An Introduction to the Discharging Method via Graph Coloring

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slides available on DBW preprint page

Based on a survey written with Daniel W. Cranston

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"An unavoidable set of reducible configurations"

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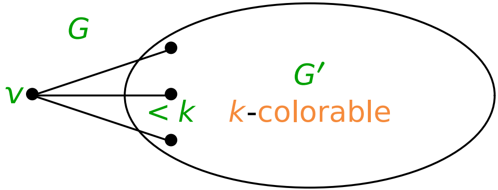

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This is sharp, since $\mathrm{Mad}(K_k) = k-1$ and $\chi(K_k) = k$.

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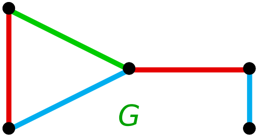

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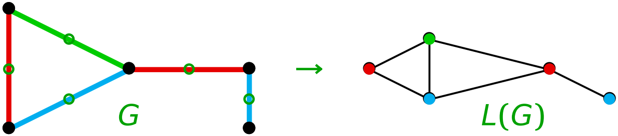

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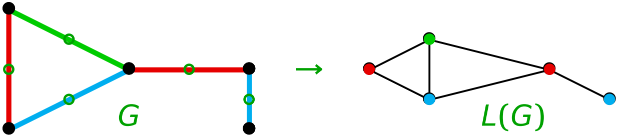

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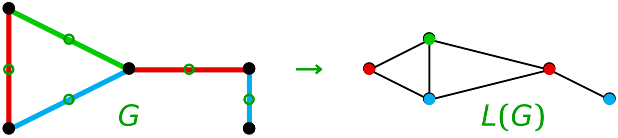

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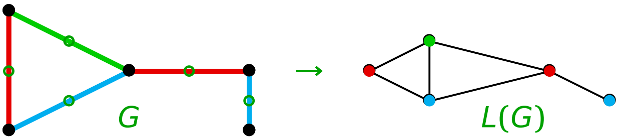

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Pf. An edge with weight at most $k+1$ in G has degree at most $k-1$ as a vertex in $L(G)$. ∎

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Avoiding S $\Rightarrow$ total charge $\ge bn$ $\Rightarrow$ avg degree $\ge b$.
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We want configurations in S to be reducible for the desired graph property (such as $\chi'(G) \le k$).

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H  G

$|V(H)| = n$, $|E(H)| = 3n$, $|V(G)| = 4n$, $|E(G)| = 6n$, $\overline{d}(G) = 3$.

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Need $d(w)[1 - \frac{b-1}{2}] \geq b$, hardest for $d(w) = k$.

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Thm. Fix $k \geq 3$. If $\overline{d}(G) < \frac{4k}{k+2}$, then G has a 1^--vertex or G has an edge with weight at most $k+1$.

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2-vertices need charge, as do 3-vertices when $k > 6$.

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We know the proof before the theorem statement!

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Since an edge of weight $\le k+1$ is reducible for k-edge-colorability, the claim holds by induction. ∎

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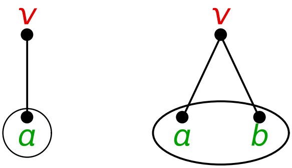

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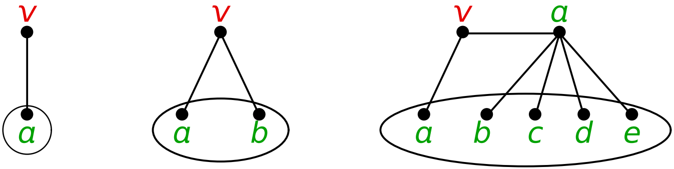

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Pf. Every subgraph of G also is planar with girth ≥ 6, so $\mathrm{Mad}(G) < 3$. ∎

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Pf. Multiply Euler's Formula by -6 or -4 and split the contribution from the number of edges:

$$-6n + 2m + 4m - 6p = -12; \quad -4n + 2m + 2m - 4p = -8.$$

Substitute the degree-sum in G or G^* for $2m$. ■

Three Types of Initial Charging

	Charge on v	Charge on f		

$$\sum_v (d(v)-6) + \sum_f (2d(f)-6) = -12$$ vertex charging

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Method: Use one of these as initial charging.

