. These results suggest a natural extremal problem for drn.

Conjecture 1.1. *If G is an n-vertex graph, then $\mathrm{drn}(G) \le n/4 + 2$ (equality for $2K_{n/4,n/4}$).*

In Sections 4–6 we study trees. Section 4 gives sufficient conditions for $\mathrm{drn}(G) = 2$ when G is a tree. These aid subsequently in computing $\mathrm{drn}(G)$ when G is a *caterpillar*, which is a tree whose non-leaf vertices form a path. If G is a caterpillar, then $\mathrm{drn}(G) = 2$ unless G is a star or the 6-vertex tree with four leaves and maximum degree 3. This is our longest and most difficult result. We consider special families of caterpillars in Section 5 and complete the general proof in Section 6. Our study of caterpillars is motivated by the following:

Conjecture 1.2. *If G is a tree, then $\mathrm{drn}(G) \le 2$, with finitely many exceptions.*

In many reconstruction arguments, reconstructibility is proved first for special subfamilies where the general argument does not work; this occurs for example in the classical argument for reconstruction of trees. Our proof for caterpillars has this form, where the proof in Section 6 works because we may exclude the special subfamilies treated in Section 5. Similarly, our result for caterpillars could be a steppingstone to a full proof of Conjecture 1.2.

Conjecture 1.2 is supported by known results about reconstruction of trees. For a family $\mathcal{F}$ of graphs, the *$\mathcal{F}$-reconstruction number* or *class reconstruction number* of a graph G in $\mathcal{F}$ is the minimum number of cards from its deck needed to determine G given the knowledge that $G \in \mathcal{F}$; that is, G is the only graph in $\mathcal{F}$ having this multiset of cards in its deck. For the family $\mathcal{T}$ of trees, Harary and Lauri [5] proved that the class reconstruction number of every tree T is at most 3 (the result of Myrvold [15] that $\mathrm{rn}(T) \le 3$ strengthens this), and they conjectured that the class reconstruction number of every tree is at most 2.

Recently, Welhan [23] obtained a structural condition on a tree T that is sufficient for the class reconstruction number of T to be 2. The condition holds, for example, for all trees having no vertices of degree 2. Furthermore, he notes that among these trees, all except a two-parameter family are class reconstructible from two cards such that one of the cards arises by deleting a leaf. If G has a dacard that is a tree with the deleted vertex having degree 1, then G must be a tree. Hence the dacards corresponding to his two cards imply that $\mathrm{drn}(G) \le 2$ when G is such a tree. This and our result for caterpillars, which can have many vertices of degree 2, together provide support for Conjecture 1.2.

We summarize terminology and notation used throughout the paper. Our graphs are "simple" (no loops or multiedges). For a graph G, the vertex set and edge set are $V(G)$ and $E(G)$. The *(open) neighborhood* $N_G(v)$ and *closed neighborhood* $N_G[v]$ of a vertex v in G are defined by $N_G(v) = \{u \in V(G)\colon\ uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$. Let $d_G(v)$ or simply $d(v)$ denote the degree in G of vertex v, which equals $|N_G(v)|$. The maximum and minimum vertex degrees are $\Delta(G)$ and $\delta(G)$. A vertex v in G is *isolated* if $d_G(v) = 0$, a *leaf* if $d_G(v) = 1$, and *dominating* if $N_G[v] = V(G)$. Given $S \subseteq V(G)$, the subgraph $G[S]$ *induced by* S is the graph with vertex set S in which two vertices are adjacent if and only if they are adjacent in G. For $v \in V(G)$, write $G - v$ for $G[V(G) - v]$.

2 Small reconstruction numbers and regular graphs

As mentioned earlier, Bollobás [2] and Myrvold [13] determined the reconstruction numbers of almost all graphs.

Theorem 2.1 ([2, 13])**.** *Almost every graph has reconstruction number 3 (and hence is reconstructible). Furthermore, for almost every graph, any two cards in the deck determine everything about the graph except whether the two corresponding deleted vertices are adjacent.*

The reconstruction number of any graph is at least 3, since $G - u$ and $G - v$ are cards for both G and G', where G and G' differ only on whether the edge uv is present. Thus, the previous result is sharp. Converting two cards to dacards adds the degree information, which determines the last unknown bit of information without introducing another dacard. This immediately yields our first observation:

Corollary 2.2. *For almost every graph G,* $\mathrm{drn}(G) \le 2$.

Our next aim is to characterize the graphs G such that $\mathrm{drn}(G) = 1$. Let $\overline{G}$ denote the complement of a graph G.

Lemma 2.3. *For any graph G,* $\mathrm{drn}(G) = \mathrm{drn}(\overline{G})$.

Proof. Let v be a vertex in an n-vertex graph G. Since $d_{\overline{G}}(v) = n - 1 - d_G(v)$ and $\overline{G - v} = \overline{G} - v$, it follows that (C, d) is a dacard of G if and only if $(\overline{C}, n - 1 - d)$ is a dacard of $\overline{G}$. Since also G and $\overline{G}$ determine each other, we conclude that the dacards of G from a vertex subset S determine G if and only if the dacards of $\overline{G}$ from S determine $\overline{G}$. □

Note that $\mathrm{drn}(G) = 1$ if and only if G has a dacard that does not occur in the dadeck of any other graph. We next determine all dacards of this type.

Theorem 2.4. *The dacard (C, d) belongs to the dadeck of only one graph (up to isomorphism) if and only if one of the following holds:*
(1) *$d = 0$ or $d = |V(C)|$;*
(2) *$d = 1$ or $d = |V(C)| - 1$, and C is vertex-transitive;*
(3) *C is complete or edgeless.*

Proof. Sufficiency. In each case listed, all graphs formed by adding to C a vertex with d neighbors in C lie in the same isomorphism class.

Necessity. If (C, d) is a dacard for only one graph, then the same isomorphism class is produced no matter what set of d vertices is chosen for the neighborhood of the added vertex v. Since isomorphic graphs have the same number of triangles, and the number of triangles after adding v is the number of triangles in C plus the number of edges in C induced by the vertices made adjacent to v, we conclude that all d-vertex induced subgraphs of C have the same number of edges. It is a well-known exercise (see Exercise 1.3.35 on page 50 of [24]) that if $1 < d < |V(C)| - 1$, then this property forces C to be complete or edgeless, as in (3).

Since $d \in \{0, |V(C)|\}$ is covered by (1), the remaining case is $d \in \{1, |V(C)| - 1\}$. Since (C, d) determines G if and only if $(\overline{C}, |V(G)| - 1 - d)$ determines $\overline{G}$, we may assume $d = 1$. We conclude that C is regular, since otherwise giving v one neighbor would make (C, d) a dacard for a graph with maximum degree $\Delta(C)$ and a graph with maximum degree $\Delta(C)+1$.

When C is regular of degree 0 or 1, it is vertex-transitive. For larger degree, every automorphism of the resulting graph G fixes v, since it is the only vertex of degree 1. Since attaching v to any vertex yields the same graph, C must therefore have automorphisms taking each vertex to any other. Hence C is vertex-transitive. □

Interpreting the statement of Theorem 2.4 in terms of the reconstructed graph, we obtain the following corollary.

Corollary 2.5. *A graph G satisfies* $\mathrm{drn}(G) = 1$ *if and only if G or $\overline{G}$ has an isolated vertex or has a leaf whose deletion leaves a vertex-transitive graph.*

Together, Corollaries 2.2 and 2.5 imply that almost always $\mathrm{drn}(G) = 2$. Graphs with vertices of degree at most 1 are rare; it is a standard elementary result about random graphs that almost all graphs (and their complements) have minimum degree at least 2. Thus almost no graphs are determined by one dacard.

Next we consider regular graphs. Every regular graph G is reconstructible, since the degree list can be determined from the deck, and then in any card the vertices of minimum degree must be the neighbors of the missing vertex. Although one dacard gives the degree of the missing vertex and hence the total number of edges, it does not give the degree list and does not determine G. Nevertheless, we obtain an upper bound on $\mathrm{drn}(G)$.

Theorem 2.6. *If G is a k-regular graph on n vertices, then* $\mathrm{drn}(G) \le \min\{k+2, n-k+1\}$.

Proof. Since the complement of a k-regular graph is $(n-1-k)$-regular, by Lemma 2.3 it suffices to prove that $\mathrm{drn}(G) \le k+2$.

Let H be a graph that shares $k+2$ dacards with G. Let (C, k) be one shared dacard, so $C = H - u$ for some $u \in V(H)$. Since C also arises by deleting one vertex from the k-regular graph G, the graph C has k vertices of degree $k-1$ and $n-1-k$ vertices of degree k.

Attaching u to the k vertices of degree $k-1$ in C forms a copy of G. If $H \ncong G$, then some vertex $v \in N_H(u)$ has degree k in C and hence degree $k+1$ in H. With $\Delta(H) = k+1 > \Delta(G)$, each vertex of degree k in H whose deletion produces a card of G must be adjacent in H to every vertex of degree $k+1$ in H. There can be at most $k+1$ such vertices, which contradicts the assumption of $k+2$ shared dacards with G. Hence $H \cong G$. □

Ramachandran [21] proved that $\mathrm{drn}(tK_{m,m}) = m+2$ when $t > 1$. Since $tK_{m,m}$ is m-regular, these graphs prove sharpness of the upper bound in Theorem 2.6. Ramachandran [21] proved for $k, t \ge 2$ that if G is a connected k-regular graph on n vertices, where $n \ge 3$, then $\mathrm{drn}(tG) \le n-k+2$.

