Degree Ramsey numbers for cycles and blowups of trees

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Abstract

Let $H \xrightarrow{s} G$ mean that every s-coloring of $E(H)$ produces a monochromatic copy of G in some color class. Let the *s-color degree Ramsey number* of a graph G, written $R_\Delta(G;s)$, be $\min\{\Delta(H)\colon\ H \xrightarrow{s} G\}$. We prove that the 2-color degree Ramsey number is at most 96 for every even cycle and at most 3458 for every odd cycle. For the general s-color problem on even cycles, we prove $R_\Delta(C_{2m};s) \le 16s^6$ for all m, and $R_\Delta(C_4;s) \ge .007s^{14/9}$. The constant upper bound for $R_\Delta(C_n;2)$ uses blowups of graphs, where the *d-blowup* of a graph G is the graph G' obtained by replacing each vertex of G with an independent set of size d and each edge e of G with a copy of the complete bipartite graph $K_{d,d}$. We also prove the existence of a function f such that if G' is the d-blowup of G, then $R_\Delta(G';s) \le f(R_\Delta(G;s),s,d)$.

1 Introduction

Given a target graph G, classical graph Ramsey theory seeks a graph H such that every 2-edge-coloring of H produces a monochromatic copy of G. Such a graph H is a *Ramsey host* for G; we then write $H \to G$ and say that H *arrows* or *forces* G. More generally, we write $H \xrightarrow{s} G$ when every s-edge-coloring of $E(H)$ produces a monochromatic copy of G. The classical *Ramsey number* $R(G;s)$ is $\min\{|V(H)|\colon\ H \xrightarrow{s} G\}$.

For any mononotone graph parameter ρ, the *ρ-Ramsey number* of G, written $R_\rho(G;s)$, is $\min\{\rho(H)\colon\ H \xrightarrow{s} G\}$. Classically, $\rho(G)$ was the number of vertices; the notion has also been studied with $\rho(G)$ being the the clique number [15, 25], the chromatic number [6, 31, 32], and the number of edges (yielding the *size Ramsey number*) [2, 3, 9, 11, 16, 27].

The *degree Ramsey number* is $R_\Delta(G;s)$, where $\Delta(G)$ denotes the maximum degree of G. Burr, Erdős, and Lovász [6] began its study, showing that $R_\Delta(K_n;s) = R(K_n;s) - 1$ and computing $R_\Delta(K_{1,m};2)$. Kinnersley, Milans, and West [20] showed more generally that $R_\Delta(K_{1,m};s)$ is $s(m-1)+1$ when m is odd and $s(m-1)$ when m is even. This yields

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sharpness of their upper bound for a class of trees: $R_\Delta(T;s) \le R_\Delta(K_{1,m};s)$ when T is a tree with one vertex of degree at most m whose other vertices have degree at most $\lceil m/2 \rceil$. Letting m be $2\Delta(T)$ gives the general upper bound $R_\Delta(T;s) \le 2s\Delta(T) - s + 1$ for all trees, which can be improved to $2s(\Delta(T) - 1)$. They showed near-sharpness of the general bound using double-stars (trees with two nonleaf vertices).

Our focus is on the degree Ramsey numbers of cycles. Always $R_\Delta(G;s) \le R(G;s) - 1$, because $K_{R(G;s)} \xrightarrow{s} G$. Although [6] showed that equality holds for complete graphs, our main result shows that when $s = 2$ the degree Ramsey numbers for cycles are much smaller than their classical Ramsey numbers. We give a small constant upper bound on the 2-color degree Ramsey numbers for all cycles, proving $R_\Delta(C_n;2) \le 96$ when n is even (Section 2) and $R_\Delta(C_n;2) \le 3458$ when n is odd (Section 4). (For exact results, [20] showed $R_\Delta(C_3;2) = R_\Delta(C_4;2) = 5$.) In contrast, the fixed-color ordinary Ramsey numbers of cycles grow linearly with n (when m is sufficiently large, $R(C_{2m};2) = 3m - 1$ and $R(C_{2m+1};2) = 4m + 1$ [14, 28], while $R(C_{2m};3) = 4m$ [4] and $R(C_{2m+1};3) = 8m + 1$ [21]).

For multicolor Ramsey or degree Ramsey numbers of cycles, it is natural to concentrate first on C_4. Chung and Graham [7] and Irving [19] showed that $R(C_4;s) \le s^2+s+1$ always; if $s-1$ is a prime power, then $R(C_4;s) > s^2-s+1$. More generally, Chung and Graham [7] gave an upper bound for $R(K_{p,q};s)$. Using that as an upper bound and the Local Lemma to prove a lower bound, we obtain $\left(\frac{1}{epq}\right)^{1/(p+q-2)} s^{(pq-1)/(p+q-2)} \le R_\Delta(K_{p,q};s) < \left\lfloor (q-1)(s+s^{1/p})^p \right\rfloor$. For $p = q = 2$, the lower bound is $\frac{1}{\sqrt{4e}} s^{3/2}$. By a more delicate argument, we improve that special case to $R_\Delta(C_4;s) \ge .007s^{14/9}$ (Section 3).

