Degree Ramsey Number of Cycles

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Joint work with
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Parameter Ramsey Numbers

**Def.** \( H \rightarrow G \) means every 2-coloring of \( E(H) \) gives a monochromatic \( G \). Ramsey’s Theorem \( \Rightarrow H \) exists. Ramsey number \( R(G) = \min\{n: K_n \rightarrow G\} \).
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- $R_\rho(G_1, G_2, G_3, \ldots, G_s; s)$ not yet much studied.
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For size Ramsey number, write $R'(G)$ for $R_\rho(G)$ when $\rho(G) = |E(G)|$; always $R'(G) \leq \binom{R(G)}{2}$.

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Similarly, size Ramsey number is linear in $n$ for cycles (Haxell–Kohayakawa–Łuczak [1995]) and bounded-degree trees (Friedman–Pippinger [1981]), but not graphs w. maxdegree 3 (Rödl–Szemerédi [2000]).
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**Lower bound:** Every 4-chromatic graph has a 2-edge-coloring with both classes bipartite.
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Equality holds in lower bound \( \iff \exists \text{ hom. } \phi: G \rightarrow C_5 \).

In particular, \( R_\chi(C_5) = 5 \).
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**Lem.** Fix $n, k$. When $p$ is suff. large, in every 2-coloring of $E(K_k[p])$ there is a copy of $K_k[n]$ such that for every two parts, all edges joining them have the same color.

**Pf.** Iterated use of the bipartite Ramsey theorem that $K_{r,r} \rightarrow K_{t,t}$ when $r$ is sufficiently large.

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**Pf.** Let $k = R(G)$ and $n = |V(G)|$. Let $H = K_k[p]$ with $p$ as in the lemma. For any 2-coloring of $H$, the copy $H'$ of $K_k(n)$ with restricted edge-coloring collapses to a coloring of $E(K_k)$. In it is a monochr. $G'$ with $\phi: G \to G'$. Partite sets in $H'$ are big enough for $G'$ to lift back into a monochromatic copy of $G$ in $H'$. 

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**Conj.** (BEL [1976]) \( \min\{R_{\chi}(G) : \chi(G)=k\} = (k-1)^2+1. \)
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**Thm.** (BEL): \( R_\Delta(K_{1,m}) = \begin{cases} 2m - 2 & \text{m even} \\ 2m - 1 & \text{m odd} \end{cases}. \)
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**Thm.** \( R_{\Delta}(K_{1,m}; s) = \begin{cases} s(m - 1) & m \text{ even} \\ s(m - 1) + 1 & m \text{ odd} \end{cases}. \)
$s(m - 1) \leq R_\Delta(K_{1,m}; s) \leq s(m - 1) + 1$

**Pf.** Upper Bound: $K_{1,s(m-1)+1} \overset{s}{\rightarrow} K_{1,m}$. 
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**Pf.** Upper Bound: \( K_{1,s(m-1)+1}^s \rightarrow K_{1,m} \).

Improves when \( m \) is even:
When \( r > k \) and \( k \) is odd, there is an \( r \)-regular graph \( H \) having no \( k \)-factor (Bollobás–Saito–Wormald [1985]).
With \( k = m - 1 \) and \( r = s(m - 1) \), \( s \)-coloring \( E(H) \) with no monochromatic \( K_{1,m} \) requires a \( k \)-factorization.
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**Lower bound:** When \( \Delta(H) \leq s(m - 1) - 1 \), Vizing’s Theorem \( \Rightarrow H \) is \( s(m - 1) \)-edge-colorable.

Put \( m - 1 \) matchings into each color.
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Petersen’s Theorem decomposes \( s(m - 1) \)-regular supergraph \( H' \) into 2-factors. Putting \( (m - 1)/2 \) in each color avoids degree \( m \) in one color at any vertex. \( \blacksquare \)
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In a red/blue coloring, each vertex is majority red or majority blue. If most of part $X$ is red, then $Y$ must have no red vertex. Hence $Y$ is all blue and $X$ is all red. Now majority of edges are blue and majority are red.
Improved upper bound ($b$ even and $a < b$)

Claim: $H \rightarrow S_{b-1,b}$ for $b$ even

$H$ is $(2b - 2)$-regular
Claim: $H \rightarrow S_{b-1,b}$ for $b$ even

Pf. Vertices are majority red or majority blue or tied. Not all are tied (would be odd regular of odd order).

