An Introduction to the Discharging Method via Graph Coloring

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Abstract

We provide a "how-to" guide to the use and application of the Discharging Method. Our aim is not to exhaustively survey results proved by this technique, but rather to demystify the technique and facilitate its wider use, using applications in graph coloring as examples. Along the way, we present some new proofs and new problems.

1 Introduction

Arguments that can be phrased in the language of the Discharging Method have been used in graph theory for more than 100 years, though that name is much more recent. The most famous application of the method is the proof of the Four Color Theorem, stating that graphs embeddable in the plane have chromatic number at most 4. However, the method remains mysterious to many. Our aim is to explain its use and make the method more widely accessible. Although we mention many applications, including stronger versions of results proved here, cataloguing applications is not our goal. Borodin [23] presents a survey of applications of discharging to coloring of plane graphs.

Discharging is most commonly used as a tool in a two-pronged approach to inductive proofs, typically for sparse graphs. In this context, it is used to prove that a global sparseness hypothesis guarantees the existence of some desired local structure. The method has been applied to many types of problems (including graph embeddings and decompositions, spread of infections in networks, geometric problems, etc.). Nevertheless, we present only applications in graph coloring (where it has been used most often), in order to emphasize the discharging techniques.

In the simplest version of discharging involves just reallocation of vertex degrees in the context of a global bound on the average degree. We view each vertex as having an initial "charge" equal to its degree. To show that average degree less than b forces the presence of

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a desired local structure, we show that the absence of such a structure allows charge to be moved (via "discharging rules") so that the final charge at each vertex is at least b. This violates the hypothesis, and hence the desired structure must occur.

In an application of the resulting structure theorem, one shows that each such local structure is "reducible", meaning that it cannot occur in a minimal counterexample to the desired conclusion. This motivates the phrase "an unavoidable set of reducible configurations" to describe the overall process.

Definition 1.1. A *configuration* in a graph G can be any structure in G (often a specified sort of subgraph). A configuration is *reducible* for a graph property Q if it cannot occur in a minimal graph not having property Q. Let $d_G(v)$ or simply $d(v)$ denote the degree (number of neighbors) of vertex v in G, and let $\overline{d}(G)$ denote the average of the vertex degrees in G. *Degree charging* is the assignment to each vertex v of an "initial charge" equal to $d(v)$.

The notion of configuration is vague to permit use in various contexts. "Minimal" refers to some partial order on the graphs being considered; usually it is just minimality with respect to taking subgraphs, and the property Q is *monotone* (preserved by taking subgraphs).

Sparse local configurations aid in inductive proofs about coloring. For example, when $\overline{d}(G) < k$ with $k \in \mathbb{N}$, the pigeonhole principle guarantees a vertex with degree less than k in G. Also, when $d(v) < k$, a proper k-coloring of $G - v$ extends to a proper k-coloring of G. (A *k-coloring* is a function that assigns labels to vertices from a set of size k, a coloring of a graph G is *proper* if adjacent vertices have distinct colors, G is *k-colorable* if it admits a proper k-coloring, and the *chromatic number* $\chi(G)$ is the least k such that G is k-colorable.)

In other words, vertices of degree less than k are reducible for the property $\chi(G) \le k$. However, guaranteeing such a vertex from the global bound $\overline{d}(G) < k$ does not need discharging. To illustrate how discharging works and interacts with reducibility, we consider another elementary example after introducing notation convenient for discussing vertex degrees.

Definition 1.2. A *j-vertex*, *j^+-vertex*, or *j^--vertex* is a vertex with degree equal to j, at least j, or at most j, respectively. A *j-neighbor* of v is a j-vertex that is a neighbor of v. We write $\delta(G)$ for the minimum and $\Delta(G)$ for the maximum of the vertex degrees in G.

Lemma 1.3. *If $\overline{d}(G) < 3$, then G has a 1^--vertex or a 2-vertex with a 5^--neighbor.*

Proof. We use degree charging; each vertex v starts with charge $d(v)$. Suppose that G has no 1^--vertex and that no 2-vertex in G has a 5^--neighbor. We move charge so that each vertex ends with charge at least 3. The 2-vertices need charge; 4^+-vertices can give charge.

Let each 2-vertex take $\frac{1}{2}$ from each neighbor. Now each 2-vertex has charge 3, since no two 2-vertices are adjacent. Vertices of degrees $3, 4, 5$ lose no charge, since we assumed that no 2-vertex has a 5^--neighbor. Every 6^+-vertex v loses charge at most $\frac{1}{2}$ to each neighbor, leaving it with charge at least $d(v)/2$, which is at least 3 when $d(v) \ge 6$. Thus $\overline{d}(G) \ge 3$ when no 2-vertex has a 5^--neighbor. □

A 2-vertex with a 5^--neighbor is a local sparseness condition, somehow more sparse than a 2-vertex with high-degree neighbors. We first consider its use for edge-coloring. (A *k-edge-coloring* of a graph G assigns labels to edges from a set of size k; it is *proper* if incident edges have distinct colors, G is *k-edge-colorable* if it has a proper k-edge-coloring, and the *edge-chromatic number* $\chi'(G)$ is the least k such that G is k-edge-colorable.)

Here we phrase the reducibility statement in more generality. The *weight* of a subgraph H of a graph G is $\sum_{v \in V(H)} d_G(v)$; we sum the degrees in the full graph G.

Lemma 1.4. *An edge with weight at most $k+1$ is a reducible configuration for the property of being k-edge-colorable.*

Proof. Let G be a graph having an edge e of weight at most $k+1$. If the graph $G-e$ is k-edge-colorable, then a color is available to extend the coloring to e, because e is incident to a total of at most $k-1$ other edges at its two endpoints. Thus a minimal graph G with $\chi'(G) > k$ cannot contain such a configuration. □

To complete an inductive proof of $\chi'(G) \le 6$ from Lemmas 1.3 and 1.4, we also need average degree less than 3 in subgraphs of G.

Definition 1.5. The *maximum average degree* of a graph G, denoted $\mathrm{mad}(G)$, is the maximum of the average degree over all subgraphs of G.

The application is now easy. Note that always $\chi'(G) \ge \Delta(G)$. In fact, Vizing's Theorem [74, 118] states that always $\chi'(G) \le \Delta(G)+1$, and distinguishing between $\chi'(G) = \Delta(G)$ and $\chi'(G) = \Delta(G)+1$ is an important and difficult problem.

Theorem 1.6. *If $\mathrm{mad}(G) < 3$ and $\Delta(G) \ge 6$, then $\chi'(G) = \Delta(G)$.*

Proof. Fix an integer k at least 6. We prove more generally that if $\mathrm{mad}(G) < 3$ and $\Delta(G) \le k$, then $\chi'(G) \le k$. That is, among graphs with $\mathrm{mad}(G) < 3$ and $\Delta(G) \le k$ there is no minimal graph satisfying $\chi' > k$. Note that the hypotheses also hold in subgraphs.

We may discard isolated vertices. By Lemma 1.3, G then has a 1-vertex or has a 2-vertex with a 5^--neighbor. The edge incident to a 1-vertex has weight at most $\Delta(G)+1$; an edge joining a 2-vertex to a 5^--neighbor has weight at most 7. In either case, the weight of this edge e is at most $k+1$, and Lemma 1.4 implies that G is not a minimal graph satisfying $\chi'(G) > k$. Hence there is no minimal counterexample. □

Before leaving Theorem 1.6, we note that many reducibility arguments for coloring problems involve deleting some parts of a graph (such as a 1-vertex or the edge e in the proof above) and then choosing colors for the missing pieces as they are replaced. Suitable choices can be made if there are enough available colors; it does not matter what the colors are. In this situation, the arguments yield stronger statements about coloring from lists.

Definition 1.7. A *list assignment* L on a graph G gives each $v \in V(G)$ a set $L(v)$ of colors, called its *list*. In a k*-uniform* list assignment, each list has size k. Given a list assignment L, an L*-coloring* of G is a proper coloring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. A graph G is k*-choosable* if G is L-colorable whenever each list has size at least k (we may assume L is k-uniform). The *list chromatic number* or *choice number* of G, written $\chi_\ell(G)$, is the least k such that G is k-choosable. Analogous language is used for edge-colorings chosen from list assignments to edges.

Since the lists can be identical, always $\chi_\ell(G) \geq \chi(G)$. Thus $\chi_\ell(G) \leq b$ is stronger than $\chi(G) \leq b$. For example, $\text{mad}(G) < k$ inductively yields $\chi_\ell(G) \leq k$. Similarly, an edge of weight at most $k+1$ is reducible for k-edge-choosability, and the proof of Theorem 1.6 yields $\chi'_\ell(G) = \Delta(G)$ when $\text{mad}(G) < 3$ and $\Delta(G) \geq 6$.

We present various classical applications, some with new proofs. We emphasize discharging arguments but include reducibility arguments to show how discharging is applied. For clarity and simplicity in illustrating the method, we often assume more restrictive hypotheses than used in the strongest known results. Often those results are proved similarly, but with more detail in the discharging arguments and more configurations to be proved reducible.

The basic idea of discharging proofs is simple, and the proofs are usually easy to follow, though they may have many details. The mystery arises in the choice of reducible configurations, the rules for moving charge, and how to find the best hypothesis. We will explain the interplay among these and suggest how the proofs are discovered, starting with the context of $\text{mad}(G) < b$ in Section 2. We include related results as exercises to aid in self-study; most exercises have relatively short solutions (items labeled "Question" are unsolved).

As we have illustrated, structural results proved by discharging when $\overline{d}(G) < b$ are applied inductively to obtain coloring conclusions under the hypothesis $\text{mad}(G) < b$. The point is that every subgraph H satisfies $\text{mad}(H) < b$. For natural hereditary families like planar graphs, bounds on $\text{mad}(G)$ are easily obtained. The families satisfying $\text{mad}(G) < b$ for all positive b provide a rich spectrum for study.

Discharging has been used to prove many results on coloring or structure of planar graphs (or planar graphs with large girth). Euler's Formula implies that (every subgraph of) a planar graph with girth at least g has average degree less than $\frac{2g}{g-2}$. Some results on such graphs in fact hold whenever $\text{mad}(G) < \frac{2g}{g-2}$, regardless of planarity, often with the same proof by discharging. Others, as discussed in Section 3, truly need planarity and may assign charge to both the faces and the vertices (the dual graph is also sparse). This is the basic reason why discharging is so useful for planar graphs. Subsequent sections will discuss additional techniques of discharging, especially with examples from "list coloring".

Finally, we note that in addition to its usefulness as a proof technique, the discharging method also has algorithmic implications, often yielding fast constructive algorithms for good colorings or embeddings. Iterative application of the structure theorem yields reductions to smaller graphs. After a good coloring of a base graph is found, the intermediate graphs receive good colorings using the reducibility arguments, until the original graph is restored and its coloring obtained (see Section 6 of [55]).

2 Structure and coloring of sparse graphs

In studying discharging, the principle and the details are simple. The mystery is the source of the discharging rule and the hypothesis on $\overline{d}(G)$. The secret is that the discharging rule is found before knowing the hypothesis of the theorem and is used to discover it. To explain such aspects of discharging, we study the forcing of local configurations with small weight.