Upper bounds for multicolor Ramsey numbers of even cycles are well known. Bondy and Simonovits [5] proved that every n-vertex graph with more than $cn^{1+1/m}$ edges contains a $2m$-cycle. When $\binom{n}{2} > scn^{1+1/m}$, this yields $K_n \xrightarrow{s} C_{2m}$. Hence $R_\Delta(C_{2m};s) < R(C_{2m};s) < c_m s^{m/(m-1)}$, where c_m is a constant depending on m. For particular m, many authors have studied these Ramsey numbers; Li and Lih [24] discussed the history and showed that $R(C_{2m};s) \in \Theta(s^{m/(m-1)})$ for $m \in \{2,3,5\}$. Note that these results are given for fixed m and growing number of colors.

Now let the number of colors be fixed. Recently, Łuczak, Simonovits, and Skokan [23] proved that $R(C_n;s) \le sn + o(n)$ when n is even and $R(C_n;s) \le s2^s n + o(n)$ when n is odd. For even cycles, we prove $R_\Delta(C_{2m};s) \le 16s^6$ (Section 2), an upper bound depending only on s. In contrast, [20] showed $R_\Delta(C_n;s) \ge 2^s + 1$ when n is odd.

For each s, an upper bound on $R_\Delta(C_n;s)$ follows from earlier work of Haxell, Kohayakawa, and Łuczak [17], in which for fixed s they obtained an upper bound linear in n on the s-color induced size Ramsey number of C_n (that is, for some graph H with at most $c_s n$ edges, every s-edge-coloring contains a monochromatic copy of C_n as an induced subgraph). Because their bound applies also to odd cycles, it grows at least exponentially in s. Since their work used the Regularity Lemma, it yields a very large upper bound even when $s = 2$.

Our constructions for upper bounds on $R_\Delta(C_n;2)$ arise from a "blowup" operation. The *d-blowup* of a graph G is the graph G' obtained by replacing each vertex of G with an independent set of size d (called a *cluster*) and replacing each edge e of G with a copy of the complete bipartite graph $K_{d,d}$ whose partite sets are the clusters arising from the endpoints of e. The resulting graph has also been called the "d-multiplication" of G.

Blowups also shed some light on a fundamental question in degree Ramsey theory.

Question 1.1. Is $R_\Delta(G;s)$ bounded by some function of $\Delta(G)$ and s?

By analogy with Gyárfás' concept of χ-bounded graph families, we say that a family $\mathcal{G}$ of graphs is *R_Δ-bounded* if there is a function f such that $R_\Delta(G;s) \le f(\Delta(G),s)$ whenever $G \in \mathcal{G}$ and $s \ge 2$. Question 1.1 asks whether the family of all graphs is R_Δ-bounded.

In Section 5, we prove the existence of a function h such that if G' is the d-blowup of G, then $R_\Delta(G';s) \le h(R_\Delta(G;s),s,d)$. As a result, if $\mathcal{G}$ is an R_Δ-bounded family of graphs, then the family of d-blowups of graphs in $\mathcal{G}$ is also R_Δ-bounded.

A *closed k-blowup* of a graph G is obtained from G by replacing each vertex with a complete graph on k vertices and each edge with a complete bipartite graph. Horn, Milans, and Rödl [18] proved that the family of closed blowups of trees is R_Δ-bounded. In particular, let k and s be integers with $k \ge 2$ and $s \ge 1$, and let $r = R(K_{2k};s)$. If T is a tree with $|V(T)| \ge 3$ and G is the closed k-blowup of T, then $R_\Delta(G;s) \le (r-1)\left(2\binom{r}{2k}(\Delta(T)-1)\right)^{\binom{r-1}{2k-1}}$. This bound is quite a bit larger than the bound we obtain for the ordinary k-blowup in Section 5.

The degree Ramsey problem is much simpler for paths than for cycles. Alon, Ding, Oporowski, and Vertigan [1] gave a short proof that $R_\Delta(P_n;s) \le 2s$ for all n. By a more difficult argument, they showed that equality holds when n is sufficiently large in terms of s. For $s=2$, Thomassen [30] showed that $R_\Delta(P_n;2) = 4$ for $n \ge 7$, but it remains open whether $R_\Delta(P_6;2)$ is 3 or 4. For short paths, an old result of Egawa, Urabe, Fukuda, and Nagoya [10] yields $R_\Delta(P_4;s) \le 2s-3$ for $s \ge 4$, and [20] shows $R_\Delta(P_4;s) \ge s+1$.

The argument of [1] is generalized in Theorem 1.2 below to give upper bounds for trees. We will use its ideas in our upper bounds on $R_\Delta(C_n;2)$. The subsequent Corollary 1.4 is relevant to our comments on R_Δ-bounded families.

Theorem 1.2. *If T is a tree and H is a graph with average degree at least $2s(\Delta(T)-1)$ and girth larger than $2|V(T)|$, then $H \xrightarrow{s} T$.*

Proof. Let $k = \Delta(T)$, and consider an s-edge-coloring of H. By the pigeonhole principle, some color class has average degree at least $2(k-1)$. It is well known that when k is an integer greater than 1, any graph with average degree at least $2(k-1)$ contains a subgraph with minimum degree at least k.

Thus H contains a monochromatic subgraph H' with minimum degree at least k. Let u be a vertex in H'. Because H' has minimum degree at least k and girth larger than $2|V(T)|$, the subgraph of H' induced by $\{v\colon \operatorname{dist}_{H'}(v,u) \le |V(T)|\}$ contains T. $\square$

The argument of Erdős and Sachs [13] for the existence of regular graphs with large degree and girth can be modified easily to yield a bipartite analogue.

Lemma 1.3. *For each g and k, there is a k-regular bipartite graph with girth at least g.*

A bound for trees follows immediately from Theorem 1.2.