In comparing drn and rn for regular G, an argument like that above yields $\mathrm{rn}(G) \le b+1$, where b is the upper bound in Theorem 2.6. We have observed that almost always $\mathrm{drn}(G) = 2 = \mathrm{rn}(G) - 1$. Nevertheless, $\mathrm{drn}(G)$ and $\mathrm{rn}(G)$ can differ greatly: for $t, m > 1$, Ramachandran [21] proved that $\mathrm{drn}(tK_m) = 3$ even though $\mathrm{rn}(tK_m) = m+2$ (Myrvold [14]).

Since two cards never determine whether the two deleted vertices are adjacent, always $\mathrm{rn}(G) \ge 3$. Hence the parameters differ by more than 1 when $\mathrm{drn}(G) = 1$. This case and the family $\{tK_m\}$ are the only infinite families we presently know consisting of graphs G such that $\mathrm{drn}(G) \ne \mathrm{rn}(G) - 1$. Isolated examples include the 4-vertex path P_4 and two small trees with degree-reconstruction number 3 presented in Section 4; Myrvold [15] proved that $\mathrm{rn}(G) = 3$ for every tree with more than two vertices other than P_4, while $\mathrm{drn}(P_4) = 2 = \mathrm{rn}(P_4) - 2$.

3 Vertex-transitive graphs

For a regular graph G that is also vertex-transitive, we obtain sharper results on $\text{drn}(G)$. A graph is vertex-transitive if and only if its cards are pairwise isomorphic. Since vertex-transitive graphs are regular, Theorem 2.6 provides an upper bound. We will prove further lower and upper bounds and give sufficient conditions for equality in the bounds.

Since $\text{drn}(G) = 2$ almost always, higher values require some sort of special structure. When the dacards are identical, the only flexibility is how many to use; one may therefore expect vertex-transitive graphs to be harder to reconstruct from dacards. As noted above, $\text{drn}(tK_{m,m}) = m + 2$ when $t > 1$, and $\text{drn}(tK_m) = 3$. By setting $t = 2$ in the latter example and applying $\text{drn}(\overline{G}) = \text{drn}(G)$, also $\text{drn}(K_{m,m}) = 3$. We prove next that 3 is a lower bound.

Definition 3.1. A *clone* of a vertex x in a graph is a vertex having the same closed neighborhood as x. When G is edge-transitive, let G^- denote the (unlabeled) graph formed by deleting any edge of G.

As we have noted, the cards of a vertex-transitive graph G are pairwise isomorphic. Given a dacard (C, d) of G, we refer to other dacards of G as "copies" of (C, d). We usually start with C obtained as $G - v$ for some $v \in V(G)$, but we may also describe the structure of C as an unlabeled graph. We use $G + H$ to denote the disjoint union of graphs G and H (in the sense of isomorphism classes).

Theorem 3.2. *If G is vertex-transitive and is not complete or edgeless, then* $\text{drn}(G) \geq 3$.

Proof. Let (C, d) be a dacard of G, where $C = G - v$. To show that $\text{drn}(G) > 2$, we construct a graph H not isomorphic to G that has at least two copies of (C, d) in its dadeck.

If every neighbor of v in G is a clone of v, then G is a disjoint union of complete graphs. That is, $G = tK_r$ with $t \geq 2$ and $r \geq 2$, where $r = d + 1$. In this case, $C = (t - 1)K_r + K_{r-1}$. Let $H = (t-2)K_r + K_{r+1}^- + K_{r-1}$. Now H has two copies of (C, d) in its dadeck, and $H \not\cong G$.

Otherwise, choose $u \in N_G(v)$ with $N_G[u] \neq N_G[v]$. Form H by adding to $G - v$ a clone u' of u. Now $H - u \cong H - u' \cong C$ and $d_H(u) = d_H(u') = d$. However, with $x \in N_G(u) - N_G[v]$ and $y \in N_G(v) - N_G[u]$, we have $d_H(x) = d + 1$ and $d_H(y) = d - 1$; hence $H \not\cong G$. $\square$

We will later give sufficient conditions for equality in the lower bound $\text{drn}(G) \geq 3$. Although the n-vertex cycle C_n does not satisfy those conditions, it does achieve the bound. This easy example (stated in Ramachandran [21] as being in the inaccessible [20]) will be useful later and illustrates the technique for proving upper bounds on $\text{drn}(G)$. Let P_n denote the n-vertex path.

Example 3.3. *If $n \geq 4$, then* $\text{drn}(C_n) = 3$. Theorem 3.2 provides the lower bound. The dacards of C_n are copies of $(P_{n-1}, 2)$. For the upper bound, let H be a graph having three such dacards; H is constructed from P_{n-1} by adding a vertex x with two neighbors in P_{n-1}. Thus H consists of a cycle plus pendant paths from at most two vertices. If there is at least one nontrivial pendant path, then there are at most two vertices whose deletion leaves a path. We conclude that $H \cong C_n$. $\square$

Since $\mathrm{drn}(tK_{m,m}) = m + 2$ when $t > 1$, Theorem 3.2 can be arbitrarily weak. We extend that example, computing $\mathrm{drn}(G)$ on a more general family of vertex-transitive graphs that contains both $tK_{m,m}$ and some connected graphs; $tK_{m,m}$ arises when the base graph is tK_2.

Definition 3.4. An *expansion* of a base graph G is a graph H obtained by replacing each vertex of G with an independent set such that two vertices of H are adjacent if and only if the vertices of G they replaced were adjacent. The *m-fold expansion* $G^{(m)}$ is the expansion of G in which each vertex expands into an independent set of size m. A *twin* of a vertex v is a vertex having the same open neighborhood as v. A *twin-set* in a graph is a maximal vertex subset consisting of vertices with identical open neighborhoods.

A twin-set in a graph is an independent set, while a set of clones is a clique.

Theorem 3.5. *Let G be a vertex-transitive graph that is not a complete graph and has no twins. If $m \geq 2$, then $\mathrm{drn}(G^{(m)}) = m + 2$.*

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$. In $G^{(m)}$, each vertex v_i of G becomes an independent set V_i. Vertices in V_i have the same neighborhood, but vertices in distinct such sets have different neighborhoods, since G has no twins. Hence $V_1, \ldots, V_n$ are the twin-sets in G. If G is k-regular (and vertex-transitive), then $G^{(m)}$ is km-regular and vertex-transitive, and its twin-sets have size m. Every vertex neighborhood in $G^{(m)}$ is a union of twin-sets.

Lower bound. Since G is not complete, it has nonadjacent vertices v_i and v_j. Fix $x \in V_i$, and let $C = G^{(m)} - x$. Construct H by adding to $G^{(m)} - x$ a vertex u with neighborhood $N(V_j)$; this makes $V_j \cup \{u\}$ a twin-set in H. Since $x \notin N(V_j)$, we have $d_H(u) = km$. In $G^{(m)}$ every set of $m + 1$ vertices contains two with distinct neighborhoods, but in H the $m + 1$ vertices in $V_j \cup \{u\}$ have the same neighborhood. Hence $H \not\cong G^{(m)}$, but the $m + 1$ dacards for $V_j \cup \{u\}$ in H are copies of (C, km). Thus $\mathrm{drn}(G^{(m)}) \geq m + 2$.

Upper bound. Again let $C = G^{(m)} - x$ for some $x \in V(G^{(m)})$. Since $m \geq 2$, there are n twin-sets and n distinct vertex neighborhoods in C; one twin-set has size $m - 1$, and the others have size m. Let H be a graph having $m + 2$ dacards that are copies of (C, km). Let u be a vertex of H yielding one such dacard, and let U be the twin-set of size $m-1$ in $H - u$. Since $|U| = m - 1$, among the vertices yielding this dacard is a vertex v not in U. Since the twin-set containing v has size m in $H - u$, in $H - \{u, v\}$ there remain n distinct twin-sets.

If $N_{H-v}(u)$ is not a vertex neighborhood in $H - \{u, v\}$, then replacing u shows that $H - v$ has more than n distinct vertex neighborhoods, which contradicts $H - v \cong C$. Thus $N_{H-v}(u)$ is a vertex neighborhood in $H - \{u, v\}$, which means that $H - v$ is obtained from $H - \{u, v\}$ by augmenting one twin-set T to form T'.

If $T \neq U$, then H is an expansion of G having twin-sets of sizes $m+1$, $m-1$, and the rest of size m. Deleting a vertex from H so that the resulting twin-sets have the same sizes as in C requires deleting a vertex of T'. Since $|T'| = m+1$, the dacard (C, km) cannot occur $m+2$ times for H. We conclude that $T = U$ and $H \cong G^{(m)}$, which implies $\mathrm{drn}(G^{(m)}) \leq m + 2$. $\square$

In a vertex-transitive graph, the twin-sets all have the same size.

Corollary 3.6. *If G is a vertex-transitive graph other than a complete multipartite graph, then* $\mathrm{drn}(G^{(m)}) = rm + 2$ *for every* $m \geq 2$*, where* r *is the size of each twin-set in* G*.*

Proof. Collapsing the twin-sets of G into single vertices yields a vertex-transitive graph G_0 having no twins, and $G = G_0^{(t)}$. Since G is not a complete multipartite graph, G_0 is not a complete graph. Hence Theorem 3.5 applies to G_0, and $\mathrm{drn}(G^{(m)}) = \mathrm{drn}(G_0^{(rm)}) = rm + 2$.□

We next study sharpness in the lower bound of Theorem 3.2. We give a sufficient condition for $\mathrm{drn}(G) = 3$ in the family of vertex-transitive graphs and show that hypercubes and some other graphs satisfy it.

Definition 3.7. A vertex-transitive graph G with card C is *coherent* if for all $x, y \in V(G)$, the only way to form a graph isomorphic to C by adding a new vertex z to $G - \{x, y\}$ is to make z adjacent to $N_{G-y}(x)$ or $N_{G-x}(y)$.