Remark 2.1. *Finding the best bound on* $\mathrm{mad}(G)$*.* Consider Lemma 1.3 more generally. When we want G with $\mathrm{mad}(G) < b$ to have a 1^--vertex or have a 2-vertex with a j^--neighbor, what is the best choice of b? Actually, we start with the proof and let it produce the statement. We must make b at most 3, since otherwise G may be 3-regular with no 2-vertex. Given that, when we exclude 1^--vertices and use degree charging, only 2-vertices will need charge. The most natural way for them to obtain it is to take it from their neighbors.

If each 2-vertex takes ρ from each neighbor, then final charge is at least b at each vertex if and only if 2-vertices obtain enough charge and vertices with degree larger than j do not lose too much. Such vertices can lose ρ to each neighbor, so we need $2 + 2\rho \ge b$ and $d - d\rho \ge b$ when $d \ge j+1$. To find the largest b that works, set $2+2\rho = (j+1)(1-\rho)$, yielding $\rho = \frac{j-1}{j+3}$ and hence $b = 2 + 2\rho = 4\frac{j+1}{j+3}$. When $j = 5$, we obtain Lemma 1.3.

What we did was find the weakest hypothesis allowing the discharging proof to work. The proof also provides sharpness examples showing that the condition $\mathrm{mad}(G) < b$ cannot be weakened. If every 2-vertex has only $(j+1)$-neighbors, every $(j+1)$-vertex has only 2-neighbors, and there are no other vertices, then all the equalities are tight, no 2-vertex has a j^--neighbor, all vertices end with charge exactly b, and the average degree is b. Hence we obtain a sharpness example by taking a $(j+1)$-regular graph and subdividing every edge.

What the discharging argument does is count part of the degree of higher-degree vertices at their 2-neighbors. In this sense discharging is “amortized counting”; the counting of the degree of a vertex is allocated to (or “charged to”) other vertices.

The discharging argument for a structure theorem guaranteeing local configurations is quite separate from the reducibility arguments used to give an inductive proof of the desired conclusion. Thus the unavoidable set resulting from a particular sparseness condition may be usable to prove other results. In practice, usually the configurations are those already known to be reducible for the desired property in the application. Nevertheless, Lemma 1.3 does apply to another coloring problem.

Definition 2.2. An *acyclic coloring* of a graph is a proper coloring such that the union of any two color classes induces an acyclic subgraph; equivalently, no cycle is 2-colored.

Theorem 2.3. *If* $\mathrm{mad}(G) < 3$*, then* G *is acyclically* 6*-choosable.*

Proof. It suffices to show that the configurations forced by Lemma 1.3 when $\mathrm{mad}(G) < 3$ are reducible for the existence of an acyclic coloring chosen from a 6-uniform list assignment L. By definition, $\mathrm{mad}(G - v) < 3$. To show reducibility, we assume an acyclic L-coloring ϕ of $G - v$ and obtain such a coloring of G. The cases appear in Figure 1.

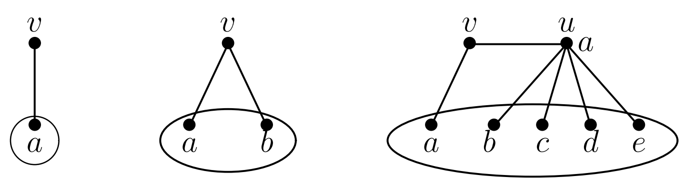

Figure 1: Reducibility for acyclic 6-coloring

If $d_G(v) \le 1$, then we extend ϕ by letting $\phi(v)$ be a color in $L(v)$ not used on the neighbor of v. If $d_G(v) = 2$ and ϕ gives distinct colors on $N_G(v)$, then again we just avoid them on v. If $d_G(v) = 2$ and ϕ gives the same color to both vertices of $N_G(v)$, then there is danger of completing a 2-colored cycle. However, since v has a 5^--neighbor u, at most four other colors appear on the neighbors of u, so a color in $L(v)$ remains available for v. □

If a structure theorem with hypothesis $\overline{d}(G) < b$ is sharp, then when $\overline{d}(G)$ exceeds b we must add other configurations to obtain a structure theorem. At $\overline{d}(G) = 3$ we may have no 2-vertices with 5^--neighbors and perhaps no 2-vertices at all. Nevertheless, when $\overline{d}(G) < 4$ the graph must have a 2^--vertex or have a 3-vertex with a 5^--neighbor (Exercise 2.1).

In the other direction, when we reduce the bound on $\overline{d}(G)$ we can impose more sparseness. By Remark 2.1, $\overline{d}(G) < \frac{12}{5}$ implies that G has two adjacent 2-vertices if it has no 1^--vertex. What sparser local configuration can we force when the average degree declines even further?

Definition 2.4. An ℓ*-thread* in a graph G is a trail of length $\ell + 1$ in G whose ℓ internal vertices have degree 2 in the full graph G.

Under this definition, an ℓ-thread contains two $(\ell - 1)$-threads, and the ends of a thread may be the same vertex.

Lemma 2.5. *If* $\overline{d}(G) < 2 + \frac{2}{3\ell - 1}$ *and* G *has no* 2*-regular component, then* G *contains a* 1^-*-vertex or an* ℓ*-thread.*

Proof. Let $\rho = \frac{1}{3\ell - 1}$, so the hypothesis is $\overline{d}(G) < 2 + 2\rho$. Use degree charging. If neither stated configuration occurs, then we redistribute charge to leave each vertex with at least

$2+2\rho$. Since G has no 1^--vertex, $\delta(G) \geq 2$. Since G has no 2-regular component, each 2-vertex lies in a unique maximal thread. Redistribute charge as follows:

(R1) Each 2-vertex v takes charge ρ from each end of its maximal thread.

Since each 2-vertex lies on a unique maximal thread, it ends with charge $2+2\rho$. Since ℓ-threads are forbidden, each j-vertex with $j \geq 3$ gives charge to at most $\ell-1$ vertices along the thread started by each incident edge, losing at most $j(\ell-1)\rho$. To show that its final charge is at least $2+2\rho$, we compute

$$j - j(\ell-1)\rho \geq 3\left[1 - \frac{\ell-1}{3\ell-1}\right] = 2 + \frac{2}{3\ell-1}.$$

Hence avoiding the specified configurations requires $\overline{d}(G) \geq 2 + \frac{2}{3\ell-1}$. □

Remark 2.6. Once again, the hypothesis of Lemma 2.5 is discovered from the proof, and the structure theorem is sharp. When using degree charging with $\overline{d}(G) < 2+2\rho$, only 2-vertices need charge (once we restrict to $\delta(G) \geq 2$), and the natural (local) sources of charge are the nearest vertices of larger degree. This yields the discharging rule, taking ρ from each.

We choose ρ by finding the weakest hypothesis that avoids taking too much from 3^+-vertices. The inequality $j - j(\ell-1)\rho \geq 2+2\rho$ implies that the proof guarantees an ℓ-thread when $\rho \leq \frac{j-2}{j(\ell-1)+2}$ for $j \geq 3$. Thus setting $\rho = \frac{1}{3\ell-1}$ both makes the proof work and gives the weakest hypothesis where it works.

Furthermore, to achieve sharpness in the proof, all vertices should have degree 2 or 3. Replace each edge of any 3-regular graph with an $(\ell-1)$-thread. By the discharging computation, the average degree is $2 + \frac{2}{3\ell-1}$, and there are no ℓ-threads.

Discovering a discharging argument can be fun, but its value is in applications. To apply our result on threads inductively to a coloring problem, we replace the bound on $\overline{d}(G)$ by the same bound on $\text{mad}(G)$. The condition $\text{mad}(G) < 3$ already implies 3-colorability, and graphs having any odd cycle require at least three colors, so we need another coloring model to allow a stronger bound on $\text{mad}(G)$ to have a chance to improve on 3-colorability.

Definition 2.7. A *t-fold coloring* of a graph G assigns each vertex a set of t colors so that adjacent vertices receive disjoint sets. The *t-fold chromatic number* $\chi_t(G)$ is the least k such that G has a t-fold coloring using subsets of $[k]$ (where $[n] = \{1, \ldots, n\}$). The *fractional chromatic number* $\chi^*(G)$ of G is $\inf_t \chi_t(G)/t$. The *odd girth* of G, written $g_o(G)$, is the length of a shortest odd cycle in G (infinite when G is bipartite).

An ordinary proper coloring is a 1-fold coloring, so always $\chi^*(G) \leq \chi(G)$. The *independence number* $\alpha(G)$ of a graph G is the maximum size of an independent set of vertices. When G has n vertices, always a t-fold coloring of G needs at least $nt/\alpha(G)$ colors, since each color can only be used on an independent set. Hence $\chi^*(G) \geq n/\alpha(G)$; equality holds for vertex-transitive graphs using all automorphic images of a largest independent set. In particular, $\chi^*(C_{2t+1}) = 2 + \frac{1}{t}$, where C_n denotes the n-vertex cycle.

Theorem 2.8. *If $g_o(G) \geq 2t+1$ and $\mathrm{mad}(G) < 2 + \frac{1}{3t-2}$, then G has a t-fold coloring with $2t+1$ colors, and hence $\chi^*(G) \leq 2 + \frac{1}{t}$.*

Proof. We have noted that $g_o(G) \geq 2t+1$ is needed. By Lemma 2.5, it suffices to show that 1^--vertices and $(2t-1)$-threads (which may be contained in longer threads) are reducible for t-fold $(2t+1)$-colorability. If $d(v) \leq 1$, then a such a coloring ϕ of $G - v$ easily extends to v, choosing $\phi(v)$ from the complement of the set assigned to its neighbor when $d(v) = 1$.

When G contains a $(2t-1)$-thread with endpoints u and v, let G' be the graph obtained by deleting its internal vertices. The hypotheses hold for G', so G' admits a t-fold coloring ϕ using $2t+1$ colors. We want to extend ϕ along the thread. When two t-sets in $[2t+1]$ differ by one element, a unique t-set lies in the complement of both. Hence in two steps we can switch any color in a t-set to any missing color. In fact, this is the only change achievable in two steps (we can also return to the same t-set). Since there are $2t$ steps along the thread from u to v, we can thus extend ϕ along the thread to obtain the desired coloring of G. □

We introduced fractional coloring to illustrate the use of threads in sparse graphs. Similar results are available in connection with another variation on coloring.

Definition 2.9. A (p, q)-*coloring* ϕ of G colors $V(G)$ by elements of $\{0, \ldots, p-1\}$ so that adjacent vertices receive colors cyclically at least q apart; that is, $q \leq |\phi(u) - \phi(v)| \leq p - q$ when $uv \in E(G)$. A graph having a (p, q)-coloring is (p, q)-*colorable*. The *circular chromatic number* of G, written $\chi_c(G)$, is $\inf\{\frac{p}{q} \colon G \text{ is } (p, q)\text{-colorable}\}$.

A $(k, 1)$-coloring is just a proper k-coloring, so $\chi_c(G) \leq \chi(G)$. A (p, q)-coloring can be viewed as a q-fold coloring with p colors, where the q-sets used are segments of q cyclically consecutive colors. Thus circular coloring is a restricted form of fractional coloring, and always $\chi_c(G) \geq \chi^*(G)$. In fact, always $\lceil \chi_c(G) \rceil = \chi(G)$, so $\chi_c(G)$ can be viewed as a refinement of $\chi(G)$. Zhu [133, 134] surveyed results on circular coloring.