If no desired configuration occurs, then discharging rules move charge so that every vertex and face has nonnegative charge (is happy).

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Balanced charging: When $\delta(G), \delta(G^*) \geq 3$, only 3-vertices and 3-faces need charge.

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For $\delta(G) \geq 2$, if $\mathrm{Mad}(G) < \frac{8}{3}$ then $\exists$ 2-vert with 3^{-}-nbr.

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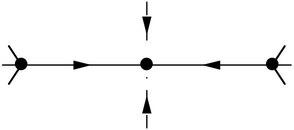

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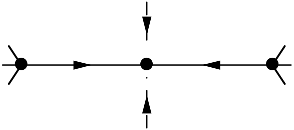

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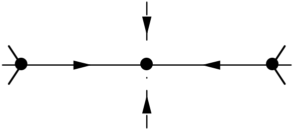

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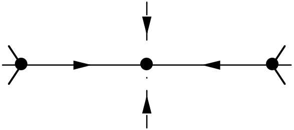

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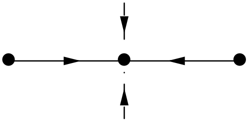

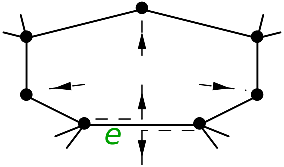

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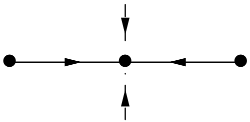

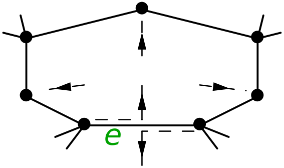

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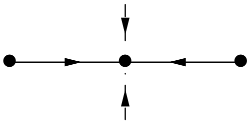

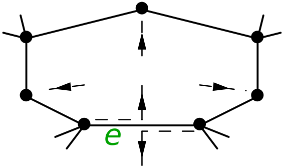

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- Discharging may take too much! Maybe add more discharging or show that the offending configuration is also reducible.

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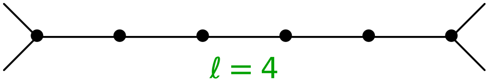

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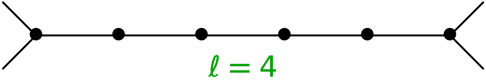

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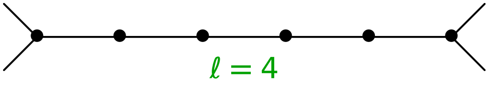

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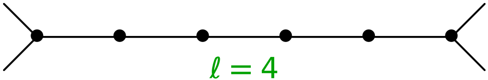

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Like light edges, long threads are reducible for coloring properties, such as bounds on the circular chromatic number, a refinement of chromatic number.

Some History

Thm. Appel–Haken [1976] 4CT G planar $\Rightarrow$ $\chi(G) \le 4$.

RSST [1996]: 633 configurations, via 32 discharging rules.

In planar graphs, 4^--vertices are reducible for $\chi(G) \le 4$.

In a planar triangulation with $\delta(G) = 5$,
some edge has weight ≤ 11 (Wernicke [1904]),
some 5-vertex has two 6^--neighbors (Franklin [1922]),
some triangle has weight ≤ 17 (Borodin 1989]).

Thm. Borodin [1989] In planar G with $\delta(G), \delta(G^*) \ge 3$,
$\exists$ edge with weight ≤ 11 or
$\exists$ a 4-cycle with two 3-vertices and a 10^--vertex.

All these results use vertex charging:
5^--vertices need charge from 7^+-vertices.