Coherence prevents the deletion of vertices x and y from G in such a way that a graph isomorphic to the card C can be recreated by adding a vertex adjacent to some subset of $N_G(x) \cup N_G(y)$ other than the full neighborhood of x or y.

Theorem 3.8. *Let G be a vertex-transitive graph that is not complete or edgeless. If G is coherent and has no clones or twins, then* $\mathrm{drn}(G) = 3$.

Proof. Let k be the degree of each vertex in G, and let $C = G - x$. Given the lower bound in Theorem 3.2, it suffices to show that if some graph H has vertices u, v, and w of degree k whose deletion yields cards isomorphic to C, then $H \cong G$.

Let S be the set of vertices of degree $k - 1$ in $H - u$. Since $H - u \cong C = G - x$, we may name the vertices of $H - u$ so that $H - u = G - x$ yielding $N_G(x) = S$. Now $H - \{u, v\} = G - \{x, v\}$. The card $H - v$ is obtained by adding u and appropriate edges to $H - \{u, v\}$; doing this adds u and appropriate edges to $G - \{x, v\}$ to produce a graph isomorphic to C. By coherence, $N_{H-v}(u)$ is $N_{G-v}(x)$ or $N_{G-x}(v)$.

If $N_{H-v}(u) = N_{G-x}(v)$, then $|N_H(u) \cap N_H(v)|$ is $k - 1$ or k, depending on whether $v \in N_G(x)$. Since $d_H(u) = k$, this makes u and v clones or twins in H and hence also in $H - w$. Since $H - w \cong C = G - x$, adding a vertex x' and appropriate edges to $H - w$ yields a graph isomorphic to G. Since $d_{H-w}(u) = d_{H-w}(v)$ and G is regular, x' must be made adjacent to neither or both of $\{u, v\}$. Now u and v are clones or twins in a graph isomorphic to G, which is forbidden.

Thus $N_{H-v}(u) = N_{G-v}(x)$. Since $d_H(u) = d_G(x)$, we have $N_H(u) = S$ and $H \cong G$. □

Although $tK_{m,m}$ and tK_m are coherent, $tK_{m,m}$ has twins and tK_m has clones. Since $\mathrm{drn}(tK_m) = 3$, the condition in Theorem 3.8 is not a necessary condition. Similarly, the cycle C_n has no clones or twins and satisfies $\mathrm{drn}(C_n) = 3$ (Example 3.3), but it is not coherent for $n \geq 6$. For vertices x and y separated by distance at least 3 in C_n, adding adding a vertex adjacent to one neighbor of each of $\{x, y\}$ in distinct components of $C_n - \{x, y\}$ is an "incoherent" way to obtain the card P_{n-1}.

Thus Theorem 3.8 does not apply to these graphs. Before applying it to other graphs, we show that coherence is preserved by repeated disjoint union.

Proposition 3.9. *If G is a coherent connected vertex-transitive graph, then tG is coherent.*

Proof. Every connected vertex-transitive graph is 2-connected. If x and y lie in the same component of tG, then the needed property follows from the coherence of G. If they do not, then what remains of each of those components is connected, since G is 2-connected. Thus a vertex added to turn $tG - \{x, y\}$ into a graph isomorphic to a card of tG must restore one of the components of G, which requires it to be adjacent to the neighborhood of the vertex deleted from that component. $\square$

We apply coherence to several natural examples.

Example 3.10. *If G is the Petersen graph, then* $\operatorname{drn}(G) = 3$. Nonadjacent vertices in G have exactly one common neighbor, and adjacent vertices have none; hence G has no twins or clones. It therefore suffices to check coherence. Let C be a card. There are only two types of vertex pairs in G; adjacent or nonadjacent.

Deleting adjacent vertices x and y leaves four vertices with degree 2. Any two of them that had no common neighbor in $\{x, y\}$ have a common remaining neighbor. Adding a vertex adjacent to both of them creates a 4-cycle and hence cannot form C.

Deleting nonadjacent vertices x and y leaves one vertex w with degree 1 and four vertices with degree 2 that induce $2K_2$. A vertex added to form C must be adjacent to w and to one vertex from each edge of this $2K_2$. To avoid creating a 4-cycle, only two of the four such choices are allowable, and these yield the vertex neighborhoods of x and y. $\square$

We next consider the k-dimensional hypercube Q_k, the graph with vertex set $\{0, 1\}^k$ in which two vertices are adjacent if and only if they differ in exactly one coordinate. From the definition, vertices at distance 2 in Q_k have exactly two common neighbors.

Theorem 3.11. *If $k \geq 2$, then* $\operatorname{drn}(Q_k) = 3$.

Proof. The lower bound follows from Theorem 3.2. Since $Q_2 \cong K_{2,2}$, we have $\operatorname{drn}(Q_2) = 3$, so we may assume $k \geq 3$. Since Q_k has no clones or twins, it suffices by Theorem 3.8 to show that Q_k is coherent. Let C be a card of Q_k. Given $x, y \in V(Q_k)$, let $F = Q_k - \{x, y\}$, and let $S = N_{Q_k - y}(x)$ and $S' = N_{Q_k - x}(y)$. Let z be a vertex added to F to obtain a graph C' isomorphic to C; we must show that $N_{C'}(z) \in \{S, S'\}$.

The vertex z cannot have neighbors in both partite sets of F, since C is bipartite. Also it has no neighbor with degree k in F, since $\Delta(C) \leq k$. Hence $N_{C'}(z) \in \{S, S'\}$ when x and y lie in opposite partite sets.

Now consider x and y in the same partite set. Since $\delta(C) = k - 1$ and $\Delta(C) = k$, we have $S \cap S' \subseteq N_{C'}(z) \subseteq S \cup S'$. If $N_{C'}(z) \notin \{S, S'\}$, then z has neighbors in both $S - S'$ and $S' - S$. Since $d_{C'}(z) = k = |S| = |S'|$, there also exist $w \in S - S'$ and $w' \in S' - S$ outside $N_{C'}(z)$. Now $d_{C'}(w) = d_{C'}(w') = k - 1$. Since $C' \cong C$, adding to C' a vertex adjacent to all vertices of degree $k - 1$ produces a graph Q' isomorphic to Q_k. Since $d_{Q'}(w, w') = 2$, these vertices have two common neighbors in Q', and only one remains in C'. Since $w, w' \notin N_{C'}(z)$, the common neighbor lies in F. However, since $F = Q_k - \{x, y\}$, the distance between w

and w' as vertices of Q_k is 2. Since $w \in S - S'$ and $w' \in S' - S$, neither x nor y is a common neighbor of w and w'. Hence w and w' still have two common neighbors in F. The contradiction yields $N_C(z) \in \{S, S'\}$. $\square$

The *cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ such that (u, v) and (u', v') are adjacent precisely when $u = u'$ and $vv' \in E(H)$ or when $v = v'$ and $uu' \in E(G)$. The hypercube Q_k is the cartesian product of k factors isomorphic to K_2. We have not generalized Theorem 3.11 to all cartesian products of complete graphs but can prove it for $K_k \square K_2$. A *k-clique* in a graph is a set of k pairwise adjacent vertices.

Theorem 3.12. *If $k \geq 2$, then* $\mathrm{drn}(K_k \square K_2) = 3$.

Proof. Again the lower bound is from Theorem 3.2. Since $K_2 \square K_2 = C_4$, Example 3.3 yields $\mathrm{drn}(K_2 \square K_2) = 3$. Since the complement of $K_3 \square K_2$ is C_6, and always $\mathrm{drn}(\overline{G}) = \mathrm{drn}(G)$ by Lemma 2.3, Example 3.3 also yields $\mathrm{drn}(K_3 \square K_2) = 3$.

Hence we may assume that $G \cong K_k \square K_2$ with $k \geq 4$. Let C be a card of G. Since G has no clones or twins, by Theorem 3.8 it suffices to show that G is coherent. Given $u, v \in V(G)$, let $F = G - \{u, v\}$, and let $S = N_{G-v}(u)$ and $S' = N_{G-u}(v)$. Let z be a vertex added to F to form a graph C' isomorphic to C; we need $N_{C'}(z) \in \{S, S'\}$. Let A and B be the two k-cliques in G. Note that G is k-regular. By symmetry, we have two cases.

Case 1: $u, v \in A$. Vertices remaining in A have degree $k-2$ in F, and the neighbors of u and v in B have degree $k-1$ in F. Since $\delta(C) = k-1$ and $\Delta(C) = k$, we conclude that $N_{C'}(z)$ contains all of $A - \{u, v\}$ and the neighbor of u or v in B. Hence $N_{C'}(z) \in \{S, S'\}$.

Case 2: $u \in A$, $v \in B$. Since $k \geq 4$, the only $(k-1)$-cliques in F are $A - u$ and $B - v$. Since C has a k-clique, z must be adjacent to all of $A - u$ or $B - v$. Since C has exactly k vertices of degree $k-1$, z has no other neighbor if $uv \in E(G)$ and is adjacent to the remaining vertex of degree $k-2$ in F if $uv \notin E(G)$. In either case, $N_{C'}(z) \in \{S, S'\}$. $\square$

Similar arguments can be made for other families of vertex-transitive graphs. For example, $\mathrm{drn}(C_k \square K_2) = 3$ for $k \geq 3$.