Corollary 1.4. *If T is a tree, then $R_\Delta(T; s) \le 2s(\Delta(T) - 1)$.*

The lower bound of Alon *et al.* [1] that $R_\Delta(P_n; s) \ge 2s$ shows that Corollary 1.4 is sharp when T is a path. In [20], it is shown that for each $\varepsilon > 0$, there is an s such that for all sufficiently large k, there exists a tree T with $\Delta(T) = k$ such that $R_\Delta(T; s) \ge (2-\varepsilon)s(k-1)$.

By Corollary 1.4, within the family of trees, the degree Ramsey number is bounded by a function of the number of colors and the maximum degree. Section 5 extends this statement to larger families.

2 Upper Bound for Even Cycles

As noted earlier, the result of Haxell, Kohayakawa, and Łuczak [17] yields a very large constant c such that $R_\Delta(C_n; 2) \le c$ for all n. In this section, we prove that $R_\Delta(C_{2n}; 2) \le 96$.

For a connected bipartite graph F with partite sets of size ℓ, an F*-blowup* of a graph G is a graph H formed by replacing each vertex in G with an independent set of size ℓ and expanding each edge in G into a copy of F. By allowing the vertices of the cluster from a vertex u in a regular graph G to represent each vertex of F equally often as the edges at u expand into copies of F, we can keep the maximum degree of the blowup small.

Lemma 2.1. *Let F be a connected bipartite graph with partite sets of size ℓ and average degree d. Let G be a k-regular graph, where (1) k is a multiple of 2ℓ or (2) G is bipartite and k is a multiple of ℓ. If kd is an integer, then there is a kd-regular F-blowup of G.*

Proof. We construct a blowup G' of G, expanding each $w \in V(G)$ into a cluster S_w of size ℓ and each edge of G into a copy of F. Each edge of G becomes ℓd edges in G'. Since G' has $\ell|V(G)|$ vertices and $\ell dk|V(G)|/2$ edges, it suffices to do this so that G' is regular. Let A and B be the partite sets of F. Index the vertices of A, B, and each cluster using indices $1, \ldots, \ell$, treating the indices modulo ℓ.

In Case (1), G decomposes into 2-factors $H_1, \ldots, H_k$, by Petersen's Theorem [26], providing a k-coloring of $E(G)$ that uses each color twice at each vertex. Orient each H_i consistently, so that each vertex has indegree 1 and outdegree 1 in H_i. In Case (2), the k-regular bipartite graph G decomposes into 1-factors, providing a k-coloring of $E(G)$ using each color once at each vertex. Orient all the edges from one partite set of G to the other.

For an edge $uv \in E(G)$ with color i, oriented from u to v, replace uv with a copy of F as follows: for $1 \le j \le \ell$, view the jth vertices of S_u and S_v as the $(j+i)$th vertices of A and B. Over all the colors, each vertex of G' serves as each vertex of the appropriate partite set in F the same number of times. Hence we have expanded the edges of G into copies of F so that the resulting graph G' is regular. $\square$

To obtain monochromatic even cycles, it will suffice to obtain a monochromatic P_4-blowup of a long path. Let $\langle v_1, \ldots, v_k \rangle$ denote the path with vertices $v_1, \ldots, v_k$ in order.

Lemma 2.2. *If H is a P_4-blowup of P_{t+1}, then H contains C_{2t}.*

Proof. For the path $\langle z_1, \ldots, z_{t+1} \rangle$, let the corresponding clusters be $Z_1, \ldots, Z_{t+1}$ in H; each Z_i has size 2. Joining Z_j and Z_{j+1} is a copy of P_4 and hence a matching of size 2. Linking these matchings from Z_2 through Z_t yields two disjoint copies of P_{t-1}. Because there is a copy of P_4 joining Z_1 and Z_2, some vertex in Z_1 is adjacent to both vertices in Z_2. Similarly, some vertex in Z_{t+1} is adjacent to both vertices in Z_t. These vertices extend the union of the two copies of P_{t-1} to a $2t$-cycle. $\square$

We next show how to force monochromatic P_4-blowups of long paths. An (X,Y)-*bigraph* is a bipartite graph with partite sets X and Y.

Lemma 2.3. *Let F be a bipartite graph with parts of size ℓ such that $F \xrightarrow{s} P_4$. Let H_0 be a $2s\binom{\ell}{2}^2$-regular (X,Y)-bigraph with girth more than $2t$. If H is an F-blowup of H_0, then every s-edge-coloring of H contains a monochromatic P_4-blowup of a t-vertex path in H_0.*

Proof. We use an s-edge-coloring of H to construct an $s\binom{\ell}{2}^2$-edge-coloring of H_0. For each vertex u in H_0, fix an indexing $\{u^1, \ldots, u^\ell\}$ of the cluster in H corresponding to u. Consider an edge $uv \in E(H_0)$ with $u \in X$ and $v \in Y$; we assign a color to uv. Let F_{uv} be the copy of F in H arising from the edge uv in H_0.

Since $F \xrightarrow{s} P_4$, in F_{uv} we find a monochromatic copy P of P_4. We color uv with a 3-tuple. The first coordinate is the color of $E(P)$. The second coordinate is the pair of indices $\{i,j\}$ such that the vertices of P in the cluster for u are u^i and u^j. Similarly, the third coordinate records the indices of the vertices of P in the cluster for v. With s choices for the first coordinate and $\binom{\ell}{2}$ choices for the second and third, $s\binom{\ell}{2}^2$ colors are used on $E(H_0)$.