No $S_{b-1,b} \Rightarrow$ all nbrs (via red) of maj red are maj blue.

A maj red vertex forces a maj blue in each direction; after 5 steps, one set has a maj red and a maj blue.

Now its neighboring sets together need $b$ maj blue and $b$ maj red vertices, but they have only $2b - 2$ total.
Thm. $R_\Delta(P_n) = \begin{cases} 
  n - 1 & n \leq 4 \\
  3 & n \in \{4, 5\} \\
  3 \text{ or } 4 & n = 6 \quad \leftarrow \text{Open} \\
  4 & n \geq 7
\end{cases}$
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**Thm.** (Alon–Ding–Oporowski–Vertigan [2003])
\[ R_\Delta(P_n; s) = 2s \text{ for } n > n_0(s). \quad R_\Delta(P_n; s) \leq 2s \text{ always.} \]
Longer Paths

**Thm.** Thomassen [1999] Every 3-regular graph has a 2-edge-coloring with each monochromatic component contained in $P_6$. $\therefore R_\Delta(P_n) \geq 4$ for $n \geq 7$.

**Thm.** (Alon–Ding–Oporowski–Vertigan [2003]) $R_\Delta(P_n; s) = 2s$ for $n > n_0(s)$. $R_\Delta(P_n; s) \leq 2s$ always.

**Lower bound:** $\exists$ function $g$ such that if $\Delta(H) = 2s - 1$, then $H$ has an $s$-edge-coloring where all monochromatic components have at most $g(s)$ edges. $\therefore n_0(s) = g(s)$ suffices.
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Upper bound:
Let \( H \) be 2s-regular with girth \( \geq n \), and \( m = |V(H)| \).
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Let $H$ be $2s$-regular with girth $\geq n$, and $m = |V(H)|$.

$s$ colors on $sm$ edges puts $\geq m$ in some color class. Since $|V(H)| = m$, this color class contains a cycle. Since girth$(H) \geq n$, this color class contains $P_n$. 

**Thm.** (Jiang) If $G$ is a tree, then $R_{\Delta}(G; s) \leq 2s\Delta(G)$. 
Trees

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Ques. Is $R_{\Delta}(G)$ bounded by a function of $\Delta(G)$?
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Ques. Is $R_{\Delta}(G)$ bounded by a function of $\Delta(G)$? For which parameters $\rho$ is $R_{\rho}(G)$ bounded by a function of $\rho(G)$? - True for clique number, chromatic number, ...
Lower Bounds for Cycles

**Thm.** $R_{\Delta}(C_4) \geq 5$. 
Lower Bounds for Cycles

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**Thm.** (Kinnersley) If \( \Delta(G) \leq 4 \), then \( E(G) \) can be 2-colored so that each color class forms a subgraph with girth at least 5.
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By Brooks’ Theorem, every connected $H$ with $\Delta(H) = 4$ (other than $K_5$) has a proper coloring $f : V(H) \rightarrow [4]$.  

Color each edge $uv$ “odd” or “even” by whether $f(u) - f(v)$ is odd or even.  Both classes bipartite.
Upper Bounds for Cycles

**Ques.** Is $R_\Delta(C_n)$ bounded?
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Bigger question: Is there a function $f$ such that $R_\Delta(G) \leq f(\Delta(G))$?
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$s$-color induced size Ramsey # of $C_n$ is linear in $n$.