Question 3.13. *Which vertex-transitive graphs are coherent? Which vertex-transitive graphs have coherent cartesian products with K_2?*

4 Trees

When G is not vertex-transitive, the problem of determining $\mathrm{drn}(G)$ becomes harder in two ways, because there are more choices of r-sets of dacards. To prove $\mathrm{drn}(G) \leq r$, we must find such a set that determines G; for vertex-transitive graphs there was only one choice. This increases the difficulty of finding the proof but not necessarily its length. To prove $\mathrm{drn}(G) \geq r$, on the other hand, the larger family of $(r-1)$-sets of dacards does increase the length of proof, because we must ensure for every choice of $r-1$ dacards of G that some graph not isomorphic to G also has those dacards.

With this in mind, we turn next to the study of trees. We conjectured in Section 1 that only finitely many trees other than stars fail to have degree-reconstruction number 2 (Prince [17] proved the weaker statement that $\mathrm{drn}(T) = 2$ for almost every tree T). Since Corollary 2.5 implies that stars are the only trees determined by one dacard, we do not encounter the difficulty of proving lower bounds, and the task is only to provide for each tree a pair of dacards that determine it.

We noted in Section 1 that the recent results of Welhan [23] do this for trees with no vertices of degree 2. In the remainder of this paper we prove it for caterpillars, which may have many such vertices.

Reconstructing a tree includes showing that every reconstruction of the given cards (or dacards) is a tree. We noted in Section 1 that a dacard $(G - v, 1)$ in which $G - v$ is a tree forces G to be a tree. Since we will reconstruct from two dacards, it is useful also to have a condition on two dacards that forces every reconstruction to be a tree.

Lemma 4.1. *Let G be a graph with dacards $(F, 2)$ and $(F', 2)$. If F and F' are forests with two components, and the components of F do not have the same sizes as those of F', then G is a tree.*

Proof. Among the four trees in F and F', by symmetry we may assume that the largest is in F'. By the hypothesis, those in F are strictly smaller. If G is not a tree, then G arises from F by adding a vertex adjacent to two vertices in the same component of F. Now G has no induced subtree as large as the larger tree in F', contradicting that F' is a card. $\square$

The condition of distinct sizes is important. For example, P_n has two copies of the dacard $(P_a+P_{n-1-a}, 2)$ with $a < n/2$. When $n \geq 4$, the graph P_a+C_{n-a} is a non-tree reconstruction from these two dacards. (Recall that $G + H$ denotes the disjoint union of G and H.)

Armed with Lemma 4.1, we start by proving $\mathrm{drn}(P_n) \leq 2$. We need this when proving the bound for more general families of caterpillars, because some of the general arguments that work for those families are not valid for the special case of paths. In particular, we may want to use a dacard from a leaf, but that does not work for paths.

Proposition 4.2. *If $n \geq 4$, then $\mathrm{drn}(P_n) \leq 2$, and only ϵ pairs of dacards determine P_n, where $\epsilon = 1$ when n is even and $\epsilon = 2$ when n is odd.*

Proof. For $n = 4$, we use dacards $(P_3, 1)$ and $(P_1+P_2, 2)$. The first forces every reconstruction to be a tree. Hence the vertex missing from the second has a neighbor in each component, and P_4 is the only reconstruction. (When two copies of the same dacard are used, $K_{1,3}$ or $P_1 + K_3$ is an alternative reconstruction.)

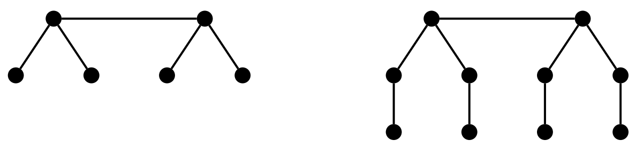
For $n \geq 5$, the dacards $(P_{n-1}, 1)$ and $(P_a+P_{n-1-a}, 2)$ do not determine P_n for any a (except $a = 2$ when $n = 5$), since the tree formed by appending one edge to P_{n-1} at an appropriate place shares these dacards. Consider the dacards $(P_a+P_{n-1-a}, 2)$ and $(P_b+P_{n-1-b}, 2)$, where $1 \leq a \leq b \leq \lfloor (n-1)/2 \rfloor$. We have noted the alternative reconstruction $P_a + C_{n-a}$ when $a = b$, so consider $a < b$. Let G be a graph having these dacards corresponding to vertices u and v, respectively. By Lemma 4.1, G is a tree.

If G is not a path, then the dacards imply that G has a vertex w of degree 3, and every such vertex is adjacent to both u and v. Hence G consists of three paths emanating from w. If $a+b \le n-4$, then making the paths from w through u and v have lengths $a+1$ and $b+1$ provides an alternative reconstruction, since there remains a vertex for the third path.

With $a < b \le \lfloor (n-1)/2 \rfloor$, we have $a+b \le n-3$ when n is even, $a+b \le n-2$ when n is odd. If $a+b \ge n-3$, then there do not remain enough vertices to give w a third neighbor, so the alternative reconstruction does not exist. Hence the unique reconstruction is P_n for $b = \lfloor (n-1)/2 \rfloor$ and $a = b-1$, and also for $b = (n-1)/2$ and $a = b-2$ when n is odd. $\square$

We considered all pairs of dacards, instead of just presenting one pair that works, in order to emphasize that our later general arguments fail for P_n; it must be treated separately.

For the desired bound $\mathrm{drn}(T) \le 2$ for trees, we actually know of only two exceptions, the trees H_1 and H_2 in Figure 1.



H_1 $\qquad$ H_2

Figure 1: Two trees requiring three dacards.

Example 4.3. $\mathrm{drn}(H_1) = 3$. Since Myrvold [15] proved that every tree with at least three vertices other than P_4 has reconstruction number 3 (in fact, $\mathrm{rn}(P_4) = 4$), the observation that always $\mathrm{drn}(G) \le \mathrm{rn}(G)$ provides the upper bound.

For the lower bound, we show that any two dacards of H_1 are dacards for another graph. Each dacard is a copy of $(P_3 + 2K_1, 3)$ or $(S, 1)$, where S arises from $K_{1,3}$ by subdividing one edge. There are three ways to take two dacards; two of the first, two of the second, and one of each. For these three cases, respectively, other graphs having the same two dacards are the graph obtained from $2K_1 + K_4$ by deleting one edge, the tree obtained from $K_{1,4}$ by subdividing one edge, and the tree obtained from $K_{1,3}$ by subdividing one edge twice. $\square$

The argument for H_2 is similar but longer, since it has three types of dacards. We omit it, since our goal is to prove that H_1 is the only caterpillar T such that $\mathrm{drn}(T) > 2$.

In the remainder of this section, we develop a sufficient condition for $\mathrm{drn}(T) \le 2$ when T is a tree (Theorem 4.6). This will help in the proof for caterpillars, because we will not need to select dacards explicitly for caterpillars satisfying this condition.

The *weight* $w(u)$ of a vertex u in a tree T is the maximum number of vertices in a single component of $T - u$; all leaves in an n-vertex tree have weight $n-1$. A *centroid* of a tree is a vertex of minimum weight. Myrvold [15] used centroids extensively in studying the reconstruction number of trees. To keep our presentation self-contained, we include short proofs of some elementary observations.

Lemma 4.4 ([15]). *An n-vertex tree T has one centroid or two adjacent centroids. It has one when the minimum vertex weight is less than $n/2$, two when it equals $n/2$.*

Proof. If $w(v) > n/2$, then the neighbor of v in the largest component of $T - v$ has smaller weight, so centroids have weight at most $n/2$.

If $w(v) < n/2$, then the neighbor of v in each component of $T - v$ has weight greater than $n/2$, as do all other vertices of those components, so v is the only centroid.

If $w(v) = n/2$, then the neighbor of v in the largest component of $T - v$ also has weight $n/2$, and all other vertices have larger weight. □

A tree is *unicentroidal* or *bicentroidal* when it has one or two centroids, respectively.

Lemma 4.5 ([15]). *Let v be the centroid in a unicentroidal tree T. If ℓ is a leaf in T, then v is a centroid in $T - \ell$.*

Proof. Let T have n vertices. By Lemma 4.4, $w(v) < n/2$. The weight of v in $T - \ell$ is at most $(n-1)/2$, since deleting ℓ just reduces one component of $T - v$. By Lemma 4.4, v is a centroid in $T - \ell$. □

Theorem 4.6. *If T is a unicentroidal tree having a leaf ℓ adjacent to the centroid, and $T - \ell$ is unicentroidal, then* $\mathrm{drn}(T) \le 2$.

Proof. Let $T' = T - \ell$, and let $\hat{T}$ be the card obtained by deleting the centroid of T. We use the dacards $(T', 1)$ and $(\hat{T}, d)$. Note that ℓ is an isolated vertex in $\hat{T}$. By Lemma 4.5, the degree of the centroid in T' is $d - 1$.

Let G be a graph having these dacards, from vertices u and v, respectively. By the first dacard, G is a tree. Lemma 4.4 and the sizes of components in $\hat{T}$ then make G unicentroidal with centroid v. By Lemma 4.5, v is also the centroid in T'. Thus v has degree $d - 1$ in T', and $\hat{T}$ has d components (including an isolated vertex), so G arises from T' by adding u adjacent to the centroid. Hence $G \cong T$. □

5 Caterpillars of special form

To show that $\mathrm{drn}(T) \le 2$ when T is a caterpillar other than H_1, we find for each such caterpillar two dacards that determine it. There is a particular choice of two dacards that generally works (generated by a particular leaf and its neighbor), but this choice fails for paths. The general choice also fails for several other classes of caterpillars. In this section we find special pairs of dacards to permit reconstruction for caterpillars in these classes. These choices also fail for paths, which is why we treated paths separately.

This approach of successively excluding special subfamilies until a general argument handles the remaining graphs in the desired family is typical of reconstruction arguments. It is the method in the original proof by Kelly [8] of reconstructibility of trees. It may be that caterpillars themselves similarly form a special class whose exclusion permits a general argument for reconstructibility of trees from two dacards.