The hypotheses of Theorem 2.8 also suffice for the stronger conclusion $\chi_c(G) \leq 2 + \frac{1}{t}$, since again a $(2t-1)$-thread is reducible (Exercise 2.4). In fact, we can further strengthen the result by obtaining the same conclusion when $\mathrm{mad}(G)$ is allowed to be somewhat larger. We have seen that $\mathrm{mad}(G) = 2 + \frac{1}{3t-2}$ does not force $(2t-1)$-threads, but then another structure is forced that also is reducible for $\chi_c(G) \leq 2 + \frac{1}{t}$. We prove the discharging part.

Lemma 2.10. *If $\overline{d}(G) < 2 + \frac{1}{2t-1}$ and G has no 2-regular component, then G contains (1) a 1^--vertex, or (2) a 3-vertex with at least $4t-3$ vertices of degree 2 on its maximal incident threads, or (3) a 4^+-vertex incident to a $(2t-1)$-thread.*

Proof. Let $\rho = \frac{1}{2}\frac{1}{2t-1}$, so $\overline{d}(G) < 2 + 2\rho$. Use degree charging. We may assume $\delta(G) \ge 2$ and that G is connected. Redistribute charge using the same rule as before.

(R1) Each 2-vertex v takes charge ρ from each end of its maximal thread.

As in Lemma 2.5, each 2-vertex ends with charge $2 + \rho$. If no 3-vertex has enough 2-vertices on its incident threads, then each 3-vertex v loses charge at most $(4t-4)\rho$ and retains at least $3 - \frac{2t-2}{2t-1}$, which equals $2 + 2\rho$.

Now let v be a 4^+-vertex. With no incident $(2t-1)$-thread, v gives charge to at most $2t-2$ vertices along the thread starting at each incident edge. The minimum remaining charge $d(v)[1-(2t-2)\rho]$ is minimized when $d(v) = 4$. We compute $4[1-(2t-2)\rho] = 4-2\frac{2t-2}{2t-1} = 2+4\rho$.

Every vertex ends with charge at least $2 + 2\rho$, so avoiding the specified configurations requires $\overline{d}(G) \ge 2 + \frac{1}{2t-1}$. □

Showing that the configurations in Lemma 2.10 are reducible for $\chi_c(G) \le 2 + \frac{1}{t}$ (Exercise 2.4) proves the following result.

Theorem 2.11. *If* $g_o(G) \ge 2t + 1$ *and* $\mathrm{mad}(G) < 2 + \frac{1}{2t-1}$, *then* $\chi_c(G) \le 2 + \frac{1}{t}$.

The bound on $\mathrm{mad}(G)$ in Theorem 2.11 is still not sharp for $\chi_c(G) \le 2 + \frac{1}{t}$. Borodin–Hartke–Ivanova–Kostochka–West [29] proved for triangle-free graphs that $\mathrm{mad}(G) < \frac{12}{5}$ implies $\chi_c(G) \le \frac{5}{2}$, while Theorem 2.11 with $t = 2$ requires $\mathrm{mad}(G) < \frac{9}{4}$ to obtain $\chi_c(G) \le \frac{5}{2}$. Sharpness of their result follows from $t = 2$ in the following construction.

Example 2.12. Let G_t consist of two $(2t+1)$-cycles sharing a single edge, plus a $(2t-2)$-thread joining the vertices opposite the shared edge on the two cycles; Figure 2 shows G_2 and G_3. Note that $G_1 \cong K_4$ and $\overline{d}(G_2) = \frac{12}{5}$.

Consider a possible $(2t+1, t)$-coloring of G_t. Once colors are chosen on the edge shared by the two $(2t+1)$-cycles in G_t, the colors on the other two 3-vertices are forced to be the same. The coloring does not extend to all of G_t, since a thread of odd length at most $2t-1$ cannot have the same color on its endpoints. In fact, $\chi_c(G_t) = 2+\frac{1}{t-1/2}$ and $\overline{d}(G_t) = 2+\frac{2}{3t-1}$. For $t = 2$, we have $\chi_c(G_2) = \frac{8}{3}$ and $\overline{d}(G_2) = \frac{12}{5}$.

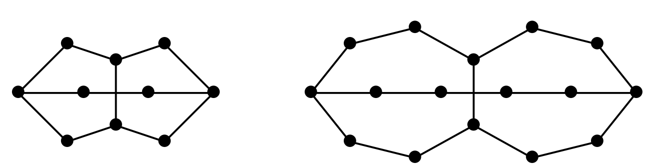

Figure 2: Graphs G_2 and G_3 for Example 2.12

We offer a conjecture; Example 2.12 shows that it would be sharp.

Conjecture 2.13. *If* $g_o(G) \ge 2t + 1$ *and* $\mathrm{mad}(G) < 2 + \frac{2}{3t-1}$, *then* $\chi_c(G) \le 2 + \frac{1}{t}$.

The conjecture is trivial for $t = 1$. The proof for $t = 2$ in [29] used "long-distance" discharging, moving charge along special long paths.

Remark 2.14. A weaker version of Conjecture 2.13 is a special case of the following result from Borodin–Kim–Kostochka–West [40]: If G has girth at least $6t - 2$ and $\mathrm{mad}(G) < 2 + \frac{3}{5t-2}$, then $\chi_c(G) \le 2 + \frac{1}{t}$. Their proof uses discharging and reducible configurations involving multiple threads, like the 3-vertex in Lemma 2.11. The result is motivated by the conjecture of Jaeger [82] that every $4t$-edge-connected graph has "circular flow number" at most $2 + \frac{1}{t}$. When G is planar, this statement for the dual graph G^* becomes the conjecture that $\chi_c(G) \le 2 + \frac{1}{t}$ when G is planar with girth at least $4t$.

Lovász–Thomassen–Wu–Zhang [95] proved a weaker form of Jaeger's Conjecture, replacing $4t$ by $6t$. Thus $\chi_c(G) \le 2 + \frac{1}{t}$ when G is planar with girth at least $6t$. By Euler's Formula, $\mathrm{mad}(G) < \frac{2g}{g-2}$ when G is planar with girth g. Thus $\mathrm{mad}(G) < 2 + \frac{2}{2t-1}$ when G is planar with girth at least $4t$. Conjecture 2.13 in some sense proposes a trade-off: by further restricting to $\mathrm{mad}(G) < 2 + \frac{2}{3t-1}$, the girth requirement can be relaxed to $g_o(G) \ge 2t + 1$ and still yield $\chi_c(G) \le 2 + \frac{1}{t}$, even without requiring planarity.

We have considered only sparse graphs and small $\chi_c(G)$, but the problem is more general. Always $\chi(G) \ge \omega(G)$, where $\omega(G)$ is the maximum number of pairwise adjacent vertices in G, called the *clique number* of G. The *circular clique* $K_{p:q}$ is the graph whose vertices are the congruence classes modulo p, adjacent when they differ by at least q. The *circular clique number*, written $\omega_c(G)$, is $\max\{p/q\colon\ K_{p:q} \subseteq G\}$; always $\chi_c(G) \ge \omega_c(G)$.

Question 2.15. For graphs G with $\omega_c(G) \le s$, what is the largest ρ such that $\mathrm{mad}(G) < \rho$ implies $\chi_c(G) \le s$? (Note: the answer is s when s is an integer.)

Next we apply a special case of Lemma 2.10 to a further restriction of acyclic coloring (Definition 2.2); like acyclic coloring, it was introduced by Grünbaum [72].

Definition 2.16. A *star coloring* is an acyclic coloring where the union of any two color classes induces a forest of stars; equivalently, no 4-vertex path is 2-colored. The *star chromatic number* $s(G)$ (also written $\chi_s(G)$) is the minimum number of colors in such a coloring.

Every star coloring is an acyclic coloring. All trees are acyclically 2-colorable, but trees with diameter at least 3 are not star 2-colorable. Extensive early results about star colorings and their relationships to other parameters appear in papers by Fertin, Raspaud, and Reed [67] and by Albertson et al. [5], without discharging. Our focus here is on a structure that yields an upper bound on $s(G)$.

Definition 2.17. A set I of vertices is a 2-*independent set* if the distance between any two vertices of I exceeds 2. An I,F-*partition* of a graph G, introduced by Albertson et al. [5], is a partition of $V(G)$ into sets I and F such that I is a 2-independent set and $G[F]$ is a forest.

Lemma 2.18. *Every forest is star 3-colorable. Hence if a graph G has an I,F-partition, then $s(G) \le 4$.*

Proof. In a tree T, choose a root r and color each vertex v with $d_T(v,r)$, reduced modulo 3. Each connected 2-colored subgraph is a star consisting of a vertex and its children. Using a fourth color on a 2-independent set I cannot complete a 2-colored 4-vertex path, since no two vertices with that color have a common neighbor. □

Theorem 2.19 (Timmons [115]). *If* $\mathrm{mad}(G) < \frac{7}{3}$*, then G has an I,F-partition.*

Proof. We may assume that no component is a cycle, because in such a component it suffices to put one vertex into I. Without 2-regular components, Lemma 2.10 with $t = 2$ implies that G has a 1^--vertex, a 3-thread, or a 3-vertex with at least five 2-vertices on its incident threads. It therefore suffices to prove these configurations reducible for the existence of I,F-partitions. A 1^--vertex v can be added to the forest in any such partition of $G - v$.

Let $\langle v,w,x,y,z \rangle$ be a 3-thread in G; vertices w,x,y each have degree 2. Let I',F' be an I,F-partition of $G - \{w,x,y\}$. If v or z is in I', then add $\{w,x,y\}$ to F' to form an I,F-partition of G. Otherwise, add x to I' and $\{w,y\}$ to F', as in Figure 3.

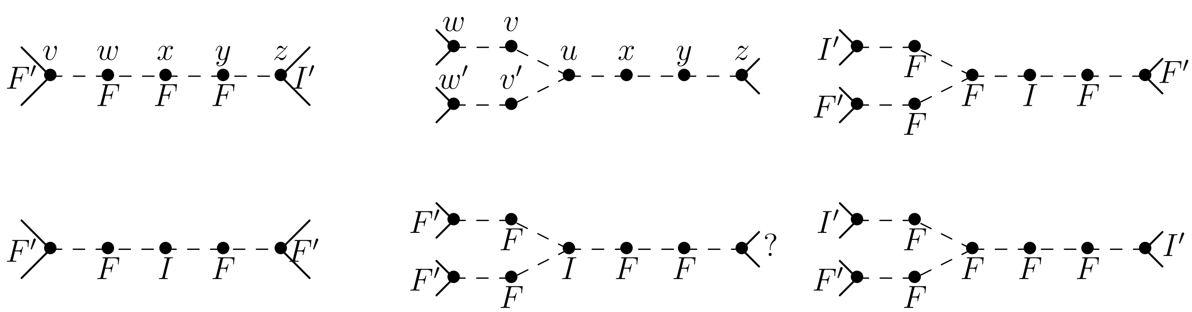

Figure 3: Reducibility cases for Theorem 2.19

Finally, consider a 3-vertex u with at least five 2-vertices on its incident threads. If one of the threads has three 2-vertices, then G has a 3-thread. Otherwise, u has at least two incident 2-threads plus a 2-neighbor on the third incident thread. It suffices to consider a 2-thread $\langle u,x,y,z \rangle$ and two 1-threads $\langle u,v,w \rangle$ and $\langle u,v',w' \rangle$ incident to u (see Figure 3). Let $R = \{w,w'\}$ and $S = \{u,x,y,v,v'\}$, and let $G' = G - S$. Let I',F' be an I,F-partition of G'. If $R \cap I' = \varnothing$, then add u to I' and the rest of S to F'. If $R \cap I' \ne \varnothing$ and $z \notin I'$, then add x to I' and the rest of S to F'. If $R \cap I' \ne \varnothing$ and $z \in I'$, then add all of S to F'. In each case, the resulting sets form an I,F-partition of G. □

With more detailed discharging, Bu et al. [45] proved that $\mathrm{mad}(G) < \frac{26}{11}$ yields an I,F-partition, and then Brandt et al. [43] proved that $\mathrm{mad}(G) < \frac{5}{2}$ suffices. This result is sharp, as infinitely many examples with average degree $\frac{5}{2}$ have no I,F-partition (Exercise 2.10).