Since H_0 has girth larger than $2t$, Theorem 1.2 implies $H_0 \xrightarrow{s'} P_t$, where $s' = s\binom{\ell}{2}^2$. A monochromatic copy of P_t in H_0 yields a monochromatic P_4-blowup of P_t in H. $\square$

Theorem 2.4. *If there is a bipartite s-color Ramsey host for P_4 with parts of size ℓ and average degree d, then $R_\Delta(C_{2n}; s) \le 2sd\binom{\ell}{2}^2$.*

Proof. Let F be an s-color Ramsey host for P_n with parts of size ℓ and average degree d. Let H_0 be a $2s\binom{\ell}{2}^2$-regular bipartite graph with girth more than $2(n+1)$. By Lemma 2.1, there is a $2sd\binom{\ell}{2}^2$-regular F-blowup H of H_0. We claim that $H \xrightarrow{s} C_{2n}$. By Lemma 2.3, any s-edge-coloring of H contains a monochromatic P_4-blowup of P_{n+1}. By Lemma 2.2, such a coloring contains a monochromatic copy of C_{2n}. □

Corollary 2.5. $R_\Delta(C_{2m}; s) \le 16s^6$.

Proof. A bipartite graph contains P_4 unless it is a forest of stars. A forest of stars in $K_{2s,2s}$ has at most $4s-2$ edges, but every s-edge-coloring puts at least $4s$ edges in some color class. Hence $K_{2s,2s} \xrightarrow{s} P_4$. Now apply Theorem 2.4 with $\ell = d = 2s$. □

The exponent in Corollary 2.5 seems weak in light of $R_\Delta(C_{2m}; s) \le R(C_{2m}; s) \le c_m s^{m/(m-1)}$, but the latter is for fixed m as s grows, while Corollary 2.5 is a statement for all m and s. In particular, for fixed s it yields a constant upper bound for all even cycles.

For $s \le 3$, our main result reduces that constant. A graph G is *edge-transitive* if whenever $uv, xy \in E(G)$, there is an automorphism of G mapping $\{u, v\}$ to $\{x, y\}$. For an edge-transitive graph G, let G^- denote the graph obtained by deleting any one edge.

Corollary 2.6. $R_\Delta(C_{2n}; 2) \le 96$ *and* $R_\Delta(C_{2n}; 3) \le 864$.

Proof. The bounds follow from Theorem 2.4 after showing $K_{3,3}^- \to P_4$ and $K_{4,4} \xrightarrow{3} P_4$. (Unfortunately, $K_{4,4}^-$ is not a 3-color Ramsey host for P_4. There is a 3-edge-coloring in which the graph in each color is $2P_3 + P_2$.)

A bipartite graph contains P_4 unless it is a forest of stars. In $K_{3,3}$ a forest of stars has at most four edges. Hence $K_{3,3}^- \to P_4$ unless the graph in each color is $2P_3$. The centers of the two stars in one color have only each other as a possible remaining neighbor; hence they are nonadjacent in $K_{3,3}^-$. Now the remaining graph for the other color is C_4, not $2P_3$.

Since $K_{4,4}$ has 16 edges, in any 3-edge-coloring of $K_{4,4}$ some color class has at least six edges. The largest forest of stars has six edges, occurring only as $2K_{1,3}$. In this case, the graph for the two remaining colors contains $K_{3,3}$, forcing P_4. □

3 Lower Bound for Bipartite Graphs

We have noted that upper bounds for $R(G; s)$ yield upper bounds for $R_\Delta(G; s)$. In particular, Chung and Graham [7] proved $R(K_{p,q}; s) \le (q-1)(s + s^{1/p})^p$. For $p = 2$, this was sharpened to $R(K_{2,q}; s) \le (q-1)s^2 + s + 2$, with strict inequality when $p = q = 2$. We use the Local Lemma to provide lower bounds on degree Ramsey numbers. The bound is asymptotically stronger than the lower bound on $R(K_{p,q}; s)$ in [7].

Theorem 3.1. $\left(\frac{1}{epq}\right)^{1/(p+q-2)} s^{(pq-1)/(p+q-2)} \le R_\Delta(K_{p,q}; s) < \lfloor (q-1)(s+s^{1/p})^p \rfloor$.
With a stronger upper bound, $\frac{1}{\sqrt{4e}} s^{3/2} \le R_\Delta(C_4; s) \le s^2 + s + 1$.

Proof. The upper bounds follow from $R_\Delta(G; s) < R(G; s)$. For the lower bound, consider any graph H with maximum degree less than $cs^{(pq-1)/(p+q-2)}$, where $c = \left(\frac{1}{epq}\right)^{1/(p+q-2)}$. Color the edges of H independently at random from s colors. The probability that any copy of $K_{p,q}$ is monochromatic is $1/s^{pq-1}$.