The *skeleton* of a tree T is the subtree obtained by deleting all leaves from T. Caterpillars are the trees whose skeletons are paths, and the skeleton of a caterpillar is called its *spine*.

We use $\langle v_1, \ldots, v_s \rangle$ to denote a path with vertices $v_1, \ldots, v_s$ in order. We use $C(a_1, \ldots, a_s)$ to denote a caterpillar with spine $\langle v_1, \ldots, v_s \rangle$ formed by attaching a_i leaf neighbors to v_i for each $i \in \{1, \ldots, s\}$. We call $(a_1, \ldots, a_s)$ the *spine list*. Note that $C(a_1, \ldots, a_s) \cong C(a_s, \ldots, a_1)$ and that always a_1 and a_s are both positive. Where convenient, we denote a repeated string in this notation by enclosing it in parentheses and writing its multiplicity as an exponent. For example, $C(a, b, c, d, b, c, d, b, c, d, e, f) = C(a, (b, c, d)^3, e, f)$.

Our aim in this section is to prove that $\mathrm{drn}(T) \le 2$ when T is a caterpillar having the form $C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$. Note that every path has this form. We begin with a technical lemma that will restrict the form of caterpillars with special symmetry properties. A *palindrome* is a list unchanged under reversal.

Lemma 5.1. *Let $B = (b_1, \ldots, b_s)$. If $(b_1, \ldots, b_s)$ and $(b_3, \ldots, b_s)$ are palindromes, then either B is constant, or s is odd and B alternates two values. If $(b_1, \ldots, b_{s-1})$ and $(b_2, \ldots, b_s)$ are palindromes, then either B is constant, or s is even and B alternates two values.*

Proof. In the first case, alternating use of the palindrome requirements for $(b_1, \ldots, b_s)$ and $(b_3, \ldots, b_s)$ yields $b_1 = b_s = b_3 = b_{s-2} = b_5 = b_{s-4} = \cdots$, and similarly $b_2 = b_{s-1} = b_4 = b_{s-3} = b_6 = b_{s-5} = \cdots$. If s is even, then the two lists index the same (all) positions, in opposite order, and hence B must be constant. For odd s, the two sets are disjoint and may have different values.

The proof of the second case is similar. □

In the remainder of the paper, $T = C(a_1, \ldots, a_s)$, with spine $\langle v_1, \ldots, v_s \rangle$, where v_i is adjacent to a_i leaves in T. In the rest of this section, $a_1 = a_s = 1$ and $a_2 = a_{s-1} = 0$. By Proposition 4.2, $\mathrm{drn}(P_{s+2}) = 2$. Since P_{s+2} is the subcase $a_3 = \cdots = a_{s-2} = 0$, we may exclude that and let $r = \min\{i\colon\ a_i > 0 \text{ and } 3 \le i \le s-2\}$. To show $\mathrm{drn}(T) \le 2$, we present two dacards that determine T. Let D_1 and D_2 be the dacards for leaves adjacent to v_1 and v_r; we have $D_1 = (C_1, 1)$ and $D_2 = (C_2, 1)$, where

$$\begin{aligned} C_1 &= C(1, 0^{r-3}, a_r, \ldots, a_{s-2}, 0, 1), \\ C_2 &= C(1, 0^{r-2}, a_r - 1, a_{r+1}, \ldots, a_{s-2}, 0, 1). \end{aligned}$$

Let G be a graph reconstructed from dacards D_1 and D_2, with vertices u and v being the corresponding deleted vertices. Since $d_G(u) = d_G(v) = 1$, either card forces G to be a tree. We show that $G \cong T$, with some exceptions where we will later use other dacards. We write $\mathrm{diam}(G)$ for the *diameter* of G, which is the maximum distance between vertices in G.

Lemma 5.2. *If $T = C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$ and T is not a path, then the dacards D_1 and D_2 determine T unless T satisfies one of the following conditions:*
(1) $T = C(1, 0^p, 1, 0^q, 1)$ *with $p, q \ge 1$;*
(2) $T = C(1, 0^{p+1}, k, (\alpha), k-1, 0^p, 1)$ *with $k \ge 1$, $p \ge 0$, and (α) a palindrome.*

Proof. From D_2 it follows that G is a tree with diameter at least $s+1$. Since $\mathrm{diam}(G-u) = s$ and $s \ge 5$, it follows that u is adjacent in G to an endpoint of a longest path in $G-u$.

Hence G is T or is $C(L)$ with $L = (1, 0^{r-3}, a_r, \ldots, a_{s-2}, 0, 0, 1)$. Suppose the latter. Since $G - v \cong C_2$, and both G and C_2 have spines with s vertices, decreasing one term of L yields the spine list L' for C_2 or its reverse, L''. We use subscripts to index terms in these lists.

Case 1: Decreasing some L_i by 1 yields L'.

$$\begin{aligned}
\text{index} &= \quad 1, \ \ldots, r-1, \ r, \ \ r+1, \ldots, s-3, s-2, s-1, s \\
T \quad &= C(1, 0^{r-3}, \ \ 0, \ \ a_r, \ \ a_{r+1}, \ldots, a_{s-3}, a_{s-2}, \ \ 0, \ \ 1) \\
G = C(L) &= C(1, 0^{r-3}, \ \ a_r, a_{r+1}, a_{r+2}, \ldots, a_{s-2}, \ \ \ 0, \ \ \ 0, \ \ 1) \\
C_2 = C(L') &= C(1, 0^{r-3}, \ \ 0, \ a_r{-}1, a_{r+1}, \ldots, a_{s-3}, a_{s-2}, \ \ 0, \ \ 1)
\end{aligned}$$

Since $L_{r-1} = a_r > 0 = L'_{r-1}$, changing L into L' by decreasing one L_i requires $i = r-1$ and $a_r = 1$. Since no other change is allowed, we obtain $a_r - 1 = a_{r+1} = \cdots = a_{s-2} = 0$. Hence $T = C(1, 0^{r-2}, 1, 0^{s-r-1}, 1)$, as in (1).

Case 2: Decreasing some L_j by 1 yields L''.

$$\begin{aligned}
\text{index} &= \quad 1, \ \ldots, r-1, \ r, \ \ r+1, \ldots, s-3, s-2, s-1, s \\
T \quad &= C(1, 0^{r-3}, \ \ 0, \ \ a_r, \ \ a_{r+1}, \ldots, a_{s-3}, a_{s-2}, \ \ 0, \ \ 1) \\
G = C(L) &= C(1, 0^{r-3}, \ \ a_r, a_{r+1}, a_{r+2}, \ldots, a_{s-2}, \ \ \ 0, \ \ \ 0, \ \ 1) \\
C_2 = C(L'') &= C(1, 0, a_{s-2}, \ldots, \ a_{r+1}, \ \ a_r - 1, \quad 0^{r-3} \ \ , \ \ 0, \ \ 1) \\
\text{index} &= \quad 1, 2, \ \ 3, \ \ \ldots, s-r, \ s-r+1, \ \ \ldots \ \ , s-1, s
\end{aligned}$$

We first restrict j. By construction, $3 \le r \le s-2$. Since $L_i = a_{i+1}$ for $2 \le i \le s-2$, we have $L_{r-1} + L_{s-r+1} = a_r + a_{s-r+2}$. Since $L''_{s-r+1} = a_r - 1$, and $L''_i = L'_{s+1-i} = a_{s+1-i}$ for $i \ne s-r+1$, we have $L''_{r-1} + L''_{s-r+1} \le a_{s-r+2} + a_r - 1$. Hence $j \in \{r-1, s-r+1\}$.

Since $L_i = 0$ for $2 \le i \le r-2$, we have $j \ge r-1$. Since only position j changes, $L''_i = L_i = 0$ for $2 \le i \le r-2$. If $s-r+1 \le r-2$, then $a_r - 1 = L''_{s-r+1} = 0$ and $T = C(1, 0^{r-2}, 1, 0^{s-1-r}, 1)$, which satisfies description (1).

If $s-r+1 = r-1$, then $a_{s+1-i} = L''_i = 0$ for $2 \le i \le r-2$. We obtain $T = C(1, 0^{r-2}, a_r, 0^{r-3}, 1)$ and $G = C(1, 0^{r-3}, a_r, 0^{r-2}, 1)$, and hence $G \cong T$.

Finally, consider $s-r+1 > r-1$. Now $a_{i+1} = L_i = L''_i = a_{s+1-i}$ for $r \le i \le s-r$. Hence $(a_{r+1}, \ldots, a_{s-r+1})$ is a palindrome; write it as (α). If $j = r-1$, then $a_{s-r+2} = L_{s-r+2} = L''_{s-r+2} = a_r - 1$, but if $j = s-r+1$ then $a_{s-r+2} = a_r$. Also $a_{i+1} = L_i = L''_i = 0$ for $s-r+2 \le i \le s-3$. Thus $T = C(1, 0^{r-2}, k, (\alpha), k', 0^{r-3}, 1)$ and $G = C(1, 0^{r-3}, k, (\alpha), k', 0^{r-2}, 1)$, where $k = a_r \ge 1$ and $k' \in \{k, k-1\}$. If $k' = k$, then $G \cong T$; otherwise, description (2) holds. $\square$

Since $C(a_1, \ldots, a_s) \cong C(a_s, \ldots, a_1)$ for every caterpillar by reversing the spine, we have shown that a caterpillar of the form $C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$ is determined by the stated choice of dacards taken from one end or the other unless under both directions the caterpillar has one of the exceptional forms listed. In these cases, the dacards used in Lemma 5.2 do not determine T. Our argument to handle these exceptional forms has exceptions itself, covered in the next three results. In all exceptional cases, we find two dacards that work. We consider first the type (1) exceptional form in Lemma 5.2.