However, the optimal value of $\text{mad}(G)$ implying star 4-colorability is not known. Chen, Raspaud, and Wang [48, 49] proved that $s(G) \le 8$ when $\text{mad}(G) < 3$ and that $s(G) \le 6$ when $\Delta(G) = 3$ (the latter is sharp). Unfortunately, no bound on $s(G)$ (or $a(G)$) can be implied by $\text{mad}(G) < 4$. The constructions for this in Exercise 2.11 have average degree tending to 4; what happens for bounds between 3 and 4 remains open.

Question 2.20. For $3 < b < 4$, is it true that $s(G)$ or $a(G)$ is bounded when $\text{mad}(G) < b$? Given k, what is the minimum number of vertices in a graph G with $\text{mad}(G) < 4$ and $s(G) > k$ (or $a(G) > k$)?

Exercise 2.1. Given $\overline{d}(G) < 4$, prove that G has a 2^--vertex or a 3-vertex with a 5^--neighbor. Explain why we cannot place any bound on the smallest degree of a neighbor of a 2-vertex, both by construction and by explaining how the proof would fail.

Exercise 2.2. Given $0 \le j < k$, let G be a graph with $\delta(G) = k$. Determine the largest ρ such that $\overline{d}(G) < k + \rho$ guarantees that G has a k-vertex having more than j neighbors of degree k.

Exercise 2.3. Show that Lemma 2.10 is sharp. For each $t \in \mathbb{N}$ construct infinitely many examples with average degree $2 + \frac{1}{2t-1}$ in which none of the specified configurations occurs.

Exercise 2.4. Prove that the configurations in Lemma 2.10 are reducible for $\chi_c(G) \le 2 + \frac{1}{t}$. Conclude that $\chi_c(G) \le 2 + \frac{1}{t}$ when $g_o(G) \ge 2t + 1$ and $\text{mad}(G) < 2 + \frac{1}{2t-1}$

Exercise 2.5. (Cranston–Kim–Yu [56]) Let G be a connected graph with at least four vertices. Prove that if $\overline{d}(G) < \frac{5}{2}$ and $\delta(G) \ge 2$, then G contains a 2-thread or a 3-vertex having three 2-neighbors, one of which has a second 3-neighbor.

Exercise 2.6. (Cranston–Jahanbekam–West [54]) Prove that if $\overline{d}(G) < \frac{5}{2}$ and G is connected, then G contains a 3^--vertex with a 1-neighbor, a 4^--vertex with two 2^--neighbors, or a 5^+-vertex v with at least $\frac{d(v)-1}{2}$ 2^--neighbors. (Comment: These configurations are reducible for the "1, 2-Conjecture" of Przybyło and Woźniak [102]. Although that proves the conjecture when $\text{mad}(G) < \frac{5}{2}$, [102] proved the stronger result that the conjecture holds for 3-colorable graphs.)

Exercise 2.7. Prove that if $\overline{d}(G) < k + \rho$ with $0 < \rho \le \frac{k}{k+1}$, then G contains a $(k-1)^-$-vertex, two adjacent k-vertices, or a $(k+1)$-vertex with more than $(\frac{1}{\rho} - 1)k$ neighbors having degree k. Construct sharpness examples with $\overline{d}(G) = k + \rho$ when $\rho = \frac{1}{2}$ and when $\rho = \frac{k}{k+1}$ (the latter may have maximum degree $k+1$ or $k+2$).

Exercise 2.8. Prove that if $\Delta(G) = k \ge 3$ and $\overline{d}(G) < k - \frac{2}{k^2+1}$, then G contains one of the following configurations: (C1) a $(k-2)^-$-vertex, (C2) two adjacent $(k-1)$-vertices, (C3) a k-vertex with two $(k-1)$-neighbors, (C4) two adjacent k-vertices each having a $(k-1)$-neighbor, or (C5) a k-vertex having three k-neighbors such that each is adjacent to a $(k-1)$-vertex.

Exercise 2.9. Argue that the configuration consisting of two 3-vertices each incident to two 2-threads is not reducible for the existence of I, F-partitions.

Exercise 2.10. Obtain G from an n-cycle by attaching to each vertex a new edge whose other endpoint lies on a new triangle ($|V(G)| = 4n$). Prove that G has no I, F-partition, despite $\text{mad}(G) = \frac{5}{2}$.

Exercise 2.11. Let G_n be the graph obtained from the complete graph K_n by subdividing every edge once. Determine $\text{mad}(G_n)$. Prove $s(G_n) > k$ for $n > k^2$ and $a(G_n) > k$ for $n > 2k^2$.

3 Discharging on Plane Graphs

The context of bounded $\mathrm{mad}(G)$ remains valid when we study planar graphs. Euler's Formula for connected graphs embedded in the plane ("plane graphs") is $n - m + p = 2$, where n, m, and p count the vertices, edges, and faces ("points" of the dual graph). When loops and multiedges are forbidden, each face boundary has length at least 3, yielding $m \le 3n - 6$. Since the degree-sum is twice the number of edges, we obtain $\mathrm{mad}(G) < 6$. When the girth (minimum cycle length) is g, the inequality generalizes to $m \le \frac{g}{g-2}(n-2)$. Deleting edges cannot create short cycles, so $\mathrm{mad}(G) < \frac{2g}{g-2}$ when G is a planar graph with girth g.

Some results on planar graphs or planar graphs with large girth hold more generally for graphs satisfying the corresponding bound on $\mathrm{mad}(G)$. However, planar graphs form a highly restricted subfamily, and often stronger results hold when planarity is also required.

The discharging method is well suited to exploit planarity. The distinctive feature of discharging for a plane graph G is that charge can also be assigned to faces, which are vertices in the dual graph G^*. Since G^* is also planar, $\mathrm{mad}(G^*) < 6$ and we can use discharging on both G and G^*. Even stronger is to use the two graphs in combination via Euler's Formula. This leads to three common natural ways to assign charge on plane graphs.

Proposition 3.1. *Let $V(G)$ and $F(G)$ be the sets of vertices and faces in a plane graph G, and let $\ell(f)$ denote the length of a face f. The following equalities hold for G.*

$$\begin{aligned}
\textstyle\sum_{v\in V(G)}(d(v)-6) + \sum_{f\in F(G)}(2\ell(f)-6) &= -12 \qquad \text{vertex charging}\\
\textstyle\sum_{v\in V(G)}(2d(v)-6) + \sum_{f\in F(G)}(\ell(f)-6) &= -12 \qquad \text{face charging}\\
\textstyle\sum_{v\in V(G)}(d(v)-4) + \sum_{f\in F(G)}(\ell(f)-4) &= -8 \qquad \text{balanced charging}
\end{aligned}$$

Proof. Multiply Euler's Formula by -6 or -4 and split the term for edges to obtain the three formulas below.

$$-6n + 2m + 4m - 6p = -12; \quad -6n + 4m + 2m - 6f = -12; \quad -4n + 2m + 2m - 4f = -8.$$

Substitute $\frac{1}{2}\sum_{v\in V(G)} d(v)$ for the first occurrence of m and $\frac{1}{2}\sum_{f\in F(G)} \ell(f)$ for the second in each equation, and then collect the contributions by vertices and by faces. □

The initial charge assigned to a vertex or face is the corresponding term in the equation being used. The initial charges are not degree or length, but rather an adjustment of those quantities based on Euler's Formula. A vertex or face now is "happy" when it reaches nonnegative charge. When specified configurations are assumed not to occur, making every vertex and face happy provides a contradiction in the same way as with degree charging; it makes the left side nonnegative, while the right side is negative.

In principle, any result provable by one of these discharging methods can also be proved by the others. However, depending on the context, one type of discharging may lead to simpler proofs than the others. For triangulations, such as in the Four Color Problem, vertex charging is appropriate. All the faces have charge 0, and often they can be ignored. In this situation, vertex charging is very much equivalent to degree charging, and such proofs can be phrased equally well using either approach. For 3-regular plane graphs, face charging is appropriate, with each vertex given charge 0. Under balanced charging when G and its dual G^* are simple, 3-vertices and 3-faces are the only objects needing charge; those with degree or length at least 5 have spare charge to give away.

In subsequent sections we will present some results about planar graphs that use additional tools of discharging. Here we emphasize the classical topic of "light subgraphs", that is, subgraphs of small weight (small degree-sum). As we have seen, light subgraphs can be reducible configurations for coloring probems; the topic was surveyed by Jendrol' and Voss [84]. We prove some results by balanced charging or face charging that were originally proved by vertex charging.

The best-known result on light edges is Kotzig's Theorem [93]: every 3-connected planar graph has an edge of weight at most 13. A *normal plane map* is a plane multigraph such that every vertex degree and face length is at least 3; in particular, every plane graph with minimum degree at least 3 is a normal plane map. Jendrol' [83, 84] gave a short proof that every normal plane map G has an edge with weight at most 11 or a 3-vertex with a 10^--neighbor. We modify the proof by Jendrol' to obtain the earlier stronger extension of Kotzig's Theorem by Borodin [17].

Lemma 3.2 (Borodin [17])**.** *Every normal plane map G has an edge with weight at most 11 or a 4-cycle through two 3-vertices and a common 10^--neighbor.*

Proof. Assume that G has no light edge. If any face F has length more than 3, then adding a chord joining the neighbors along F of a vertex on F with least degree cannot introduce a light edge. Hence we may assume that every face has length 3. Use vertex charging.

(R1) Every 5^- vertex v takes charge $\frac{6-d(v)}{d(v)}$ from each 7^+-neighbor.

Since G is a triangulation with no edges of weight at most 11, a k-vertex loses charge to at most $\lfloor \frac{k}{2} \rfloor$ vertices. Since a 7-vertex loses charge only to 5-vertices, it loses at most $3 \cdot \frac{1}{5}$ and remains positive. An 8-vertex sends charge only to 4^+-vertices and loses at most $4 \cdot \frac{1}{2}$, remaining nonnegative. For $d(v) \geq 9$, neighbors of v may have degree 3 and take charge 1; but the final charge is at least $\left\lceil \frac{d(v)}{2} \right\rceil$ 6, which is nonnegative when $d(v) \geq 11$.

Finally, suppose $d(v) \in \{9, 10\}$. Since faces have length 3, the neighbors of v form a closed walk of length $d(v)$ when followed in order, and each 3-vertex appears only once in this walk. With light edges and the specified 4-cycles forbidden, 3-vertices must be separated by at least three steps along this walk.