Borodin's result can be used to prove that a planar G with $\Delta(G) \ge 9$ decomposes into $\lceil (\Delta(G)+1)/2 \rceil$ linear forests (maxdegree 2, no cycles) (Wu [1999]).

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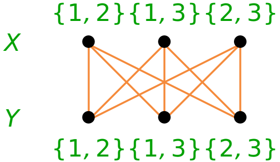

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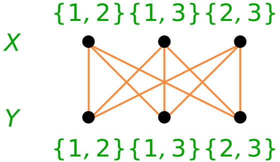

Choosing colors from all lists requires using k colors in each part, but then some color is used in both parts. ∎

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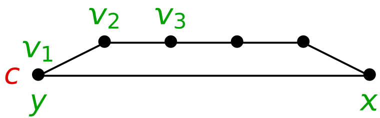

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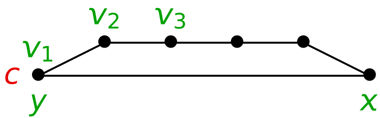

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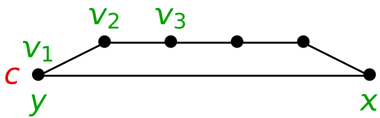

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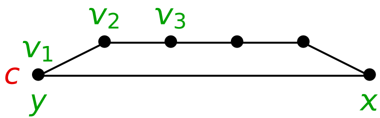

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How does this relate to discharging?

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Proof: iterated discharging (Woodall [2010]).

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Since edges of weight $\leq k + 2$ are reducible for $\chi_\ell'(G) \leq k + 1$, none such exist in G. Hence $\delta(G) \geq 3$, and every nbr of a j-vertex has degree $\geq k + 3 - j$.

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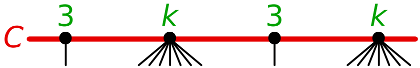

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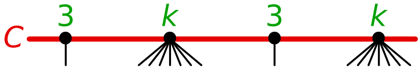

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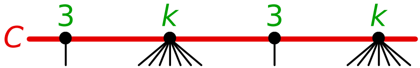

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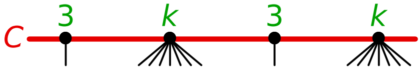

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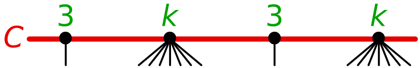

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Thus H is acyclic and has fewer than $n_3 + n_k$ edges. It also has $3n_3$ edges, so $3n_3 < n_3 + n_k$.

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Reducible: light edges and $3, k$-alternating even cycles.

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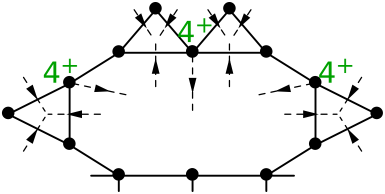

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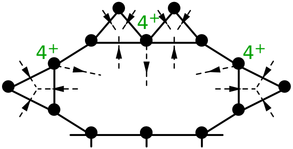

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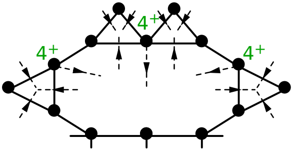

Vertices: only 4^+ vertices lose charge.

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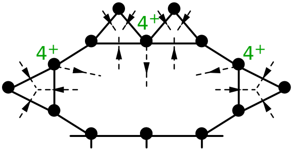

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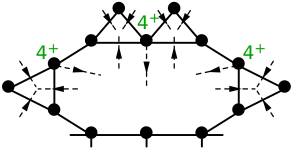

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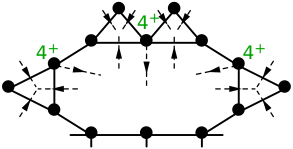

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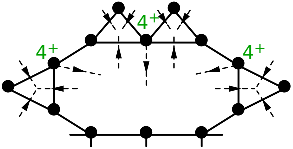

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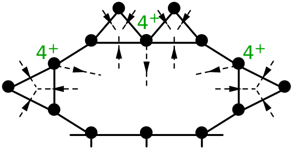

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For $j = 10$, losing more than 4 requires five paths where f loses 1. They must be single edges sharing no vertices, so all vertices on f are 3-vertices. ■

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Hence 10-face via 3-vertices is reducible for $\chi(G) \le 3$. ∎

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$$2e_{5,5} + e_{5,6} + \frac{2}{7}e_{5,7} \geq 60,$$

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Since the total charge is -12 and there is no negative charge elsewhere, in units of $\frac{-1}{5}$ the claim holds.