• The proof shows that $R_\Delta(C_n) \leq c$ (where $c$ is huge).
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**Thm.** (Jiang-Milans-West [2009+]) $R_\Delta(C_{2k+1}) \leq 3890$. 
Tools for Cycles

**Idea:** Force a long even cycle by forcing a blowup of a long monochromatic path.
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**Lem.** $K_{3,3} \rightarrow P_4$
**Idea:** Force a long even cycle by forcing a blowup of a long monochromatic path.

\[
\begin{align*}
\text{Lem. } K_{3,3} & \rightarrow P_4 \\
\text{Pf. } & \text{ Each vertex of } X \text{ has a majority in some color. Two vertices have majority in the same color, say red. Since } |Y| = 3, \text{ they have a common neighbor in } Y.
\end{align*}
\]
Even Cycles

**Thm.** \( R_\Delta(C_{2k}) \leq 118. \)
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**Pf.** Let \( G \) be a 36-regular \( X,Y \)-bigraph with girth \( \geq k \). Let \( H = G[\overline{K}_3] \); this is 118-regular. Claim: \( H \rightarrow C_{2k} \).
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**Thm.** $R_{\Delta}(C_{2k}) \leq 118$.

**Pf.** Let $G$ be a $36$-regular $X,Y$-bigraph with girth $\geq k$. Let $H = G[K_3]$; this is $118$-regular. Claim: $H \rightarrow C_{2k}$.

Consider 2-coloring $f$ of $E(H)$. Each edge $xy \in E(G)$ becomes $K_{3,3}$ with parts $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$. 

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Say that \( xy \) has Type \((c; i, j)\) if the resulting \( P_4 \) in the copy of \( xy \) has color \( c \) and omits \( x_i \) and \( y_j \).
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The 18 Types yield an 18-coloring of \( E(G) \). Some color class has average degree at least 2.
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The 18 Types yield an 18-coloring of \( E(G) \).
Some color class has average degree at least 2.

This class contains a cycle; length is \( \geq k \); contains \( P_k \).
Since the edges have the same Type, in \( H \) they yield pasted copies of \( P_4 \) in the same color \( c \).
This yields monochr. \( C_{2k} \) in \( H \).
Comments

Same type $\Rightarrow$ pasting

![Graph](image)
Cor. If $F$ is bipartite and $\Delta(F) = 2$, then $R_{\Delta}(F) \leq 118$. 
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Bipartite $G$ helps pasting but gives only even cycles. In fact, $H$ was bipartite.
Odd Cycles

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Odd Cycles

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**Pf.** Let \( G \) be a 36-regular \( X,Y \)-bigraph with girth > 2\( k \). Let \( H = G^2[K_3] \); this is 3890-regular (\( 3 \times 36^2 + 2 \)). Claim: \( H \rightarrow C_{2k-1} \).
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Claim: \( H \to C_{2k-1} \).

Consider 2-coloring \( f \) of \( E(H) \). Again make 18-coloring of \( E(G) \) (not \( G^2 \)); it has monochromatic \( P_{2k} \), say red.
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**Thm.** $R_{\Delta}(C_{2k-1}) \leq 3890$.

**Pf.** Let $G$ be a $36$-regular $X,Y$-bigraph with girth $> 2k$. Let $H = G^2[K_3]$; this is $3890$-regular ($3 \times 36^2 + 2$). Claim: $H \to C_{2k-1}$.

Consider 2-coloring $f$ of $E(H)$. Again make 18-coloring of $E(G)$ (not $G^2$); it has monochromatic $P_{2k}$, say red.

Any red edge "inside" $\Rightarrow$ red $C_{2k-1}$. 
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![Graph Diagram](image)

Any red edge "inside" \( \Rightarrow \) red \( C_{2k-1} \).

If all are blue, consider the added edges of \( G^2 \) joining alternate pairs along the path. Any red \( \Rightarrow \) red \( C_{2k-1} \).
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If all are blue, then we have a blue \( C_{2k-1} \).
The On-line Ramsey Problem

Graph Ramsey theory = a game
Builder presents a graph; Painter 2-colors the edges.
Builder wins if a monochromatic $G$ is produced.
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“Arrow” $H \rightarrow G$ $\iff$ Builder wins by playing $H$. 
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Builder presents one edge at a time; Painter colors it. Builder wins if a monochromatic $G$ is produced.
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Idea: Restrict Builder to a hereditary family $\mathcal{H}$.
After every move, the graph presented so far lies in $\mathcal{H}$. 
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After every move, the graph presented so far lies in $\mathcal{H}$.