Proposition 5.3. *If* $T = C(1, 0^p, 1, 0^q, 1)$*, where* $p, q \geq 0$*, then* $\mathrm{drn}(T) \leq 2$*.*

Proof. Here T has one vertex of degree 3, and it has one leaf neighbor. Use the resulting dacards $(P_{p+2} + K_1 + P_{q+2}, 3)$ and $(P_{p+q+5}, 1)$. Let G be a graph with these dacards generated by vertices v and u, respectively. First $(P_{p+q+5}, 1)$ forces G to be a tree. Since $G - u$ is a path, v is the only vertex of degree 3 in G. Hence v has a neighbor in each component of $P_{p+2} + K_1 + P_{q+2}$, and that neighbor cannot have degree 3 in G. We obtain $G \cong T$. □

Among the type (2) lists $(1, 0^{p+1}, k, (\alpha), k-1, 0^p, 1)$, we consider several special cases.

Proposition 5.4. *If* $T = C(1, 0^{p+1}, (2, 0)^q, 1, 0^p, 1)$*, where* $p, q \geq 1$*, then* $\mathrm{drn}(T) \leq 2$*.*

Proof. Let $j = p + 3 + 2\lfloor q/2 \rfloor$. The spine vertex v_j has degree 4. Consider the dacards obtained by deleting v_j or an adjacent leaf ℓ. Since $T - \ell$ is a tree with $2p + 4q + 6$ vertices, every reconstruction G is a tree with $2p + 4q + 7$ vertices. Note that $T - v_j$ consists of two isolated vertices and two caterpillars. Regardless of the parity of q, the caterpillars have $p + 3 + 2q$ and $p + 1 + 2q$ vertices.

Let u and v be the leaf and non-leaf vertices deleted from G to obtain these dacards. Since $p + 3 + 2q < (2p + 4q + 7)/2$, Lemma 4.4 implies that v is the centroid of G. The tree $G - u$ has $2p + 4q + 6$ vertices and is bicentroidal, with v_j and one of its neighbors each having weight $p + 2q + 3$. By Lemma 4.5, v is one of these two vertices. Since these vertices have degrees 3 and 2 in $G - u$, and $d_G(v) = 4$, the graph G is obtained by making u adjacent to the one centroidal vertex having degree 3 in $G - u$, which is v_j. Thus $G \cong T$. □

Proposition 5.5. *If* $T = C(1, 0^p, 1^q, 0^p, 1)$*, where* $p \geq 1$ *and* $q \geq 0$*, then* $\mathrm{drn}(T) \leq 2$*.*

Proof. If $q = 0$, then T is a path, and Proposition 4.2 applies. If $q = 1$, then Proposition 5.3 applies. Now consider $q \geq 2$. Note that $s = 2p + q + 2$, so $\mathrm{diam}(T) = 2p + q + 3$.

Let x be the leaf adjacent to v_{p+2}. Consider the dacards obtained by deleting v_p (with degree 2) and x. Note that $T - x = C(1, 0^{p+1}, 1^{q-1}, 0^p, 1)$ and $T - v_p = P_p + C(2, 1^{q-1}, 0^p, 1)$. Let G be a reconstruction from these two dacards, with $G - u \cong T - x$ and $G - v \cong T - v_p$. As usual, the leaf dacard forces G to be a tree. Since $\mathrm{diam}(G - u) = 2p + q + 3 = \mathrm{diam}\, T$, the two neighbors of v in G must be endpoints of longest paths in the two components of $G - v$. Hence $G \cong T$ or $G = C(2, 1^{q-1}, 0^{2p+1}, 1)$, depending on which end of the longest path in the non-path component in $G - v$ is adjacent to v.

In the latter case, since the spine endpoints in $G - u$ each have only one leaf neighbor, u must be adjacent in G to the spine vertex having two leaf neighbors. Now $G - u \cong C(1^q, 0^{2p+1}, 1)$. Since $p \geq 1$ and $q \geq 2$, this graph is not isomorphic to $T - x$, a contradiction. Hence this case does not arise, and $G \cong T$. □

We now have the tools to prove the main result of this section.

Theorem 5.6. *If* $T = C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$*, then* $\mathrm{drn}(T) = 2$*.*

Proof. By Proposition 4.2, we may assume that T is not a path. In Lemma 5.2, we proved that the dacards for the leaves adjacent to v_1 and the next spine vertex having a leaf neighbor determine T unless both T and its reverse description $C(a_s, \ldots, a_1)$ have the forms specified in Lemma 5.2. If the description is as in (1) of Lemma 5.2, then T is a path plus one edge, and Proposition 5.3 yields $\mathrm{drn}(T) \le 2$.

Hence we may assume that both T and the reverse description T' are as in (2) of Lemma 5.2. Letting L be the spine list of T, we thus have

$$L = (1, 0^{p+1}, k, (\alpha), k-1, 0^p, 1) = (1, 0^q, \ell - 1, (\beta), \ell, 0^{q+1}, 1)$$

for some palindromes (α) and (β) and integers p, q, k, ℓ such that $p, q \ge 0$ and $k, \ell \ge 1$.

If $k \ge 2$, then the last nonzero entry of L before a_s is both a_{s-p-1} and a_{s-q-2}, so $q = p-1$ and $\ell = k-1$. Hence

$$L = (1, 0^{p+1}, k, (\alpha), k-1, 0^p, 1) = (1, 0^{p-1}, k-2, (\beta), k-1, 0^p, 1),$$

which implies that $(a_{p+4}, \ldots, a_{s-p-2})$ and $(a_{p+2}, \ldots, a_{s-p-2})$ are both palindromes and that $k = 2$. Since $a_{p+2} = 0 \ne 2 = a_{p+3}$, Lemma 5.1 yields $T = C(1, 0^{p+1}, (2,0)^{s/2-p-2}, 1, 0^p, 1)$, where s is even and $p \ge 1$. Since L contains at least one 2, Proposition 5.4 yields $\mathrm{drn}(T) \le 2$.

By reversing L, the same argument holds when $\ell \ge 2$. Finally, when $k = \ell = 1$,

$$L = (1, 0^{p+1}, 1, (\alpha), 0^{p+1}, 1) = (1, 0^{q+1}, (\beta), 1, 0^{q+1}, 1).$$

Since $a_{p+3} = 1$ and $a_2 = \cdots = a_{q+2} = 0$, we have $p \ge q$. Since $a_{s-q-2} = 1$ and $a_{s-p-1} = \cdots = a_{s-1} = 0$, we have $q \ge p$. Thus $p = q$, and $(a_{p+4}, \ldots, a_{s-p-2})$ and $(a_{p+3}, \ldots, a_{s-p-3})$ are palindromes. Since $a_{p+3} = a_{s-p-2} = 1$, Lemma 5.1 implies that $a_{p+3} = \cdots = a_{s-p-2} = 1$, so $T = C(1, 0^{p+1}, 1^{s-2p-4}, 0^{p+1}, 1)$. By Proposition 5.5, again $\mathrm{drn}(T) \le 2$. □

6 General caterpillars

Having shown that $\mathrm{drn}(T) \le 2$ whenever T has the form $C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$, we may exclude such caterpillars (and stars) from our study of general caterpillars. In the general case, we will use the dacards obtained by deleting the first spine vertex v_1 and one of its leaf neighbors. These determine T except in some cases. Again we handle the exceptional cases separately, using other dacards. The next three propositions handle these cases. Note that setting $k = 0$ in the first would yield a path.

Proposition 6.1. *If $T = C(k+1, k^m, k+1)$, where $k, m \ge 1$, then* $\mathrm{drn}(T) = 2$.

Proof. The cards obtained by deleting leaf neighbors of v_1 and v_2 are $C(k^{m+1}, k+1)$ and $C(k+1, k-1, k^{m-1}, k+1)$. Let G share these dacards, with u and v being the respective added vertices of degree 1; G must be a tree. Since the ends of the spine in $G - v$ both have degree $k+2$, G has two vertices at distance $m+1$ having degree at least $k+2$. In $G - u$ there is only one vertex of degree $k+2$ at distance $m+1$ from a vertex of degree at least $k+1$; these two are the ends of the spine in $G - u$. Hence G must arise from $G - u$ by making u adjacent to the spine endpoint with lower degree, yielding $G \cong C(k+1, k^m, k+1)$. □

Note that for $C(k+1, k^m, k+1)$, the dacards for a spine endpoint and one of its leaf neighbors are shared also by $C(k^m, k+1, k+1)$. Similarly, in the next proposition, the dacards for $C(2, 0^{s-2}, 2)$ generated by a spine endpoint and one of its leaf neighbors are shared also by $C(1, 0^{s-4}, 1, 0, 2)$.

A *branch vertex* is a vertex with degree at least 3. Let the *broom* B_k be the caterpillar formed by giving two leaf neighbors to one end of P_k, and call the other end of the path the *tip* of B_k when $k > 1$. Note that $B_1 \cong P_3$, and that T below reduces to H_1 when $s = 2$.

Proposition 6.2. *If* $T = C(2, 0^{s-2}, 2)$*, where* $s \geq 3$*, then* $\mathrm{drn}(T) = 2$*.*

Proof. Let $p = \lceil s/2 \rceil$. Note that v_p is a centroid and v_{p-1} is not. We use their dacards.