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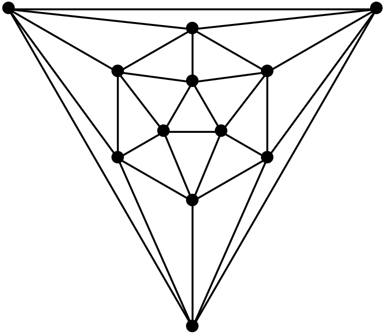

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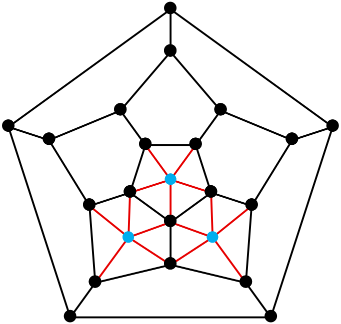

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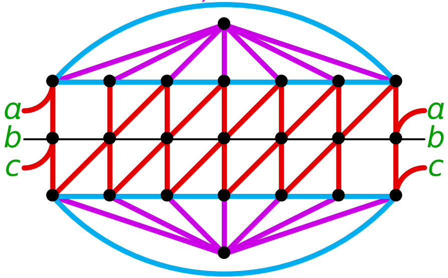
The graph below has $2e_{5,5} = e_{5,6} = 28$ and $e_{5,7} = 14$; hence the coefficient of $e_{5,7}$ cannot be reduced. ■



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- Discharging has been used in many contexts: decomposition into forests, domination number, sparse 2-connected subgraphs, graph reconstruction, geometric graphs, the firefighter problem, etc.

Exercises 1

Ex. 1 Find the largest b such that if $\delta(G) = k$ and $\overline{d}(G) < b$, then G must have two adjacent vertices of degree k.
Find the least g such that all planar graphs with girth at least g have adjacent 2-vertices. Compare the answers.

Ex. 2 Let G be a connected plane graph with $\delta(G), \delta(G^*) \geq 3$.
(a) Use balanced charging to prove that G has a 3-vertex on a 5^--face or a 5^--vertex on a 3-face.
(b) Strengthen (a): prove that G contains a 3-vertex on a 5^--face, a 4-vertex on a 3-face, or a 5-vertex incident to at least four 3-faces.

Ex. 3 Construct a triangulation where all lightest edges have weight 11 and another where all lightest edges have weight 13.

Ex. 4 For $\overline{d}(G) < 4$, prove that $\delta(G) \leq 2$ or G has a 3-vertex with a 5^--neighbor.

Ex. 5 Let G be a graph with $\delta(G) = 3$ and $\overline{d}(G) < \frac{10}{3}$. Prove that G has a 3-vertex whose neighbors have degree-sum at most 10.

Ex. 6 Suppose $\delta(G) = k$. For $0 \leq j < k$, find the largest ρ such that $\overline{d}(G) < k + \rho$ guarantees a k-vertex with more than j k-neighbors.

Exercises 2

Ex. 7 Determine whether every planar graph with girth at least 4 and minimum degree 3 has a 3-vertex with a 4^--neighbor. Construct a plane graph G_k with girth 4 and $\delta(G_k) = 3$ having distance at least k between 3-vertices. Construct a planar graph H_k with $\delta(H_k) = 5$ having distance at least k between 5-vertices.