The number of copies of $K_{p,q}$ containing a given edge of H is at most $\binom{\Delta(H)-1}{p-1}\binom{\Delta(H)-1}{q-1}$, which is less than $\Delta(H)^{p+q-2}$. Thus the event that a particular copy of $K_{p,q}$ is monochromatic is mutually independent of a set of all but fewer than $pq[\Delta(H)]^{p+q-2}$ other events. Since $\Delta(H) \le \left(\frac{1}{epq} s^{pq-1}\right)^{1/(p+q-2)}$, the Local Lemma guarantees an s-coloring of $E(H)$ with no monochromatic copy of $K_{p,q}$. □

With a more delicate argument, we can improve the lower bound on $R_\Delta(C_4; s)$ for large s. For convenience, we will call an edge-coloring that contains no monochromatic C_4 a *C_4-free coloring*. Recall that if $k-1$ is a prime power, then $R(C_4; k) \ge k^2 - k + 2$ ([7, 19]; see [22] for a slight improvement). This implies (with room to spare) that K_n has a C_4-free edge-coloring using at most $\lfloor 4\sqrt{n} \rfloor$ colors.

Lemma 3.2. *Every bipartite graph with maximum degree at most d has a C_4-free edge-coloring using fewer than $8.7d^{9/14}$ colors.*

Proof. Let H be such a graph, with partite sets X and Y. We decompose H into two subgraphs H' and H'', which will be colored using disjoint sets of colors. The claim clearly holds when $d \le 2$, so we may assume $d \ge 3$.

We form H' by iteratively including certain subgraphs with "large" minimum degree. Let $H_1 = H$. As long as H_i has an edge that lies in at least $2d^{27/14}$ 4-cycles, we will extract the edges of a subgraph F_i, letting $H_{i+1} = H_i - E(F_i)$. Eventually the process must end with a subgraph having no such edge, say H_m. We then let $H' = \bigcup_{i=1}^{m-1} F_i$ and $H'' = H_m$.

While $i < m$, determine F_i as follows. Let xy be an edge in at least $2d^{27/14}$ 4-cycles in H_i. These 4-cycles correspond to edges in the subgraph L_{xy} induced by $N(x) \cup N(y) - \{x, y\}$. Since $\Delta(H) \le d$, there are fewer than $2d$ vertices in L_{xy}, so its average degree is at least $2d^{13/14}$. Every nontrivial graph G has a subgraph whose minimum degree is at least half the average degree in G. Let F_i be a subgraph of L_{xy} having minimum degree at least $d^{13/14}$.

We color the edges of the residual graph H'' as in the proof of Theorem 3.1. By construction, each edge of H'' lies in fewer than $2d^{27/14}$ 4-cycles in H''. In a random s-coloring, each 4-cycle has probability $1/s^3$ of being monochromatic. The event that a particular 4-cycle is monochromatic is mutually independent of a set of all but fewer than $4(2d^{27/14})$ other such events. When $e(8d^{27/14})/s^3 \le 1$, the Local Lemma ensures that some outcome has no monochromatic 4-cycle. The inequality holds for some integer s less than $3d^{9/14}$ when $d \ge 3$.

Now consider H', the union of $F_1, \ldots, F_{m-1}$. We have noted that $|V(F_i)| < 2d$. As noted before the proof, F_i therefore has a C_4-free edge-coloring c_i using at most $\lfloor 4\sqrt{2d} \rfloor\}$ colors. Since $\delta(F_i) \geq d^{13/14}$, each vertex of H' lies in at most $d^{1/14}$ of these subgraphs. Consider $uv \in E(H')$, where $u \in X$, $v \in Y$, and $uv \in E(F_i)$. Let $\phi(uv) = (a, b, c_i(uv))$, where $a = |\{j\colon\ u \in V(F_j) \text{ and } j < i\}|$ and $b = |\{j\colon\ v \in V(F_j) \text{ and } j < i\}|$. The number of colors used by the edge-coloring ϕ of H' is less than $(d^{1/14})^2 \cdot 4\sqrt{2d}$, which is less than $5.7d^{9/14}$.

We claim that ϕ is C_4-free. If the first two coordinates of ϕ agree on two incident edges e and e', then the common vertex of these edges has appeared in the same number of previous subgraphs, so e and e' belong to the same subgraph F_i. Since c_i is not constant on any 4-cycle in F_i, also ϕ is not.

Thus ϕ and the coloring obtained for H'' together define a C_4-free edge-coloring of H using fewer than $8.7d^{9/14}$ colors. □

Lemma 3.3. *Every graph with maximum degree at most d has a C_4-free edge-coloring using fewer than $24.2d^{9/14}$ colors.*

Proof. We use induction on d. By Theorem 3.1, H has a C_4-free s-coloring when $d < (1/\sqrt{4e})s^{3/2}$, or equivalently when $s > (4e)^{1/3}d^{2/3}$. If $d \leq (10.9)^{42}$, then $\lceil (4e)^{1/3}d^{2/3} \rceil < \lfloor 24.2d^{9/14} \rfloor$, and the desired coloring exists.

For larger d, let H_1 be a largest bipartite subgraph of H; let X_1 and Y_1 be its partite sets. By the choice of H_1, each vertex of H has at least as many neighbors in the opposite partite set as in its own. Hence $H[X_1]$ and $H[Y_1]$ both have maximum degree at most $d/2$.

By the induction hypothesis, each of $H[X_1]$ and $H[Y_1]$ has a C_4-free edge-coloring using fewer than $24.2(d/2)^{9/14}$ colors. Since they share no vertices, their union has such a coloring. Since H_1 is bipartite with maximum degree at most d, Lemma 3.2 gives H_1 a C_4-free edge-coloring using fewer than $9d^{9/14}$ colors. The two colorings together form a C_4-free edge-coloring of H using fewer than $9d^{9/14} + 24.2(d/2)^{9/14}$ colors. Since $9 + 24.2/2^{9/14} < 24.2$, the proof is complete. □

Theorem 3.4. $R_\Delta(C_4; s) \geq .007s^{14/9}$.