	$C_1 = T - v_p$	$C_2 = T - v_{p-1}$	D_1	D_2
$s \geq 5$	$B_{p-1} + B_{s-p}$	$B_{p-2} + B_{s-p+1}$	$(C_1, 2)$	$(C_2, 2)$
$s \in \{3, 4\}$	$P_3 + B_{s-2}$	$2K_1 + B_{s-1}$	$(C_1, 2)$	$(C_2, 3)$

Let G have dacards D_1 and D_2, where $C_1 = G - u$ and $C_2 = G - v$. If $s \geq 5$, then Lemma 4.1 implies that G is a tree. For $s \in \{3, 4\}$, again G is a tree, because D_1 forbids isolated vertices, and then D_2 gives v a neighbor in each component of $G - v$. It remains to determine which tree G is.

Case 1: $s \geq 5$ and $uv \in E(G)$. Since $d_G(v) = 2$, vertex v is a leaf in $G - u$, and $G - v$ arises from $G - u$ by deleting the leaf v in $B_{p-1} + B_{s-p}$ and attaching u to one vertex in the other component of $G - u$. Since $G - v = B_{p-2} + B_{s-p+1}$, we conclude that v is the tip of B_{p-1}, and u is adjacent to the tip of B_{s-p} in $G - u$. Now $G \cong T$.

Case 2: $s \geq 5$ and $uv \notin E(G)$. Since $uv \notin E(G)$, we have $d_{G-u}(v) = 2$. Hence $G - u - v$ has three components: a path P and two brooms. Since $G - v$ consists of two brooms, with different sizes from those in $G - u$, we conclude that u is adjacent in $G - u - v$ to one end of P and the tip of the broom not containing v. Since $p = \lceil s/2 \rceil$, the components of $G - v$ and $G - u$ differ in size by at most 2. Therefore, P is a single vertex, and u is adjacent to the tips of both brooms in $G - u$. Again $G \cong T$.

Case 3: $s = 3$. Here $C_1 = 2P_3$, so $\Delta(G) \leq 3$. Hence v cannot be adjacent to the center of B_{s-p+1} (which equals $K_{1,3}$). With v adjacent to a leaf of B_{s-p+1}, we have $G \cong T$.

Case 4: $s = 4$. Here $T = C(2, 0, 0, 2)$, with $C_1 = P_3 + K_{1,3}$ and $C_2 = 2K_1 + B_3$. Since the neighbor of u in component $K_{1,3}$ of $G - u$ cannot attain degree 3 in G, vertex v must be the center of P_3 in $G - u$, adjacent to u. If u now is adjacent to the center of $K_{1,3}$, then $\Delta(G - v) = 4$. Hence u is a adjacent to a leaf of $K_{1,3}$ in $G - u$, and $G \cong T$. □

Proposition 6.3. *If* $T = C(k+2, (0, k)^m, 0, k+2)$*, with* $k \geq 0$ *and* $m \geq 1$*, then* $\mathrm{drn}(T) \leq 2$*.*

Proof. The case $k = 0$ is given by Proposition 6.2, so we may assume that $k \geq 1$. Now T is unicentroidal and has a leaf adjacent to the centroid whose deletion leaves a unicentroidal subtree. By Theorem 4.6, $\mathrm{drn}(T) = 2$. □

Most caterpillars are determined by the dacards corresponding to an end of the spine and one of its leaf neighbors. Our final lemma proves that this holds except for caterpillars in four special classes. The proof of the theorem then uses the earlier lemmas to handle the exceptional classes. We have noted explicitly that the general choice fails for $C(2, 0^{s-2}, 2)$, which has Type 2 below, and for $C(k+1, k^m, k+1)$, which has Type 3.

Lemma 6.4. *If $T = C(a_1, \ldots, a_s)$, then the dacards for an endpoint of the spine and one of its leaf neighbors determine T unless T is one of the following four types:*
(1) $T = C(1, 0, a_3, \ldots, a_s)$ *with* $s \geq 3$*;*
(2) $T = C(2, (0,0)^m, (1,0)^n, 2)$ *with* $m, n \geq 0$*;*
(3) $T = C(k+1, k^m, (k+1)^n)$ *with* $k, m, n \geq 1$*;*
(4) $T = C(k+2, (0,k)^m, (0,k+1)^n, 0, k+2)$ *with* $k, n \geq 0$ *and* $m \geq 1$.

Proof. Since $\mathrm{drn}(K_{1,t}) = 1$, we may assume that $s \geq 2$. Recall that $a_1, a_s \geq 1$. Specify the dacards by deleting a leaf neighbor ℓ of v_1 and by deleting v_1. Let $T_1 = T - \ell$, and let T_2 be the nontrivial component of $T - v_1$. The dacards are $(T_1, 1)$ and $(a_1 K_1 + T_2, a_1 + 1)$. Let G be a graph sharing these dacards, with u and v being the corresponding deleted vertices. The dacard $(T_1, 1)$ implies that G is a tree. Let x be the neighbor of u in G.

We list four events; always (U1 or U2) and (V1 or V2) occurs. Note that if U1 and V1 occur, then T is Type 1, so we may exclude this event.

U1:	$a_1 = 1$,	$\operatorname{diam} T_1 = s$,	$T_1 = C(a_2+1, a_3, \ldots, a_s)$.
U2:	$a_1 > 1$,	$\operatorname{diam} T_1 = s+1$,	$T_1 = C(a_1-1, a_2, \ldots, a_s)$.
V1:	$a_2 = 0$,	$\operatorname{diam} T_2 = s-1$,	$T_2 = C(a_3+1, a_4, \ldots, a_s)$.
V2:	$a_2 > 0$,	$\operatorname{diam} T_2 = s$,	$T_2 = C(a_2, a_3, \ldots, a_s)$.

We prove that $G \cong T$ unless T has one of the specified Types.

Claim: G is a caterpillar. Suppose otherwise. Since $G - u$ is the caterpillar T_1, vertex x is a leaf neighbor in $G - u$ of an internal vertex of the spine of $G - u$, so $\operatorname{diam} G = \operatorname{diam} T_1$. If $v = x$, then $\operatorname{diam} T_2 = \operatorname{diam} T_1$, so U1 and V2 occur. Thus $T_1 = C(a_2+1, a_3, \ldots, a_s)$, and we obtain T_2 from T_1 by deleting x, which decreases some value in $\{a_3, \ldots, a_{s-1}\}$. Allowing for reversal, we now have $\{a_2+1, a_s\} = \{a_2, a_s\}$, which is impossible.

If $v \neq x$, then reducing $G - v$ to a caterpillar plus isolated vertices requires v to be an endpoint of the spine of T_1, with x being a leaf neighbor of the spine neighbor of v. Again $\operatorname{diam} T_2 = \operatorname{diam} T_1$, so $a_1 = 1$ and $G - v \cong G - x$. We obtain the same contradication.

Since $d_G(v) = a_1 + 1 > 1$ and $G - v$ is a caterpillar plus isolated vertices, v is an endpoint of the spine of G. We consider cases depending on the diameter of T_1 and whether $\operatorname{diam} G = \operatorname{diam} T_1$. We also consider the location of x and v relative to the description of $G - u$ as T_1. These give a spine list L for T_2, which we compare with the spine list L' for T_2 from event V1 or V2. The two lists must be the same ($L = L'$) or reversed ("$L \| L'$").

Case 1: $\operatorname{diam} G > \operatorname{diam} T_1 = s+1$. Here U2 occurs, so $a_1 > 1$. Since $\operatorname{diam} G > \operatorname{diam} T_1$, x is a leaf neighbor of an endpoint of the spine of T_1. Hence $G = C(1, a_1-2, a_2, \ldots, a_s)$ or $G = C(a_1-1, a_2, \ldots, a_{s-1}, a_s-1, 1)$. Since $d_G(v) > a_1 > 1$ and v is an endpoint of the spine, the second description is forbidden, and $v \neq x$. Now $d_G(v) = a_s + 1$, so $a_s = a_1$.

Since deleting v can only reduce the diameter of G by 2, and $\operatorname{diam} T_2 \le s$, we have $a_{s-1} = 0$, and $\operatorname{diam} T_2 = s$. Now $L = (1, a_1 - 2, a_2, \ldots, a_{s-2} + 1)$, and $L' = (a_2, \ldots, a_s)$. If $L \| L'$, then $1 = a_s = a_1 > 1$, a contradiction.

If $L = L'$, then $1 = a_2 = a_4 = \cdots$ and $a_1 - 2 = a_3 = a_5 = \cdots$. Since $a_{s-1} = 0$, we cannot have s odd. With s even and $a_1 = k + 2$, the spine list of T is $(k + 2, (1, k)^{s/2-1}, 2)$, but now $a_s = a_1$ requires $k = 0$, so T is Type 2 with $m = 0$ and $n = s/2 - 1$.

Case 2: $\operatorname{diam} G > \operatorname{diam} T_1 = s$. Here U1 occurs, so $a_1 = 1$ and $T_1 = C(a_2 + 1, a_3, \ldots, a_s)$. Again x is a leaf neighbor of an endpoint of the spine of T_1. Hence $G = C(1, a_2, a_3, \ldots, a_s)$ or $G = C(a_2 + 1, a_3, \ldots, a_{s-1}, a_s - 1, 1)$. In the first case, already $G \cong T$. In the second case, since $d_G(v) = 2$, we may have $v = x$; otherwise, $a_2 = 0$. If $a_2 = 0$, then T is Type 1.

Hence $v = x$ and $a_2 > 0$ (Event V2), so $L' = (a_2, \ldots, a_s)$ and $\operatorname{diam} T_2 = s$. Now deleting v from G reduces the diameter only by 1, so $a_s > 1$ and $L = (a_2 + 1, a_3, \ldots, a_{s-1}, a_s - 1)$. The first entry forbids $L = L'$. If $L \| L'$, then $a_2 = a_s - 1$ and $(a_3, \ldots, a_{s-1})$ is a palindrome. Now the spine list for G is the reverse of the spine list for T, so $G \cong T$.