This defines the on-line Ramsey game $(G, \mathcal{H})$.
Can Builder playing on $\mathcal{H}$ force a monochromatic $G$?
On-Line Ramsey Parameters

**Def.** For a monotone graph parameter $\rho$, the on-line $\rho$-Ramsey number $\hat{R}_\rho(G)$ of $G$ is $\min\{k: \text{Builder wins } (G, \mathcal{F}_k)\}$, where $\mathcal{F}_k = \{H: \rho(H) \leq k\}$.
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Grytczuk–Hałuszczak–Kierstead [2004] (E-JC)
\( \mathcal{R}_\chi(G) = \chi(G) \) (Builder wins \( G \) on \( \chi(G) \)-colorable graphs)
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** Conj. (GHK) ** When \( \mathcal{H} = \{ \text{planar} \} \), Builder wins \( (G, \mathcal{H}) \) if and only if \( G \) is outerplanar.
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Beck [1993] - introduced on-line size Ramsey number
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**Def.** on-line degree Ramsey number $\hat{R}_\Delta(G) = \min\{k: \text{Builder wins } (G, S_k)\}$, where $S_k = \{H: \Delta(H) \leq k\}$.
On-line Degree Ramsey Results

**Obs.** $\tilde{R}_\Delta(G) \leq R_\Delta(G)$ for all $G$. (Always $\tilde{R}_\rho(G) \leq R_\rho(G)$.)
On-line Degree Ramsey Results

**Obs.** $\hat{R}_\Delta(G) \leq R_\Delta(G)$ for all $G$. (Always $\hat{R}_\rho(G) \leq R_\rho(G)$.)

**Thm.** $\hat{R}_\Delta(G) \leq 3$ $\iff$ each component of $G$ is a path or each component is a subgraph of $K_{1,3}$. 
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**Obs.** \( \hat{R}_\Delta(G) \leq R_\Delta(G) \) for all \( G \). (Always \( \hat{R}_\rho(G) \leq R_\rho(G) \).)

**Thm.** \( \hat{R}_\Delta(G) \leq 3 \iff \) each component of \( G \) is a path or each component is a subgraph of \( K_{1,3} \).

**Thm.** \( \hat{R}_\Delta(G) \leq 2\Delta(G) - 1 \) when \( G \) is a tree, sharp if \( \exists \) adjacent vertices of maximum degree. (Also, \( \hat{R}_\Delta(S_{a,b}) = a + b - 1 \).)
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**Obs.** \( \hat{R}_\Delta(G) \leq R_\Delta(G) \) for all \( G \). (Always \( \hat{R}_\rho(G) \leq R_\rho(G) \).)

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**Thm.** \( 4 \leq \hat{R}_\Delta(C_n) \leq 5 \).
On-line Degree Ramsey Results

**Obs.** \( \hat{R}_\Delta(G) \leq R_\Delta(G) \) for all \( G \). (Always \( \hat{R}_\rho(G) \leq R_\rho(G) \).)

**Thm.** \( \hat{R}_\Delta(G) \leq 3 \iff \) each component of \( G \) is a path or each component is a subgraph of \( K_{1,3} \).

**Thm.** \( \hat{R}_\Delta(G) \leq 2\Delta(G) - 1 \) when \( G \) is a tree, sharp if \( \exists \) adjacent vertices of maximum degree. (Also, \( \hat{R}_\Delta(S_{a,b}) = a + b - 1 \).)

**Thm.** \( 4 \leq \hat{R}_\Delta(C_n) \leq 5 \).

**Thm.** \( \hat{R}_\Delta(C_n) = 4 \) if \( n \) is even or large or \( 3 \).
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**Ques.** Is \( \hat{R}_\Delta(G) \) bounded by a function of \( \Delta(G) \)? (Weaker than for \( R_\Delta(G) \).)