Ex. 8 Prove that if $\overline{d}(G) < \frac{5}{2}$ and G is connected, then G contains a 3^--vertex with a 1-neighbor, a 4^--vertex with two 2^--neighbors, or a 5^+-vertex v with at least $\frac{d(v)-1}{2}$ neighbors of degree at most 2.

Ex. 9 An ℓ-thread in G is a path whose ℓ internal vertices have degree 2 in G. Prove that if $\overline{d}(G) < 2 + \frac{1}{3t-2}$ and no component is a cycle, then G has a 1^--vertex or a $(2t-1)$-thread. Build infinitely many graphs with average degree $2 + \frac{1}{3t-2}$ but no $(2t-1)$-thread.

Ex. 10 Let G be a connected graph with at least four vertices. Prove that if $\overline{d}(G) < \frac{5}{2}$ and $\delta(G) \geq 2$, then G contains a 2-thread or a 3-vertex with three 2-neighbors, one having another 3-neighbor.

Ex. 11 Prove that every planar triangulation with minimum degree 5 has a vertex with at least two 6^--neighbors.

Exercises 3

Ex. 12 Let G be a planar graph with $\delta(G), \delta(G) \geq 3$. Prove that G has an edge with weight at most 11 or a 4-cycle having two 3-vertices and a 10^{-}-vertex.

Ex. 13 Prove that every planar graph with minimum degree 5 contains two 3-faces sharing an edge with weight at most 11.

Ex. 14 Prove that every plane triangulation with minimum degree 5 has two 3-faces sharing an edge such that the non-shared vertices have degree-sum at most 11.

Ex. 15 Let G be a plane graph where no two 3-faces share an edge. Prove that if $\Delta(G) \leq k$, where $k \geq 7$, then G has an edge with weight at most $k+2$. Conclude that if G is a plane graph with $\Delta(G) \geq 7$ where no two 3-faces share an edge, then $\chi_{\ell}{}'(G) \leq \Delta(G)+1$.

Ex. 16 For $\delta(G) \geq 2$, prove that $\Delta(G) \leq 6$ and $\overline{d}(G) < \frac{7}{2}$ guarantee an edge with weight ≤ 7 or a cycle alternating 2-vertices and 6-vertices. Conclude that $\Delta(G) \leq 6$ and $\mathrm{Mad}(G) < \frac{7}{2}$ imply $\chi_{\ell}'(G) \leq 6$.

Ex. 17 Let G be planar with $\delta(G) = 5$, having $e_{i,j}$ edges with endpoints of degrees i and j. Prove that $e_{5,6} \geq 60$ when $e_{5,5} = 0$.

Exercises 4

Ex. 18 Use balanced charging to prove that every planar graph with girth at least 7 and minimum degree at least 2 has a 2-vertex adjacent to a 3^--vertex. Prove that the conclusion does not always hold when $\mathrm{Mad}(G) < \frac{14}{5}$ (thus planarity is needed).

Ex. 19 Let G be a plane graph having no 4-cycle and no face length in $\{4, \ldots, k\}$. Use discharging to prove that the average face length is at least $6 - \frac{18}{k+4}$. Conclude $\mathrm{Mad}(G) < 3 + \frac{9}{2k-1}$. In particular, $\mathrm{Mad}(G) < 4$ when G is a plane graph with no 4-face or 5-face.

Ex. 20 (+) Let G be a plane graph with $\delta(G) \geq 2$. Prove that G contains an edge with weight at most 15 or a cycle that alternates between 2-vertices and higher-degree vertices.
Conclude that $\chi'(G) = \Delta(G)$ when G is planar and $\Delta(G) \geq 14$.

Ex. 21 (+) *Light triangles.* Let G be a planar graph with $\delta(G) = 5$.
(a) Prove that G contains a 3-face with weight at most 17.
(b) Construct one graph to show that smaller weight cannot be guaranteed. Show that $\delta(G) = 5$ is needed by constructing for each $k \in \mathbb{N}$ a planar graph with minimum degree 4 having no 3-face with weight at most k.