Case 3: $\operatorname{diam} G = \operatorname{diam} T_1 = s$. Here x is on the spine of T_1, and $a_1 = 1$. If $a_2 = 0$, then G is Type 1, so we may assume $a_2 > 0$. Hence V2 occurs, so $\operatorname{diam} T_2 = s$. However, this contradicts $G - v \cong T_2$, since deleting an endpoint v of the spine of a caterpillar with diameter s reduces the diameter of the nontrivial component below s.

Case 4: $\operatorname{diam} G = \operatorname{diam} T_1 = s + 1$. Here x is on the spine of T_1, and $a_1 > 1$. The spine list for G is obtained from $(a_1 - 1, a_2, \ldots, a_s)$ by increasing position j by 1, for one value j. If $j = 1$, then $G \cong T$, so assume $j > 1$. Now the endpoint of the spine that corresponds to position 1 in the spine list has degree a_1 in G, so this vertex cannot be v.

Hence we may assume that v corresponds to the other end of the spine list of G. If $j \le s - 1$, then $d_G(v) = a_s + 1 = a_1 + 1$, so $a_1 = a_s$. The spine list L' for T_2 obtained from T is A: $(a_2, a_3, \ldots, a_s)$ or B: $(a_3 + 1, a_4, \ldots, a_s)$ with $a_2 = 0$. The spine list L for T_2 obtained from G starts with $a_1 - 1$, so $L \| L'$ requires $a_1 - 1 = a_s$, contradicting $a_1 = a_s$. Hence $L = L'$.

The spine list L for T_2 obtained from G arises from A: $(a_1 - 1, a_2, \ldots, a_{s-1})$ or from B: $(a_1 - 1, a_2, \ldots, a_{s-3}, a_{s-2} + 1)$ (with $a_{s-1} = 0$) by increasing position j by 1, except that $j = s - 1$ and $a_{s-1} = 0$ yields $L = (a_1 - 1, a_2, \ldots, a_{s-2}, 1)$, which is forbidden since $L = L'$ then requires $1 = a_s = a_1$, contradicting $a_1 > 1$.

Since L and L' have equal length, both are A or both are B. In case A, $a_1 - 1 = a_2 = \cdots = a_j = a_{j+1} - 1 = \cdots a_s - 1$, and T is Type 3 with $k = a_1 - 1$, $m = j - 1$, and $n = s - j$. In case B with j odd, $a_1 - 2 = a_3 = a_5 = \cdots = a_j = a_{j+2} - 1 = \cdots$ and $0 = a_2 = a_4 = \cdots$. If s is even, then $0 = a_2 = a_{s-2} = a_s - 1 = a_1 - 1$, and G is Type 1. If s is odd, then the first equality ends with $a_s - 2$, and T is Type 4 with $a_1 - 2 = k$, $m = (j - 1)/2$, and $n = (s - j)/2 - 1$. With j even, instead we have $a_1 - 2 = a_3 = \cdots$ and $0 = a_2 = a_4 = \cdots = a_j = a_{j+2} - 1 = \cdots$. If s is odd, then $a_1 - 2 = a_s - 1$, contradicting $a_s = a_1$. If s is even, then by both strings of equalities and $a_s = a_1$, we conclude that T is Type 2 with $m = j/2$ and $n = (s - j)/2 - 1$.

Finally, if $j = s$, then $v = x$ and $a_1 = d_G(v) = a_s + 1$, and the spine list L for T_2 obtained from G is $(a_1 - 1, a_2, \ldots, a_{s-1})$ or is $(a_1 - 1, a_2, \ldots, a_{s-3}, a_{s-2} + 1)$ with $a_{s-1} = 0$. The lengths of L' and L must be equal. Thus in the first case $L = L'$ yields $a_1 - 1 = a_2 = \cdots = a_s$ and $G \cong T$ (by reversal), but $L \| L'$ requires $a_1 - 1 = a_s + 1$, which contradicts $a_1 = a_s + 1$. In the second case, $L = L'$ yields $0 = a_2 = a_4 = \cdots$ and $a_1 - 2 = a_3 = a_5 \cdots$. If s is even, then $0 = a_2 = a_{s-2} = a_s - 1 = a_1 - 2 = a_{s-1}$, and T is Type 1 (reversed). If s is odd, then $a_1 - 2 = a_{s-2} = a_s - 1$ and $0 = a_2 = a_{s-1}$, and $G \cong T$ (reversed). Finally, $L \| L'$ confirms $a_1 - 1 = a_s$ and requires $a_2, \ldots, a_{s-1}$ to be a palindrome; again $G \cong T$ (by reversal). $\square$

Theorem 6.5. *If T is a caterpillar that is neither H_1 nor a star, then* $\mathrm{drn}(T) = 2$.

Proof. Let $T = C(a_1, \ldots, a_s)$. Recall that $T \cong T'$, where $T' = C(a_s, \ldots, a_1)$. Hence the choice of two dacards taken from one end or the other in T uniquely determines T unless both T and T' have a Type listed in Lemma 6.4.

Suppose first that T is Type 1. If T' also is Type 1, then $T = C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$, and $\mathrm{drn}(T) \leq 2$ by Theorem 5.6. Since all other Types have $a_s > 1$, the reversal of a Type 1 caterpillar cannot be Type 2, 3, or 4. This completes the proof when T (or T') is Type 1.

Suppose next that T is Type 2. Since s has different parity in Type 2 and Type 4, T' is not of Type 4. If T' is Type 2 or Type 3, then either $T = C(2, 2)$ and $T \cong H_1$, or $T = C(2, (0, 0)^m, 2)$ with $m \geq 1$, in which case $\mathrm{drn}(T) \leq 2$ by Proposition 6.2. This completes the proof when T (or T') is Type 2.

If T and T' are both Type 3, then $T = C(k+1, k^m, k+1)$ with $k, m \geq 1$, and $\mathrm{drn}(T) \leq 2$ by Proposition 6.1. Since the spine list is all positive for Type 3 and not for Type 4, T and T' cannot be one of each.

Finally, if T and T' are Type 4, then $n = 0$. Now $\mathrm{drn}(T) \leq 2$ by Proposition 6.3. $\square$

Building on our result, one could seek a choice of two dacards that determines T when T is not a caterpillar, with few exceptions or exceptional families that can be reconstructed from other pairs of dacards. This could be a route to a proof of Conjecture 1.2.

Acknowledgment

We thank several referees for careful reading and insightful comments that led to substantial improvement in the presentation of our results.

References

[1] K. J. Asciak and J. Lauri, On disconnected graph with large reconstruction number, Ars Combin. 62 (2002), 173–181.

[2] B. Bollobás, Almost every graph has reconstruction number three, J. Graph Theory 14 (1990), 1–4.

[3] J. A. Bondy, A graph reconstructor's manual, in *Surveys in Combinatorics (Guildford, 1991)*, Lond. Math. Soc. Lec. Notes 166 (Cambridge U. Press, 1991), 221–252.

[4] J. A. Bondy and R. L. Hemminger, Graph reconstruction—a survey, J. Graph Theory 1 (1977), 227–268.

[5] F. Harary and J. Lauri, On the class-reconstruction number of trees. Quart. J. Math. Oxford Ser. (2) 39 (1988), no. 153, 47–60.

[6] F. Harary and M. Plantholt, The graph reconstruction number, J. Graph Theory 9 (1985), 451–454.

[7] P. J. Kelly, On isometric transformations, PhD Thesis, University of Wisconsin–Madison, 1942.

[8] P. J. Kelly, A congruence theorem for trees, Pacific J. Math. 7 (1957), 961–968.

[9] J. Lauri, Pseudosimilarity in graphs—a survey, Ars Combin. 46 (1997), 77–95.

[10] A. Maccari, O. Rueda, V. Viazzi, A survey on edge reconstruction of graphs, J. Discr. Math. Sci. Cryptog. 5 (2002), 1–11.

[11] B. McMullen and S. Radziszowski, Graph reconstruction numbers. J. Combin. Math. Combin. Comput. 62 (2007), 85–96.

[12] R. Molina, Correction of a proof on the ally-reconstruction number of a disconnected graph, Ars Combin. 40 (1995), 59–64.

[13] W. Myrvold, The ally and adversary reconstruction problems, PhD thesis, University of Waterloo, 1988.

[14] W. Myrvold, The ally-reconstruction number of a disconnected graph, Ars Combin. 28 (1989), 123–127.

[15] W. Myrvold, The ally-reconstruction number of a tree with five or more vertices is three, J. Graph Theory 14 (1990), 149–166.

[16] W. Myrvold, The degree sequence is reconstructible from $n-1$ cards. Discrete Math. 102 (1992), no. 2, 187–196.

[17] N. Prince, personal communication, 2008.

[18] S. Ramachandran, On a new digraph reconstruction conjecture, J. Combin. Theory Ser. B 31 (1981), 143–149.

[19] S. Ramachandran, N-reconstructibility of nonreconstructible digraphs, Discrete Math. 46 (1983), 279–294.

[20] S. Ramachandran, Degree associated reconstruction number of graphs and digraphs, Mano. Int. J. Mathematical Sciences 1 (2000), 41–53.

[21] S. Ramachandran, Reconstruction number for Ulam's conjecture. Ars Combin. 78 (2006), 289–296.

[22] S. M. Ulam, A collection of mathematical problems, Interscience Tracts in Pure and Applied Mathematics 8, (Interscience Publishers, 1960).

[23] M. Welhan, Reconstructing trees from two cards, J. Graph Theory 33 (2009), 243–257.

[24] D. B. West, Introduction to graph theory, (Prentice-Hall, 2001).