**Thm.** \( \hat{R}_\Delta(G) \leq 8 \) if \( \Delta(G) \leq 2 \) (maybe less).
**Def.** The greedy $\mathcal{F}$-Painter colors each edge red if the resulting red graph lies in $\mathcal{F}$; otherwise blue.
Lower Bounds – Greedy Painter

**Def.** The greedy $\mathcal{F}$-Painter colors each edge red if the resulting red graph lies in $\mathcal{F}$; otherwise blue.

**Thm.** $\hat{R}_\Delta(G) \geq \Delta(G) - 1 + \max_{uv \in E(G)} \min \{d(u), d(v)\}$.
Lower Bounds – Greedy Painter

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**Thm.** $\hat{\Delta}(G) \geq \Delta(G) - 1 + \max_{uv \in E(G)} \min \{d(u), d(v)\}$.

**Pf.** Let $m = \Delta(G)$. $S_{m-1}$-Painter never makes red $G$. An edge gets blue $\iff$ an endpt already has $m - 1$ red.
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Let $xy$ be an edge with maxmin degree in $G$. A blue $G$ has an edge for $xy$; it has $m-1$ red at one endpt and at least $\min\{d_G(x), d_G(y)\}$ blue there.
Lower Bounds – Greedy Painter

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**Cor.** $\hat{\mathcal{R}}_\Delta(P_n) \geq 3$; $\hat{\mathcal{R}}_\Delta(K_{1,m}) \geq m$; $\hat{\mathcal{R}}_\Delta(S_{a,b}) \geq a + b - 1$; $\hat{\mathcal{R}}_\Delta(G) \geq 2\Delta(G) - 1$ if $\exists$ adjacent maxdegree vertices.
Def. The greedy $F$-Painter colors each edge red if the resulting red graph lies in $F$; otherwise blue.

Thm. $\hat{\mathcal{R}}_\Delta(G) \geq \Delta(G) - 1 + \max_{uv \in E(G)} \min\{d(u), d(v)\}$.

Pf. Let $m = \Delta(G)$. $S_{m-1}$-Painter never makes red $G$. An edge gets blue $\iff$ an endpt already has $m - 1$ red. Let $xy$ be an edge with maxmin degree in $G$. A blue $G$ has an edge for $xy$; it has $m - 1$ red at one endpt and at least $\min\{d_G(x), d_G(y)\}$ blue there.

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• Lower bound for $\hat{\mathcal{R}}_\Delta(C_n) \geq 4$ comes from charzn of $\hat{\mathcal{R}}_\Delta(G) \leq 3$, which uses greedy linear-forest Painter.
Def. Painter follows a consistent strategy if the color used on a new edge depends only on the current 2-colored component(s) containing its endpoints (regardless of what else has been played).
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Thm. If $\mathcal{H}$ is an additive family (closed under disjoint unions), and $A$ is a Painter strategy on $\mathcal{H}$, then there is a consistent Painter strategy $A'$ on $\mathcal{H}$ such that for any list $E'$ presented by Builder, there is another list $E$ such that $A'(E') \subseteq A(E)$ (as 2-colored graphs).
Upper Bounds – Consistent Painter

**Def.** Painter follows a consistent strategy if the color used on a new edge depends only on the current 2-colored component(s) containing its endpoints (regardless of what else has been played).

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**Cor.** To prove that $\hat{R}_\Delta(G) \leq k$, it suffices to show that Builder can win against any consistent Painter on $S_k$. 
**Def.** Painter follows a consistent strategy if the color used on a new edge depends only on the current 2-colored component(s) containing its endpoints (regardless of what else has been played).