Since there are no light edges, each 9-vertex has at most four 5^--neighbors. If it has at least three 3-neighbors, then it has exactly three and loses charge to no other vertices. Hence a 9-vertex loses at most $\max\{3 \cdot 1, 2 \cdot 1 + 2 \cdot \frac{1}{2}\}$ and ends happy. Similarly, a 10-vertex has at most five 5^--neighbors, and if it has at least three 3-neighbors then it has at most four 5^--neighbors. It loses at most $\max\{4 \cdot 1, 2 \cdot 1 + 3 \cdot \frac{1}{2}\}$ and ends happy. □

We give an application of Lemma 3.2. A *decomposition* of a graph expresses it as a union of edge-disjoint subgraphs; its *size* is the number of subgraphs. The *arboricity* of a graph G, written $\Upsilon(G)$, is the minimum size of a decomposition of G into forests. A *linear forest* is a forest whose components are paths. The *linear arbority*, written $\mathrm{la}(G)$, is the minimum size of a decomposition into linear forests.

Trivially $\mathrm{la}(G) \geq \lceil \Delta(G)/2 \rceil$, but equality for $2r$-regular graphs would need 2-regular color classes, which would contain cycles. Akiyama, Exoo, and Harary [2, 3] conjectured that always $\mathrm{la}(G) \leq \lceil (\Delta(G)+1)/2 \rceil$. Together, [2, 3, 63, 73] proved it for $\Delta(G) \in \{1, 2, 3, 4, 5, 6, 8, 10\}$. Given $\epsilon > 0$, Alon [6] proved $\mathrm{la}(G) \leq (\frac{1}{2} + \epsilon)\Delta(G)$ for $\Delta(G)$ sufficiently large. When G is planar, the conjecture was proved for $\Delta(G) \geq 9$ by Wu [128] (presented below) and for $\Delta(G) = 7$ by Wu and Wu [129], so the proof is complete for planar graphs.

Theorem 3.3 (Wu [128]). *If G is a planar graph with $\Delta(G) \geq 9$, then* $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

Proof. We show for $t \geq 5$ that every planar graph G with $\Delta(G) < 2t$ decomposes into t linear forests. View the linear forests as t color classes in an edge-coloring.

For an edge uv with weight at most $2t+1$, consider such a decomposition of $G - uv$. Since $d_{G-uv}(u) + d_{G-uv}(v) < 2t$, fewer than t colors appear twice at u or twice at v or once at each. Thus a color is available at uv to extend the decomposition to G.

Hence we may assume weight at least $2t+2$ for every edge. Since $\Delta(G) < 2t$, this yields $\delta(G) \geq 3$. Now Lemma 3.2 yields a 4-cycle $[u, x, v, y]$ in G with $d(u) = d(v) = 3$. Let u' and v' be the remaining neighbors of u and v, respectively (see Figure 4). Note that $\{u, v\} = \{u', v'\}$ is forbidden, but $u' = v'$ is possible, requiring a similar analysis that we omit.

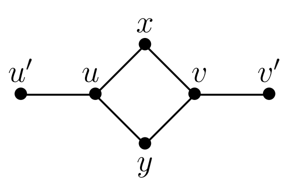

Figure 4: Reducible configuration for Theorem 3.3

The weight restrictions and $\Delta(G) \leq 2t-1$ imply that each of x, y, u', v' has degree $2t-1$ in G. Hence a linear t-decomposition of $G - \{u, v\}$ has $2t-3$ colored edges at both x and y and $2t-2$ colored edges at both u' and v'. We conclude that at most one color is missing at each of these vertices; call a vertex "bad" if a color is missing. For a vertex z, let $C(z)$ and $C'(z)$ denote the sets of colors appearing at z exactly once and at most once, respectively.

Case 1: Neither x nor y is bad. Here $|C(x)| = |C(y)| = 3$. First color uu' from $C'(u')$, then ux from $C(x)$, and then uy from $C(y)$. Make the colors on ux and uy differ. If u' is bad, then the color on uu' may be used on ux or uy, but otherwise the three edges have distinct colors. Two colors from each of $C(x)$ and $C(y)$ remain, and we ensure these remaining pairs are not equal. Now we can choose distinct colors on the three edges at v.

Case 2: One of x and y is bad. We may assume by symmetry that x is bad; still $|C(y)| = 3$. Give ux the color missing at x, and give xv the one color in $C(x)$. Color vv' from $C'(v')$, different from the color on xv if v' is not bad. Now color vy from $C(y)$ avoiding the colors on xv and vv'. Now color uu' from $C'(u')$, and choose a color for uy from $C(y)$ avoiding that and the color on vy. Note that uu' or uy may have the same color as ux.

Case 3: x and y are both bad. If the one missing color at each of x and y is the same, then use it on ux and vy. Now use the color in $C(x)$ on xv and the color in $C(y)$ on yu. If u' is bad, then its missing color can be used on uu'; otherwise, color uu' from $C(u')$ to avoid the color on yu. The symmetric argument applies to color vv'.

If the missing colors at x and y are different, then use the color missing at x on ux and xv, and use the color missing at y on uy and yv. If a color is missing at u', use it on uu', and then use any color from $C'(v')$ on vv'. By symmetry, then, only $|C(u')| = |C(v')| = 2$ remains, and it suffices to give uu' and vv' distinct colors from these sets. □

When attention is confined to planar graphs, the concern about regular graphs with large even degree vanishes. Wu [128] proved that $\mathrm{la}(G) = \lceil \frac{1}{2}\Delta(G) \rceil$ when G is planar with $\Delta(G) \geq 13$ (see Exercise 3.22). Cygan, Kowalik, and Lužar [60] proved that $\Delta(G) \geq 10$ suffices and conjectured that $\Delta(G) \geq 6$ suffices.

A famous result on light subgraphs concerns light triangles in planar graphs with minimum degree 5. It was conjectured for triangulations by Kotzig and proved in stronger form by Borodin; see Exercise 3.5 for a sharpness example. A further strengthening to a more detailed statement was proved by Borodin and Ivanova [38]. We sketch the proof here because an interesting wrinkle in the discharging rules can be viewed as "redirecting" charge.

Theorem 3.4 (Borodin [18])**.** *If G is a simple plane graph with $\delta(G) \geq 5$, then G has a 3-face with weight at most 17, and the bound is sharp.*

Proof. (sketch) For sharpness, add a vertex in each face of the dodecahedron joined to the vertices of that face. The new vertices have degree 5, and the old ones now have degree 6. Every face has one new vertex and two old vertices for total weight 17.

To prove the bound, consider an edge-maximal counterexample G. Every vertex on a 4^+-face has degree 5, since adding a triangular chord at a 6^+-vertex would create only heavy 3-faces, producing a counterexample containing G.

If there is no light triangle, then use vertex charging and move charge as follows:

(R1) Each 4^+-face gives $\frac{1}{2}$ to each incident vertex.

(R2) Each 7-vertex gives $\frac{1}{3}$ to each 5-neighbor.
(R3) Each 8^+-vertex gives $\frac{1}{4}$ through each incident 3-face to its 5-neighbors on that face, split equally if there are two such neighbors.

It suffices to show that final charges are nonnegative. The discharging rules are chosen so that 4^+-faces and 7^+-vertices do not give away too much charge. (Maximality of the counterexample puts a 7-vertex on seven triangles, and absence of light triangles then restricts a 7-vertex to have at most three 5-neighbors).

Hence the task is to prove that a 5-vertex v gains charge 1. When all incident faces are triangles, avoiding light triangles restricts v to have at most two 5-neighbors. It also forces the other neighbors to be 8^+-vertices when v has two 5-neighbors. In this and the remaining cases (such as when v lies on a 4^+-face), it is easy to check that v receives enough charge. □

In each discharging rule in Theorem 3.4, the charge given away is the most that object can afford to lose. Only 5-vertices need to gain charge. Hence it would be natural to have 8^+-vertices give $\frac{1}{4}$ to each 5-neighbor. However, this would give only $\frac{3}{4}$ to a 5-vertex having two 5-neighbors and three 8-neighbors. The 5-vertex needs to get more when it has two 8-neighbors on a single triangle. Guiding the charge from 8^+-neighbors through the incident triangles is a way to arrange that.

Our next structure theorem is a stronger version of the 5-degeneracy that follows from $\mathrm{mad}(G) < 6$ for planar graphs. It has several applications.

Lemma 3.5. *Every planar graph has a 5^--vertex with at most two 12^+-neighbors.*

Proof. We may assume that G is a triangulation, since adding an edge cannot give any vertex the desired property.

Assume that G has no such vertex, so $\delta(G) \geq 3$. Use degree charging; note that $\mathrm{mad}(G) < 6$. Every 5^--vertex has at least three 12^+-neighbors. Let each 5^--vertex u take $\frac{6-d(u)}{3}$ from each 12^+-neighbor. Now 5^--vertices are happy, and j-vertices with $6 \leq j \leq 11$ lose no charge, so it suffices to show that 12^+-vertices do not lose too much.

Let v be a 12^+-vertex. Since G is a triangulation, the neighbors of v form a cycle C, possibly having chords. Let H be the subgraph of C induced by its vertices having degree at most 5 in G. Each 5^--vertex w has at least three 12^+-neighbors, so $d_H(w) \leq d_G(w) - 3$. If all neighbors of v have degree 5, then v loses $\frac{d(v)}{3}$ and ends with $\frac{2}{3}d(v)$, which is at least 8.

Otherwise, the components of H are paths (bold in Figure 5). Combine such a k-vertex path P with the next vertex on C, which receives no charge from v. If $k = 1$, then v gives at most 1 to these two vertices. If $k > 1$, then v gives at most $\frac{1}{3}$ to internal vertices of P (degree at least 5) and at most $\frac{2}{3}$ to its endpoints (degree at least 4). Hence the $k+1$ vertices receive at most $0 + 2(\frac{2}{3}) + (k-2)(\frac{1}{3})$ from v. This equals $\frac{k+2}{3}$, which is less than $\frac{k+1}{2}$ when $k > 1$. Hence v loses in total at most $\frac{d(v)}{2}$, leaving at least 6 when $d(v) \geq 12$. □

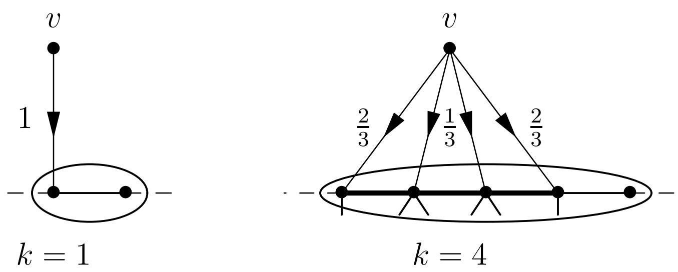

Figure 5: Final case for Lemma 3.5

For an application, consider again the *arboricity* $\Upsilon(G)$. Nash-Williams [101] famously proved that for every graph G, the arboricity equals a trivial lower bound: always $\Upsilon(G) = \max_{H\subseteq G} \frac{|E(H)|}{|V(H)|-1}$. Although there is a short general proof using matroids, for planar graphs the existence of 5^--vertices permits an inductive decomposition into three forests. We include this in Exercise 3.14 to illustrate a technique for reducibility with triangulations.