Proof. By Lemma 3.3, every graph with maximum degree at most d has a C_4-free edge-coloring using fewer than $24.2d^{9/14}$ colors. When $d = \lfloor .007s^{14/9} \rfloor$, we have $24.2d^{9/14} \leq s$. □

We have seen that $R(C_4; s)$ is approximately s^2 for infinitely many s ([7]). Although this is an upper bound for $R_\Delta(C_4; s)$, as a lower bound we have only been able to prove $\Omega(s^{14/9})$. This prompts the natural question of determining the order of growth of $R_\Delta(C_4; s)$ and also the following general question:

Question 3.5. For which graphs G is it true that $\lim_{s\to\infty} \frac{R_\Delta(G;s)}{R(G;s)} = 0$?

4 Upper Bound for Odd Cycles

To prove $R_\Delta(C_n; 2) \le 3458$ for all n, we need another simple fact about $K_{3,3}^-$ and P_4.

Proposition 4.1. *Deleting any one vertex from each part of a bipartition of $K_{3,3}^-$ leaves a copy of P_4.*

Proof. Deleting one vertex from each part yields $K_{2,2}$ or $K_{2,2}^-$, each of which contains P_4. □

To complete our proof for odd cycles, we need variations on Lemma 2.2. Let $[n]$ denote $\{1, \dots, n\}$.

Lemma 4.2. *Let H contain a P_4-blowup of a path $\langle z_1, \dots, z_t \rangle$, where each vertex z_j expands into a set Z_j of size 2. If the vertices of Z_1 are adjacent in H, then H contains cycles of every odd length from 3 to $2t - 1$.*

Proof. For $1 \le j \le t-1$, a copy of P_4 in H with vertex set $Z_j \cup Z_{j+1}$ contains a matching of Z_j into Z_{j+1}. Choose $k \in [t-1]$. Linking the matchings from Z_1 through Z_k yields two disjoint copies of P_k. In the copy of P_4 joining Z_k and Z_{k+1}, some vertex in Z_{k+1} is adjacent to both vertices in Z_k, linking the two copies of P_k to form a path of length $2k$ with both endpoints in Z_1. The edge within Z_1 completes a cycle of length $2k + 1$. □

Lemma 4.3. *Fix $t \in \mathbb{N}$ with $t \ge 5$. Let H contain a P_4-blowup of a path $\langle z_1, \dots, z_t \rangle$, where each vertex z_j expands to a set Z_j of two vertices in H. If H contains an edge e joining Z_1 and Z_3, then H contains cycles of every odd length from 7 to $2t - 3$.*

Proof. As in the proof of Lemma 4.2, the matchings from Z_1 through Z_t yield disjoint copies of P_t. Let $Z_j = \{z_j^1, z_j^2\}$, indexed so that $\langle z_1^1, \dots, z_t^1 \rangle$ and $\langle z_1^2, \dots, z_t^2 \rangle$ are paths in H.

For $4 \le k \le t$, some vertex in Z_k is adjacent to both z_{k-1}^1 and z_{k-1}^2 and completes a path of length $2k - 6$ with $\langle z_3^1, \dots, z_{k-1}^1 \rangle$ and $\langle z_3^2, \dots, z_{k-1}^2 \rangle$. It therefore suffices to prove that there is a path of length 3 or length 5 joining z_3^1 and z_3^2 in the subgraph of H induced by $Z_1 \cup Z_2 \cup Z_3$. (In the first case, we obtain paths of lengths 5 through $2t - 3$; in the second case, we obtain lengths 7 through $2t - 1$.)

If e has endpoints with different superscripts, then we may assume $e = z_1^1 z_3^2$, and the desired path is $\langle z_3^1, z_2^1, z_1^1, z_3^2 \rangle$. Otherwise, we may assume $e = z_1^1 z_3^1$. The copy of P_4 joining Z_1 and Z_2 uses either $z_1^1 z_2^2$ or $z_2^1 z_1^2$. In the first case, the desired path is $\langle z_3^1, z_1^1, z_2^2, z_3^2 \rangle$. Otherwise, it is $\langle z_3^1, z_1^1, z_2^1, z_1^2, z_2^2, z_3^2 \rangle$. □

The *square* of a graph G, denoted G^2, is the graph with vertex set $V(G)$ in which two vertices are adjacent if the distance between them in G is at most 2. Note that $G \subseteq G^2$.

Theorem 4.4. *$R_\Delta(C_n; 2) \le 3458$.*

Proof. When n is even, the result follows from Theorem 2.5. If $n \in \{3, 5\}$, then n is small enough so that a complete graph with sufficiently small maximum degree is a Ramsey host for C_n. We thus assume that n is odd and at least 7.

Let H_0 be a 36-regular (X_0, Y_0)-bigraph with girth at least $2t$, where $t > n$. Let $H_1 = H_0^2$. Since H_0 is bipartite, the extra edges in H_1 have both endpoints in the same partite set of H_0. Each vertex in H_1 is incident to 36 edges that cross the bipartition of H_0 and to $36 \cdot 35$ edges inside its partite set, since H_0 has no triangles. Thus H_1 is 1296-regular. By Lemma 2.1, there is a 3456-regular $K_{3,3}^-$-blowup H_2 of H_1. Let H be the 3458-regular graph obtained from H_2 by adding a triangle on each cluster in H_2.