**Thm.** If $\mathcal{H}$ is an additive family (closed under disjoint unions), and $\mathcal{A}$ is a Painter strategy on $\mathcal{H}$, then there is a consistent Painter strategy $\mathcal{A}'$ on $\mathcal{H}$ such that for any list $E'$ presented by Builder, there is another list $E$ such that $\mathcal{A}'(E') \subseteq \mathcal{A}(E)$ (as 2-colored graphs).

**Cor.** To prove that $\hat{R}_\Delta(G) \leq k$, it suffices to show that Builder can win against any consistent Painter on $S_k$.

- Prove upper bounds on $\hat{R}_\Delta$ for trees and cycles by algorithms for Builder to defeat a consistent Painter.
Trees

**Thm.** If $G$ is a tree, then $\hat{R}_\Delta(G) \leq 2\Delta(G) - 1$. 
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**Thm.** If $G$ is a tree, then $\hat{\mathcal{R}}_\Delta(G) \leq 2\Delta(G) - 1$.

**Pf.** Idea: Builder forces a large monochromatic complete $k$-ary tree, where $k = \Delta(G) - 1$. 
Trees

**Thm.** If $G$ is a tree, then $\hat{\Delta}(G) \leq 2\Delta(G) - 1$.

**Pf.** Idea: Builder forces a large monochromatic complete $k$-ary tree, where $k = \Delta(G) - 1$.

Candidate tree $T_R$ or $T_B$ has an active vertex $x_R$ or $x_B$ - a vertex of least depth w/o $k$ children via its own color.
**Thm.** If $G$ is a tree, then $\hat{\rho}_{\Delta}(G) \leq 2\Delta(G) - 1$.

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Candidate tree $T_R$ or $T_B$ has an active vertex $x_R$ or $x_B$ - a vertex of least depth w/o $k$ children via its own color.

Invariant: In $T_R$, each vertex other than $x_R$ either
1) is a leaf in $T_R$ with no other incident edge, or
2) has $k$ red children and at most $k$ blue incident edges.

(Symmetrically for $T_B$).
Thm. If $G$ is a tree, then $\hat{\Delta}(G) \leq 2\Delta(G) - 1$.

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(Symmetrically for $T_B$).

An active vertex becomes satisfied if it has $k$ children via its own color. dangerous if it has $k$ incident edges of the other color.
Builder Strategy

Builder plays pendant edges at active vertices (in $T_R$ or $T_B$) until Painter makes one satisfied or dangerous.
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Builder plays pendant edges at active vertices (in $T_R$ or $T_B$) until Painter makes one satisfied or dangerous.

When an active vertex is satisfied, Builder rechooses it (closest to root w/o $k$ children via its own color).
Builder Strategy

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When an active vertex is satisfied, Builder rechooses it (closest to root w/o $k$ children via its own color).

If $x_R$ and $x_B$ are both dangerous,
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When an active vertex is satisfied, Builder rechooses it (closest to root w/o $k$ children via its own color).

If $x_R$ and $x_B$ are both dangerous, Builder plays $x_R x_B$. 

![Diagram of Builder Strategy](image)
Builder Strategy

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When an active vertex is satisfied, Builder rechooses it (closest to root w/o $k$ children via its own color).

If $x_R$ and $x_B$ are both dangerous, Builder plays $x_R x_B$.

This edge enters the tree for its color, dragging the other tree with it.
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When an active vertex is satisfied, Builder rechooses it (closest to root w/o $k$ children via its own color).

If $x_R$ and $x_B$ are both dangerous, Builder plays $x_Rx_B$.

This edge enters the tree for its color, dragging the other tree with it.

Then Builder regenerates the other tree.
Even Cycles

Assume Builder plays on $S_k$ and Painter is consistent. (Weight = bound on total red + blue at a vertex.)

**Lem.** Let $F_1, F_2$ be weighted graphs Builder can force in red, with vertices $u_1, u_2$. Form $F$ from $F_1 + F_2$ by adding $u_1u_2$ and increasing weights on $u_1$ and $u_2$ by 2. If $q$ is even, then Builder can force a red $F$ or a blue $C_q$. 
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**Pf.** Builder forces $q/2$ copies of $F_1$ and $F_2$ and then adds a cycle alternating between the copies of $u_1$ and $u_2$. ■
Trees for Even Cycles

Consistent Painter makes the same monochr. $P_3$ (with weights 2) in any isolated triangle; we may assume it is red. Painter wants to avoid a monochromatic $C_q$. 