Lemma 3.5 was improved by Balogh, Kochol, Pluhar, and Yu [10] to guarantee a 5^-vertex having at most two 11^+-neighbors, proved by a much longer discharging argument than in Lemma 3.5. The result is sharp in that "11" cannot be replaced by "10" (the graph obtained by adding a vertex of degree 3 in each face of the icosahedron has three 10-neighbors for each 5^--vertex). From this they proved that every planar graph decomposes into three forests, with one having maximum degree at most 8. Lemma 3.5 allows the degree of the third forest to be bounded by 9 (Exercise 3.14). In a somewhat different direction, a much more detailed strengthening that considers a larger set of light subgraphs was proved by Borodin and Ivanova [37]; it extends or strengthens several intermediate results.

The amount by which total charge is negative can be used to prove that actually many light edges must occur. Another instance of this technique occurs in [66].

Theorem 3.6 (Borodin–Sanders [42])**.** *For any plane graph G with $\delta(G) = 5$,*

$$2e_{5,5} + e_{5,6} + \frac{2}{7}e_{5,7} \geq 60,$$

where $e_{i,j}$ is the number of edges with endpoints of degrees i and j. Furthermore, the coefficients in this inequality are sharp.

Proof. Add edges to obtain a triangulation H. No vertex degree decreases, so $\delta(H) = 5$. Also, since $\delta(G) = 5$, no edges incident to vertices having degree 5 in H are added. Hence $e_{5,j}(H) \leq e_{5,j}(G)$, and it suffices to prove the desired lower bounds when H is a triangulation with minimum degree 5.

Use vertex charging. Each 5-vertex takes $\frac{1}{5}$ from each 8^+-neighbor and $\frac{1}{7}$ from each 7-neighbor. Since $(4/5)j - 6 > 0$ when $j \geq 8$, every 6^+ vertex ends happy. Negative charge remains only at 5-vertices. Each 7^--neighbor corresponds to an edge along which a 5-vertex fails to gain $\frac{1}{5}$, thereby leaving more negative charge there. A 7-neighbor sends $\frac{1}{7}$ instead of $\frac{1}{5}$, thus contributing $\frac{-2}{35}$ to the negative charge remaining at 5-vertices. Edges counted by $e_{5,5}$ affect both endpoints. Thus edges counted by $e_{5,5}$, $e_{5,6}$ and $e_{5,7}$ contribute $\frac{-2}{5}$, $\frac{-1}{5}$, and $\frac{-2}{35}$, respectively, to the negative charge remaining at 5-vertices. Since the total charge is -12 and there is no negative charge elsewhere, in units of $\frac{-1}{5}$ we have $2e_{5,5} + e_{5,6} + \frac{2}{7}e_{5,7} \geq 60$. (Another proof sends all charge from vertices to edges so that each vertex ends with charge 0 and only light edges have negative charge.)

Equality requires that no positive charge is left anywhere, since that would require more negative charge left at 5-vertices. Hence a sharpness example must be a triangulation with maximum degree at most 7. We use different sharpness examples for different coefficients.

The 5-regular icosahedron has $e_{5,5} = 30$ and no other edges; thus the coefficient on $e_{5,5}$ cannot be reduced. The graph obtained from the 3-regular dodecahedron by inserting in each face a 5-vertex adjacent to its corners has $e_{5,6} = 60$, with all other edges joining 6-vertices; thus the coefficient on $e_{5,6}$ cannot be reduced.

The graph in Figure 6 (three edges wrap around from left to right), with $2e_{5,5} = e_{5,6} = 28$ and $e_{5,7} = 14$, shows that the coefficient on $e_{5,7}$ cannot be reduced below $\frac{2}{7}$. □

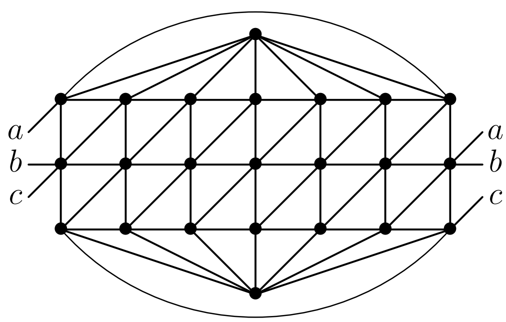

Figure 6: Sharpness example for Theorem 3.6

Let us now consider restricted families of planar graphs under girth constraints. We noted in the introduction that planar graphs with girth at least g satisfy $\mathrm{mad}(G) < \frac{2g}{g-2}$. In some cases, planarity permits a stronger result, meaning that obtaining the same conclusion using only a bound on $\mathrm{mad}(G)$ requires $\mathrm{mad}(G) < b$ for some b smaller than $\frac{2g}{g-2}$.

For example, consider a special case of Remark 2.1: every graph G with $\mathrm{mad}(G) < \frac{8}{3}$ and $\delta(G) \geq 2$ has a 2-vertex with a 3^--neighbor. Planar graphs with girth at least 8 satisfy $\mathrm{mad}(G) < \frac{8}{3}$. The result in terms of $\mathrm{mad}(G)$ is sharp, since subdividing every edge of a 4-regular graph yields a graph G with $\mathrm{mad}(G) = \frac{8}{3}$ having no such 2-vertex. However, the conclusion holds for planar graphs with girth 7, which allow $\mathrm{mad}(G)$ to be larger. The argument illustrates typical difficulties that may arise when discovering discharging arguments.

Lemma 3.7. *Every planar graph G with girth at least 7 and $\delta(G) \geq 2$ has a 2-vertex with a 3^--neighbor.*

Proof. Assume that G has no such configuration and use face charging. With initial charges $2d(v) - 6$ and $\ell(f) - 6$, when G has girth at least 7 the only objects with negative initial charge are 2-vertices. Let each 2-vertex take $\frac{1}{2}$ from each neighbor and each incident face. To complete the proof, we check that all vertices and faces end with nonnegative charge.

The discharging rule ensures that 2-vertices end with charge 0. Since 3-vertices have no 2-neighbors, their charge remains 0. For $j \geq 4$, a j-vertex may lose $\frac{1}{2}$ along each edge and ends with charge at least $2j - 6 - \frac{j}{2}$, which is nonnegative.

A j-face has at most $\lfloor \frac{j}{2} \rfloor$ incident 2-vertices, since 2-vertices are not adjacent. Hence a j-face has final charge at least $j - 6 - \frac{1}{2} \lfloor \frac{j}{2} \rfloor$, which is nonnegative for $j \geq 8$. To help the 7-faces, we add another discharging rule. When adjacent 4^+-vertices form an edge e, direct the charge $\frac{1}{2}$ that each could send to a 2-neighbor so that instead the two faces bounded by e each receive $\frac{1}{2}$. Now when a 7-face gives away $\frac{3}{2}$ to three 2-vertices, it recovers $\frac{1}{2}$ from the two adjacent 4^+-vertices on its boundary and ends with charge 0. □

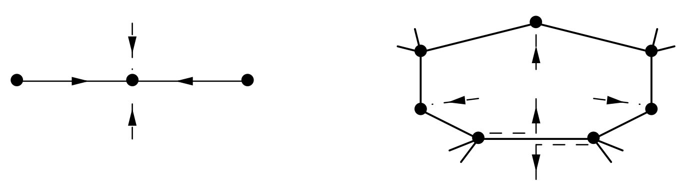

Figure 7: Discharging for Lemma 3.7

This proof illustrates both "redirection" of transmitted charge and the phenomenon of designing discharging rules initially to make deficient vertices or faces happy but discovering later that additional rules are then needed to repair others that lost too much. It turns out that balanced charging, where 2-vertices and 3-vertices both need charge, yields a simpler discharging proof of this result; see Exercise 3.16.

Lemma 3.7 yields a stronger result for planar graphs with girth 7 than is possible for the corresponding bound on $\text{mad}(G)$. A graph G is *dynamically k-choosable* if for every k-uniform list assignment L, there is a dynamic L-coloring of G, meaning a proper L-coloring with the additional property that the neighbors of a vertex cannot all have the same color if the vertex has degree at least 2. By showing that the configuration in Lemma 3.7 (a 2-vertex with a 3^--neighbor is reducible (Exercise 3.17), Kim and Park [89] showed that every planar graph with girth at least 7 is dynamically 4-choosable. This result is sharp, since it is well known that there are planar graphs that are not 4-choosable (Voigt [121]), and subdividing every edge of such a planar graph yields a planar graph with girth 6 that is not dynamically 4-choosable.

Lemma 3.7 also has an application to acyclic coloring. Configurations consisting of a 1^--vertex or a 2-vertex with a 3^--neighbor are reducible for acyclic 4-choosability, so this holds for planar graphs with girth at least 7. Grünbaum [72] conjectured that all planar graphs are acyclically 5-colorable. This was proved by Borodin [16] after successive improvements to Grünbaum's initial upper bound of 9. The bound of 5 is sharp, even among bipartite planar graphs [90] (Exercise 3.18). Borodin's proof used discharging with some 450 reducible configurations but no computers, an enormous effort.

Borodin et al. [26] conjectured the stronger statement that all planar graphs are in fact acyclically 5-choosable. Toward the conjecture, Borodin and Ivanova [35] proved that planar graphs without 4-cycles are acyclically 5-choosable. In [34], only special 4-cycles are forbidden. Montassier, Raspaud, and Wang [100] conjectured that planar graphs without 4-cycles are acyclically 4-choosable and proved this in some cases; it holds when both 4-cycles and 5-cycles are forbidden [36, 47].

Larger girth (or smaller $\mathrm{mad}(G)$) makes acyclic coloring easier. We have already observed that planar graphs with girth at least 7 are acyclically 4-choosable; Montassier [99] proved that girth at least 5 suffices, while Borodin et al. [25] proved that girth at least 7 yields acyclic 3-choosability. We saw in Theorem 2.3 that $\mathrm{mad}(G) < 3$ yields acyclic 6-choosability. The condition holds for planar graphs with girth at least 6, but using planarity allows us to relax the girth restriction. For planar graphs with girth at least 5, we prove a structure theorem that yields acyclic 6-choosability and has other applications. Note that for planar graphs with girth 5 and minimum degree 3, it guarantees an edge of weight 6.

Lemma 3.8 (Cranston–Yu [59])**.** *If G is a planar graph with girth at least 5 and $\delta(G) \geq 2$, then G has a 2-vertex with a 5^--neighbor or a 5-face whose incident vertices are four 3-vertices and a 5^--vertex.*

Proof. (sketch) Let G be such a graph containing none of the specified configurations. Assign charges by balanced charging; discharging will make all vertices and faces happy when the specified configurations do not occur.

(R1) Each 3^--vertex v takes $\frac{4-d(v)}{d(v)}$ from each incident face.
(R2) Each 6^+-vertex v gives $\frac{d(v)-4}{d(v)}$ to each incident face.

The rules immediately make each vertex happy (5-vertices end positive), and it remains only to check that each face ends happy. The configurations in Figure 8 show faces that end with charge 0; Exercise 3.19 requests the verification that other faces end happy. □

Theorem 3.9. *If G is a planar graph with girth at least 5, then G is acyclically 6-choosable.*

Proof. Since a 1^--vertex lies in no cycle, its color need only avoid that of its (possible) neighbor. Hence a 1^--vertex is reducible for $a_\ell(G) \leq 6$, and we may assume $\delta(G) \geq 2$. It

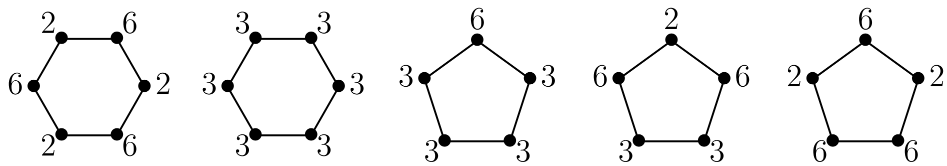

Figure 8: Sharp configurations for Lemma 3.8

therefore suffices to show that the configurations in Lemma 3.8 are reducible for $a_\ell(G) \le 6$. Let L be a 6-uniform list assignment for G.