Let X be the union of the clusters in H_2 arising from X_0, and let Y be the union of the clusters in H_2 arising from Y_0. Note that X and Y partition $V(H)$, and the subgraph of H given by the cut $[X, Y]$ is a $K_{3,3}^-$-blowup of H_0.

We claim that $H \to C_n$. Consider a 2-edge-coloring of H. Apply Lemma 2.3 to the cut $[X, Y]$ to obtain a monochromatic P_4-blowup of a path $\langle z_1, \ldots, z_t \rangle$ in H_0 that alternates between X_0 and Y_0. Let red be the color of this P_4-blowup, and let blue be the other color. For $1 \le j \le t$, let Z_j be the 3-vertex cluster in H corresponding to z_j. Let z_j^1 and z_j^2 be the vertices of Z_j used in the P_4-blowup, and let $Z_j' = \{z_j^1, z_j^2\}$.

By construction, $z_j^1 z_j^2 \in E(H)$; call these edges *internal.* Since the distance in H_0 between z_j and z_{j+2} is 2, in H_1 they are adjacent, and hence in H the clusters Z_j and Z_{j+2} are joined by a copy of $K_{3,3}^-$. By Proposition 4.1, in H there is a (not necessarily monochromatic) copy of P_4 joining Z_j' and Z_{j+2}'. Call the edges of these copies of P_4 *skip edges.*

If the internal edge in Z_j' is red, then Lemma 4.2 applies to the longer of $\langle z_j, \ldots, z_t \rangle$ and $\langle z_j, \ldots, z_1 \rangle$ to yield a red n-cycle. If some skip edge joining Z_j' and Z_{j+2}' is red, then Lemma 4.3 applies to the longer of $\langle z_j, \ldots, z_t \rangle$ and $\langle z_{j+2}, \ldots, z_1 \rangle$ to yield a red n-cycle.

Hence we may assume that every internal edge and every skip edge is blue. Now H contains a blue P_4-blowup of the path $\langle z_1, z_3, z_5, \ldots, z_n \rangle$. Because the internal edges are blue, Lemma 4.2 yields a blue n-cycle. □

5 R_Δ-bounded Families

In this section we prove that the family of all (open) blowups of trees is R_Δ-bounded. Since the family of trees is R_Δ-bounded, by Corollary 1.4, this is an immediate consequence of our main result in this section, which is a bound on the s-color degree Ramsey number of a d-blowup of G in terms of d, s, and $R_\Delta(G; s)$.

Our proof is similar to the proof of Burr, Erdős, and Lovász [6] that the chromatic Ramsey number of a graph equals the ordinary Ramsey number of its family of homomorphic images.

The main tool is the bipartite version of Ramsey's Theorem, perhaps observed first by Erdős and Rado [12]: for each d and s, there is an integer m such that $K_{m,m} \xrightarrow{s} K_{d,d}$. Let $B_s(d)$ be the least m such that $K_{m,m} \xrightarrow{s} K_{d,d}$. Currently, the best known bound for $s = 2$ is $B(d) \le (1 + o(1))2^{d+1} \log_2 d$ (Conlon [8]).

Let B_s^k be the iterated composition of B_s with itself k times. For example, $B_s^2(d) = B_s(B_s(d))$ and $B_s^0(d) = d$.

Theorem 5.1. *If $k = R_\Delta(G; s)$ and G' is the d-blowup of G, then $R_\Delta(G'; s) \le kB_s^{k+1}(d)$. In particular, the family of blowups of graphs in a R_Δ-bounded family is R_Δ-bounded.*

Proof. Let H be a graph with maximum degree k such that $H \xrightarrow{s} G$, let $m_j = B_s^j(d)$ for $j \ge 0$, and let H' be the m_{k+1}-blowup of H. We show that $H' \xrightarrow{s} G'$.

Fix an s-edge-coloring of H'. By Vizing's Theorem [29], the edges of H can be partitioned into $k+1$ matchings $M_1, \ldots, M_{k+1}$. Each edge uv in H corresponds to a copy of $K_{m_{k+1},m_{k+1}}$ in H'. Because $m_{k+1} = B_s(m_k)$, each s-edge-coloring of $K_{m_{k+1},m_{k+1}}$ contains a monochromatic copy of K_{m_k,m_k}. For each edge uv in M_{k+1}, find a monochromatic copy of K_{m_k,m_k} in its blowup in H'. In the clusters for u and v, retain only the vertices in this subgraph, and for each vertex not incident to an edge of M_{k+1}, retain an arbitrary set of m_k vertices in its cluster. Since M_{k+1} is a matching, from each cluster we discard vertices only once; the resulting induced subgraph H'_k of H' has m_k vertices from each cluster.

Iterate this process for the remaining matchings $M_k, \ldots, M_1$, obtaining successive induced subgraphs $H'_{k-1}, \ldots, H'_0$. In the final subgraph H'_0 of H', the clusters correspond to the vertices of H, each cluster consists of d vertices, and each copy of $K_{d,d}$ joining two clusters is monochromatic in the given s-edge-coloring.