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Trees for Even Cycles

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$q = 8$
Trees for Even Cycles

Consistent Painter makes the same monochr. $P_3$ (with weights 2) in any isolated triangle; we may assume it is red. Painter wants to avoid a monochromatic $C_q$.

Further extensions of the tree force any even cycle $C_q$ (just extend one half if $q \equiv 2 \mod 4$), but $C_6$ and $C_{10}$ are special.
Special Case: $C_6$

Consistent Painter makes consistent triangles.

Case 1: monochromatic
Special Case: $C_6$

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Case 1: monochromatic

Case 2: not monochromatic
Special Case: $C_6$

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Case 1: monochromatic

Case 2: not monochromatic
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Consistent Painter makes consistent triangles.

Case 1: monochromatic

Case 2: not monochromatic
Summary for Cycles; Open Questions

\[ \hat{R}_\Delta(C_{10}) = 4 \text{ done similarly; } \hat{R}_\Delta(C_3) = 4 \text{ ad hoc.} \]

**Thm.** For even \( n \) with \( n \geq 4 \), \( \hat{R}_\Delta(C_n) = 4 \).
Summary for Cycles; Open Questions

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**Thm.** For even \( n \) with \( n \geq 4 \), \[ \hat{R}_\Delta(C_n) = 4. \]

**Thm.** For all \( n \), \[ \hat{R}_\Delta(C_n) \leq 5. \]
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**Thm.** If \( 337 \leq n \leq 514 \) or \( n \geq 689 \), then \( \hat{R}_\Delta(C_n) = 4 \).
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**Ques.** \( \hat{R}_\Delta(C_n) = 4 \) for all \( n \)?
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   Is \( R_\Delta(G) \) bounded by a function of \( \hat{R}_\Delta(G) \)?

**Ques.** What is \( \hat{R}_\Delta(C_5) \)? (4 or 5)
   What is \( \hat{R}_\Delta(K_{1,3} + e) \)? (4 or 5)
   What is \( \hat{R}_\Delta(C_4 + e) \)? (5 or 6 or 7.)
Lem. Against consistent Painter, if Builder can force red $F$ or monochr. $C_q$ ($q$ odd), then Builder can force red $F+uv$ or monochr. $C_q$, with wt on $u$ and $v$ up by 2.
Odd Cycles

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**Thm.** $\hat{R}_\Delta(C_q) \leq 5$ when $q$ is odd.
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**Pf.** Force monochr. $P_q$ (say red) with weights 3.

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**Thm.** $\hat{R}_\Delta(C_q) \leq 5$ when $q$ is odd.

**Pf.** Force monochr. $P_q$ (say red) with weights 3. Grow pendant paths.
Odd Cycles

**Lem.** Against consistent Painter, if Builder can force red $F$ or monochr. $C_q$ ($q$ odd), then Builder can force red $F+uv$ or monochr. $C_q$, with wt on $u$ and $v$ up by 2.

**Thm.** $\hat{R}_\Delta(C_q) \leq 5$ when $q$ is odd.

**Pf.** Force monochr. $P_q$ (say red) with weights 3. Grow pendant paths.

![Diagram of a graph with weights on edges]
Odd Cycles

**Lem.** Against consistent Painter, if Builder can force red $F$ or monochr. $C_q$ ($q$ odd), then Builder can force red $F+uv$ or monochr. $C_q$, with wt on $u$ and $v$ up by 2.

**Thm.** $\hat{R}_\Delta(C_q) \leq 5$ when $q$ is odd.

**Pf.** Force monochr. $P_q$ (say red) with weights 3. Grow pendant paths.

Leaf distances $q - 1$ (opposite halves or to middle). Cycle through the leaves is all blue or some red.