First consider a 2-vertex v with a 5^--neighbor u. Let ϕ be an acyclic L-coloring of $G - v$. If the colors on $N_G(v)$ are distinct, then color v with a color in $L(v)$ other than those. If the colors on $N(v)$ are equal, then color v with a color not used on $N_G(v) \cup N_{G-v}(u)$. Since $|N_{G-v}(u)| \le 4$, this forbids at most five colors, and a color remains available in $L(v)$. Now there are no 2-colored cycles in $G - v$ and none through v.

For the remaining configuration, let v_1, v_2, v_3, v_4, w be the vertices on a 5-face, with each v_i of degree 3 and $d(w) \le 5$. Let x_i be the neighbor of v_i outside the 5-cycle (see Figure 9).

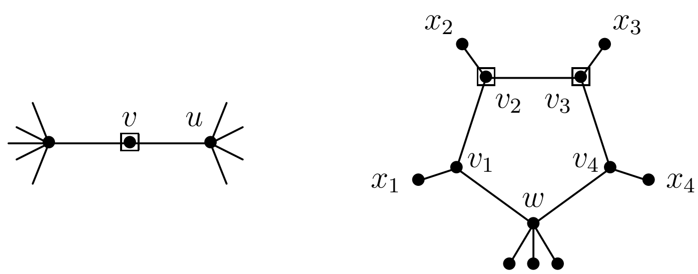

Figure 9: Reducible configurations for Theorem 3.9

Let ϕ be an acyclic L-coloring of $G - \{v_2, v_3\}$. We consider three cases, depending on whether ϕ uses one color or two colors on $N_G(v_2)$ and on $N_G(v_3)$. (a) If $\phi(v_1) \ne \phi(x_2)$ and $\phi(v_4) \ne \phi(x_3)$, then choose $\phi(v_2)$ and $\phi(v_3)$ distinct and outside $\{\phi(v_1), \phi(x_2), \phi(x_3), \phi(v_4)\}$. (b) If $\phi(v_1) = \phi(x_2)$ but $\phi(v_4) \ne \phi(x_3)$, then choose $\phi(v_2) \notin \{\phi(w), \phi(x_1), \phi(x_2)\}$ and $\phi(v_3) \notin \{\phi(v_4), \phi(x_3), \phi(v_2), \phi(v_1)\}$. (c) If $\phi(v_1) = \phi(x_2)$ and $\phi(v_4) = \phi(x_3)$, then choose $\phi(v_2) \notin \{\phi(w), \phi(v_1), \phi(x_1), \phi(v_4)\}$ and $\phi(v_3) \notin \{\phi(v_2), \phi(v_4), \phi(w), \phi(x_4)\}$. In each case, the coloring is proper, and the new vertices lie in no 2-colored cycle. □

We will not discuss the proof of the Four Color Theorem here. It is well known that after a hundred years of failed attempts, Appel and Haken (working with Koch) found "an unavoidable set of reducible configurations" using the discharging method. The discharging rules and reducibility arguments were far more complicated than anything we present here. The initial proof involved 1936 reducible configurations. The unavoidable set was generated by hand, but reducibility was checked by computer. The publication comprised nearly 140 pages in two papers [7, 9] plus over 400 pages of microfiche that became a 741-page book [8].

Some people objected to the use of computers, but the proof is now generally accepted. Robertson, Sanders, Seymour, and Thomas [103] looked for a simpler proof but eventually used the same approach. Their unavoidable set had only 633 configurations and 32 discharging rules, but they still needed a computer. With the increases in computing power and simpler arguments, their proof ran in only 20 minutes instead of the original 1200 hours.

With the Four Color Theorem proved, attention has turned to making use of it (a notable example is Robertson, Seymour, and Thomas [104] using it to prove the case $k = 6$ of Hadwiger's Conjecture, for which they won the 1994 Fulkerson Prize) and to understanding which planar graphs are 3-colorable. Computationally, testing 3-colorability of a planar graph is NP-hard [110], but many sufficient conditions are known.

The most natural condition is to increase the girth; already Grötzsch [71] proved that planar graphs with girth at least 4 are 3-colorable. There have been a number of proofs of this ([62, 92, 112, 114]), all using discharging at some point. Thomassen [114] showed that girth at least 5 suffices for 3-choosability.

Steinberg [109] conjectured that every planar graph without 4-cycles or 5-cycles is 3-colorable. Eventually, Cohen-Addad et al. [51] found counterexamples. Results on this family can be compared with the family where $\text{mad}(G) < 4$; see Exercise 3.21.

During the 40 years between [109] and [51], many papers used discharging to prove 3-colorability under various conditions excluding sets of cycle lengths. For example, Borodin et al. [28] proved that excluding cycles of lengths 4 through 7 suffices. Earlier, Borodin [21] and Sanders and Zhao [105] proved that excluding 4-cycles and faces of lengths 5 through 9 is sufficient. The traditional proof (presented in the survey [23]) uses balanced charging, but face charging yields a somewhat simpler proof.

Lemma 3.10 ([21]). *Every plane graph G with $\delta(G) \geq 3$ has two 3-faces with a common edge, or a j-face with $4 \leq j \leq 9$, or a 10-face whose vertices all have degree 3.*

Proof. Let G be a plane graph with $\delta(G) \geq 3$ having none of the listed configurations. Use face charging: assign charge $2d(v) - 6$ to each vertex v and charge $\ell(f) - 6$ to each face f. The total charge is -12.

Since no faces have lengths 4 through 9, the only objects with initial negative charge are triangles; they begin with charge -3. Each triangle takes 1 from each neighboring face. To repair faces that may lose too much, each face f takes 1 from each incident 4^+-vertex lying on at least one triangle sharing an edge with f (see Figure 10).

We have made 3-faces happy, and 3-vertices remain at charge 0. Other vertices remain happy because 3-faces do not share edges. For $j \geq 4$, a j-vertex loses charge at most $\lfloor \frac{2j}{3} \rfloor$ and ends with at least $\lceil \frac{4j}{3} \rceil - 6$, which is nonnegative for $j \geq 4$.

Now consider a j-face f for $j \geq 10$. It loses 1 for every path along its boundary such that the neighboring faces are triangles and the endpoints have degree 3; f gives 1 to each of those faces but regains 1 from each intervening vertex. If an endpoint of a maximal such path has degree at least 4, then there is no net loss. Hence the net loss for f is at most $\lfloor \frac{j}{2} \rfloor$, and the final charge is at least $\lceil \frac{j}{2} \rceil - 6$, which is nonnegative when $j \geq 11$.

Hence negative charge can occur only at 10-faces. A 10-face f must lose more than 4 to become negative. This requires five paths through which f loses 1. The paths must be single edges sharing no vertices, and all the vertices incident to f must have degree 3. $\square$

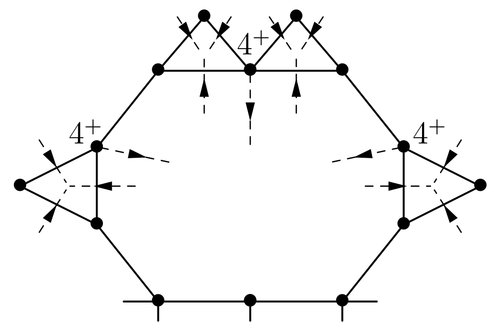

Figure 10: Discharging for Lemma 3.10

Theorem 3.11 ([21, 105]). *Every plane graph having no 4-cycle and no j-face with* $5 \leq j \leq 9$ *is 3-colorable.*

Proof. A smallest counterexample G must be 4-critical, and hence it has minimum degree at least 3 and is 2-connected. Since there is no 4-cycle, no two 3-faces share an edge. By Lemma 3.10, we may thus assume that G is embedded with at least one 10-face C, whose vertices all have degree 3. Let f be a proper 3-coloring of $G - V(C)$. Since each vertex on C has exactly one neighbor outside C, two colors remain available at each vertex of C. Since even cycles are 2-choosable, the coloring can be completed. $\square$

Exercise 3.1. Let G be a simple plane graph with $\delta(G) \geq 3$. Prove that G has a 3-vertex on a 5^--face or a 5^--vertex on a triangle.

Exercise 3.2. (Lebesgue [94]) Strengthen the previous exercise by proving that every plane graph G with $\delta(G) \geq 3$ contains a 3-vertex on a 5^--face, a 4-vertex on a 3-face, or a 5-vertex with four incident 3-faces. (Comment: Lebesgue phrased the proof only for 3-connected plane graphs.)

Exercise 3.3. Construct planar graphs to show that the bounds in Lemma 3.2 are sharp. That is, none of the values 10, 7, 6 can be reduced in the statement that normal plane maps have a 3-vertex with a 10^--neighbor, a 4-vertex with a 7^--neighbor, or a 5-vertex with a 6^--neighbor.

Exercise 3.4. Prove that planarity is needed in Lemma 3.2 by showing that a graph G with $\mathrm{mad}(G) < 6$ and $\delta(G) = 5$ need not have a 5-vertex with any 6^--neighbor.

Exercise 3.5. Prove that requiring minimum degree 5 in Theorem 3.4 is necessary, by constructing for each $k \in \mathbb{N}$ a planar graph with minimum degree 4 having no triangle with weight at most k.

Exercise 3.6. (Cranston [52]) Let G be a plane graph with $\Delta(G) \geq 7$. Prove that G has either two 3-faces with a common edge or an edge with weight at most $\Delta(G) + 2$. Conclude that if G is a plane graph with $\Delta(G) \geq 7$ and no two 3-faces sharing an edge, then G is $(\Delta(G) + 1)$-edge-choosable. (Hint: Use balanced charging. Comment: Cranston proved that the same conditions are also sufficient when $\Delta(G) \geq 6$, which implies several earlier results.)

Exercise 3.7. The argument in Remark 2.1 shows that $\frac{12}{5}$ is the largest b such that $\mathrm{mad}(G) < b$ guarantees adjacent 2-vertices in G. Note that $\mathrm{mad}(G) < \frac{12}{5}$ when G is planar with girth at least 12. Prove that planar graphs with girth at least 11 have adjacent 2-vertices, and provide a construction to show that the conclusion fails for some planar graph with girth 10.

Exercise 3.8. (Grünbaum [72]) Prove that if a planar graph has no edges joining 5-vertices, then it has at least 60 edges whose endpoints have degrees 5 and 6.

Exercise 3.9. Planar graphs with girth at least 6 satisfy $\mathrm{mad}(G) < 3$, so by Lemma 1.3 each such graph has a 2-vertex with a 5^--neighbor. Show that this is sharp even for planar graphs by constructing a planar graph with girth 6 having no edge of weight at most 6.

Exercise 3.10. Determine whether a planar graph with girth at least 4 and minimum degree 3 must have a 3-vertex with a 4^--neighbor. Construct a planar graph G_k with girth 4 and minimum degree 3 in which the distance between 3-vertices is at least k. Construct a planar graph H_k with minimum degree 5 in which the distance between 5-vertices is at least k.