The edge-colored graph H'_0 corresponds to an s-edge coloring of H by assigning uv the color of the edges in the copy of $K_{d,d}$ joining the u-cluster and v-cluster in H'_0. Because $H \xrightarrow{s} G$, a monochromatic copy of G occurs, and this copy lifts to a monochromatic copy of G' in H'_0. It follows that $H' \xrightarrow{s} G'$. □

Theorem 5.1 yields an upper bound on $R_\Delta(G'; s)$ that is a tower function of height k. Our next aim is to reduce this upper bound when G is bipartite. If G has a bipartite s-color Ramsey host H with maximum degree k, then the d-blowup G' of G satisfies $R_\Delta(G'; s) \le kd \cdot 4^{ds^{k+1}}$. This substantial improvement in the bound from Theorem 5.1 is obtained by using a highly unbalanced complete bipartite graph to force $K_{d,d}$ instead of a symmetric one. We need a simple and standard lemma; we include its proof for completeness.

Lemma 5.2. $K_{sm,\binom{sm}{m}N} \xrightarrow{s} K_{m,N}$.

Proof. Let $H = K_{sm,\binom{sm}{m}N}$. Let the partite sets of H be X and Y with $|X| = sm$ and $|Y| = \binom{sm}{m}N$. Consider an s-coloring of H, and let L be a largest color class; L has at least

$mN\binom{sm}{m}$ edges. Let λ be the number of copies of $K_{m,1}$ in L with center in Y. Using the convexity of $\binom{x}{m}$ and $\sum_{y\in Y} d_L(y) \ge mN\binom{sm}{m}$, we have

$$\lambda = \sum_{y\in Y} \binom{d_L(y)}{m} \ge \binom{sm}{m} N \cdot \binom{m}{m} = \binom{sm}{m} N.$$

Since X has $\binom{sm}{m}$ subsets of size m, some m-subset of X is the leaf set of at least N of these copies of $K_{m,1}$. Thus $K_{m,N} \subseteq L$. □

Theorem 5.3. *If a bipartite graph H is an s-color Ramsey host for G with $\Delta(H) = k$, and G' is a d-blowup of G, then $R_\Delta(G'; s) \le kd \cdot 4^{ds^{k+1}}$.*

Proof. Let $m_0 = n_0 = d$. For $i \ge 0$, let $m_{i+1} = sm_i$ and $n_{i+1} = \binom{sm_i}{m_i} n_i$. Note that $n_i \ge m_i = ds^i$. By Lemma 5.2, $K_{m_{i+1},n_{i+1}} \xrightarrow{s} K_{m_i,n_i}$. Since $n_{i+1} \le 2^{sm_i} n_i$ and $n_0 = d$ (and $s \ge 2$), we have by induction on i that $n_{i+1} \le d \cdot 4^{ds^{i+1}}$.

Let (X, Y) be a bipartition of H. Obtain H' by blowing up vertices in X into m_{k+1} copies and vertices in Y into n_{k+1} copies. Since $\Delta(H) \le k$, we have $\Delta(H') \le kd \cdot 4^{ds^{k+1}}$. The argument that $H' \xrightarrow{s} G'$ is essentially the same as in the proof of Theorem 5.1, except that at stage i we extract a monochromatic K_{m_i,n_i} from each s-colored copy of $K_{m_{i+1},n_{i+1}}$. □

For the family of blowups of trees, we can get an even better bound by combining the strategy used in Corollary 1.4 and the idea of the construction in Lemma 2.3.

Theorem 5.4. *Let T be a tree, and let G be the d-blowup of T. If $m = B_s(d)$, then $R_\Delta(G; s) \le 2sm\binom{m}{d}^2(\Delta(T) - 1)$.*

Proof. Let H_0 be an (X, Y)-bigraph with girth larger than $2|V(T)|$ that is regular of degree $2s\binom{m}{d}^2(\Delta(T) - 1)$. Let H be the m-blowup of H_0. For each cluster U in H corresponding to a vertex u in H_0, fix an arbitrary indexing $\{u_1, \ldots, u_m\}$ of U. We claim that $H \xrightarrow{s} G$.

Given an s-edge-coloring of H, we construct a $s\binom{m}{d}^2$-edge-coloring of H_0. An edge uv in H_0 with $u \in X$ and $v \in Y$ becomes a copy of $K_{m,m}$ in H whose partite sets are the clusters U and V for u and v. Because $m = B_s(d)$, some subsets $U_0 \subseteq U$ and $V_0 \subseteq V$ of size d induce a monochromatic copy F of $K_{d,d}$. We color uv in H_0 with a 3-tuple. The first coordinate records the color on $E(F)$; the second and third coordinates record the subsets U_0 and V_0 (respectively) as subsets of the indices $1, \ldots, m$. There are $s\binom{m}{d}^2$ such colors.

Since H_0 is regular of degree $2s\binom{m}{d}^2(\Delta(T)-1)$ with girth larger than $2|V(T)|$, Theorem 1.2 implies that every $s\binom{m}{d}^2$-edge-coloring of H_0 contains a monochromatic copy of T. Using the monochromatic copies of $K_{d,d}$ corresponding to edges of H_0 in this copy of T, we obtain a monochromatic copy of G in H. (The second and third coordinates in the coloring of H_0 ensure that the copies of $K_{d,d}$ come together at common vertices in the clusters.) □

A *grid* is the Cartesian product of two paths. Grids have maximum degree 4 but do not all arise as subgraphs of d-blowups of trees for any fixed d. Consequently, Theorem 5.4 does not give a constant upper bound on the degree Ramsey number of grids. Beyond trees and cycles, grids seem like a natural next family to consider.

Question 5.5. Is the family of grids R_Δ-bounded? More simply, is there a constant upper bound on $R_\Delta(G; 2)$ when G is a grid?

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