Exercise 3.11. Let G be a graph with $\delta(G) = 3$ and $\mathrm{mad}(G) < \frac{10}{3}$. Prove that G has a 3-vertex whose neighbors have degree-sum at most 10. Prove that this result is sharp even in the family of planar graphs with girth at least 5 by constructing such a graph in which no 3-vertex has three 3-neighbors. (Comment: G. Tardos constructed such a graph with 98 vertices.)

Exercise 3.12. (Borodin [22]) Prove that every planar graph with minimum degree 5 contains two 3-faces sharing an edge with weight at most 11. (Hint: Use vertex charging, with 5-vertices taking $\frac{1}{2}$ from incident 4^+-faces and the remaining needed charge from 7^+-neighbors.)

Exercise 3.13. Prove that every plane triangulation with minimum degree 5 has two 3-faces sharing an edge such that the non-shared vertices have degree-sum at most 11. (Hint: Use vertex charging; 6-vertices that give charge to 5-neighbors will need charge from 7^+-neighbors. Comment: Albertson [4] used this configuration in a proof that $\alpha(G) \geq \frac{2n}{9}$ when G is an n-vertex planar graph with no separating triangle, without using the Four Color Theorem or the language of discharging.)

Exercise 3.14. Prove inductively that every planar graph decomposes into three forests. (Hint: Reduce to triangulations, and then apply the induction hypothesis to a smaller graph obtained by deleting a light vertex and triangulating the resulting face. There are a number of cases when the deleted vertex has degree 5, depending on the usage of the two added edges.) Use Lemma 3.5 and more detailed analysis to prove that the third forest can be guaranteed to have maximum degree at most 9. (Comment: The second part of this exercise is long. See Balogh et al. [10] for maximum degree at most 8.)

Exercise 3.15. Let G be a planar graph with $\delta(G) = 5$. With $e_{i,j}$ denoting the numbers of edges with endpoints of degrees i and j, prove $\frac{26}{11}e_{5,5} + e_{5,6} \geq 60$. (Comment: Borodin and Sanders [42] proved the stronger result $\frac{7}{3}e_{5,5} + e_{5,6} \geq 60$; the coefficients are sharp.)

Exercise 3.16. Reprove Lemma 3.7 by using balanced charging to prove that every planar graph with girth at least 7 and minimum degree at least 2 has a 2-vertex adjacent to a 3^--vertex. Prove that the conclusion does not always hold when $\mathrm{mad}(G) < \frac{14}{5}$ (thus planarity is needed). Show that the conclusion does not hold for all planar graphs with girth 6.

Exercise 3.17. (Kim–Park [89]) Prove that among the planar graphs with girth at least 7, a minimal graph that is not dynamically 4-choosable cannot contain a 2-vertex with a 3^--neighbor. (Comment: With Lemma 3.7, this proves that every planar graph with girth at least 7 is dynamically 4-choosable. Note that $\mathrm{mad}(G) < \frac{14}{5}$ when G is planar with girth at least 7, but $\mathrm{mad}(G) < \frac{14}{5}$ is not sufficient for dynamic 4-choosability.)

Exercise 3.18. (Grünbaum [72], Kostochka–Mel'nikov [90]) Prove that the two graphs in Figure 11 are not acylically 4-colorable. The half-edges leaving the figure on the right lead to an additional vertex having the same neighborhood as the central vertex.

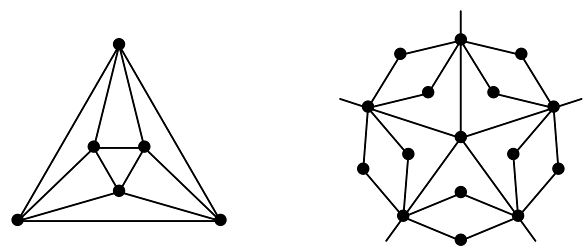

Figure 11: Graphs that are not acyclically 4-colorable

Exercise 3.19. Complete the proof of Lemma 3.8.

Exercise 3.20. (Dvořak–Kawarabayashi–Thomas [62]) Let C be the outer boundary in a 2-connected triangle-free plane graph G that is not a cycle. If C has length at most 6, and every vertex not on C has degree at least 3, then G contains a bounded 4-face or a proper 5-face, where a 5-face is *proper* if (at least) four of its vertices have degree 3 and are not on C. (Comment: This result was used in [62] to give a new proof of Grötzsch's Theorem [71] that triangle-free planar graphs are 3-colorable. The proof in [62] used vertex charging, but using face charging is simpler.)

Exercise 3.21. Let G be a plane graph having no 4-cycle and no face with length in $\{4, \ldots, k\}$. Use discharging to prove that the average face length in G is at least $6 - \frac{18}{k+4}$. Conclude that $\mathrm{mad}(G) < 3 + \frac{9}{2k-1}$. In particular, $\mathrm{mad}(G) < 4$ when G is a plane graph having no 4-face or 5-face.

Exercise 3.22. (Wu [128]) Strengthen Lemma 5.7 to show that every planar graph contains an edge of weight at most 15 or a 2-alternating cycle such that some high-degree vertex on the cycle has an additional 2-neighbor outside the cycle. Conclude that $\mathrm{la}(G) = \left\lceil \frac{1}{2}\Delta(G) \right\rceil$ for every planar graph G with $\Delta(G) \geq 13$.

Exercise 3.23. Construct a planar graph with no 5-cycle and a planar graph with no 4-cycle that are not 3-colorable.

4 List Coloring

List coloring (Definition 1.7) was invented in the 1970s by Vizing [120] and by Erdős, Rubin, and Taylor [64]. As we noted, reducibility arguments for a coloring property often extend to reducibility for the corresponding list coloring property, especially when made just by choosing colors for vertices in a particular order. For example, the famous result of Brooks [44] that $\chi(G) \le \Delta(G)$ when G is a connected graph that is not a complete graph or odd cycle was strengthened to $\chi_\ell(G) \le \Delta(G)$ for such graphs in [120] and in [64] (without discharging). For planar graphs, the beautiful result by Thomassen [111] that planar graphs are 5-choosable (sharp by [121]) also does not use discharging. Thomassen [113] also provedthat planar graphs with girth at least 5 are 3-choosable.

In this section we take a closer look at several problems involving list coloring in order to develop further techniques for discharging arguments. We begin with a useful lemma.

Lemma 4.1 ([64]). *Even cycles are* 2*-choosable.*

Proof. We show that C_{2t} is L-colorable when every list has size 2. If the lists are identical, then choose the colors to alternate. Otherwise, there are adjacent vertices x and y such that $L(x)$ contains a color c not in $L(y)$. Use c on x, and then follow the path $C_{2t} - x$ from x to y to color the vertices other than x: at each new vertex, choose a color from its list that was not chosen for the previous vertex. Such a choice is always available, and the chosen colors satisfy every edge because the colors chosen on x and y differ. □

Coloring and list-coloring have been studied extensively for squares of graphs. Given a graph G, let G^2 be the graph obtained from G by adding edges to join vertices that are distance 2 apart in G. The neighbors of a vertex v in G form a clique with v in G^2, so always $\chi(G^2) \ge \Delta(G)+1$. Proper coloring of G^2 has also been called 2-*distance coloring* of G, since vertices with the same color must be separated by distance at least 2.

Kostochka and Woodall [91] conjectured that always $\chi_\ell(G^2) = \chi(G^2)$. This was proved in special cases, but Kim and Park [88] provided counterexamples. They used orthogonal families of Latin squares to construct a graph G for prime p such that G^2 is the complete $(2p-1)$-partite graph $K_{p,\dots,p}$; on such graphs, $\chi_\ell - \chi$ is unbounded.

Thus sufficient conditions for $\chi_\ell(G^2) = \Delta(G)+1$ hold only on special classes but establish a strong property. We present such a result to show how a discharging proof is discovered. The discharging method often begins with configurations that are easy to show reducible. A discharging proof of unavoidability of a set of such configurations starts by forbidding them. When discharging, we may encounter a situation that does not guarantee the desired final charge on some vertices. Instead of trying to adjust the discharging rules, we may try to add this configuration to the unavoidable set, allowing us to assume that it does not occur. This approach succeeds if we can show that the new configuration is reducible.

We use $N_G(v)$ for the neighborhood of a vertex v in a graph G, with $N_G[v] = N_G(v) \cup \{v\}$. When L is a list assignment for G, an L-coloring of a subgraph G' of G is with respect to the restriction of L to the vertices of G'.

Lemma 4.2 (Borodin–Ivanova–Neustroeva [39])**.** *Fix $k \ge 4$. Among graphs G with $\Delta(G) \le k$, the following configurations are reducible for the property $\chi_\ell(G^2) \le k+1$.*
(A) a 1^--vertex,
(B) a 2-thread joining a $(k-1)^-$-vertex and a $(k-2)^-$-vertex,
(C) a cycle of length divisible by 4 composed of 3-threads whose endpoints have degree k.

Proof. Let L be a $(k+1)$-uniform assignment on G; Figure 12 shows (B) and (C).

If (A) occurs at a 1^--vertex v, then let $G' = G - v$. An L-coloring of G'^2 extends to an L-coloring of G^2, because at most k colors need to be avoided at v.

If (B) occurs, then G has a path $\langle x, u, v, y\rangle$ such that $d(u) = d(v) = 2$, $d(x) \le k-1$, and $d(y) \le k-2$. With distance 3 between x and y, we have $(G - \{u, v\})^2 = G^2 - \{u, v\}$. Let $G' = G - \{u, v\}$. By minimality, G'^2 has an L-coloring ϕ. In G, the color on u must avoid the colors on $\{x, y\} \cup N_{G'}(x)$. Since $d(x) \le k-1$ and $|L(u)| = k+1$, a color is available for u. Now the color on v must avoid those on $\{x, y, u\} \cup N_{G'}(y)$. Since $d(y) \le k-2$ and $|L(u)| = k+1$, a color is available for v.

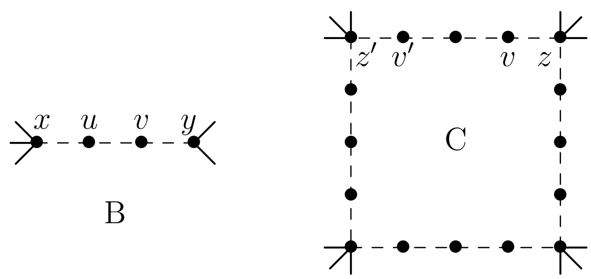

Figure 12: Reducible configurations in G for $\chi_\ell(G^2) \le k+1$ (with $k = 5$)

If (C) occurs, then obtain G' from G by deleting the 2-vertices on the given cycle C. Again G'^2 is the subgraph of G^2 induced by $V(G')$. Let v be a deleted vertex having a k-neighbor z in G. The color on v must avoid those on z and all $k-2$ neighbors of z in G'. Since $|L(v)| = k+1$, at least two colors are available for v. These neighbors of k-vertices on C induce an even cycle in G^2. By Lemma 4.1, we can extend the coloring of G' to these vertices. Finally, the 2-vertices at the centers of the 3-threads have only four neighbors in G^2, all of which are now colored. Since $k \ge 4$, a color remains available at each such vertex. □

The discharging argument to guarantee these reducible configurations is our first encounter with a global notion of discharging. We introduce a *pot* of charge. By allowing vertices to contribute charge to the pot or draw charge from it, we permit charge to move long distances in the graph. The pot starts with charge 0 and must end with nonnegative charge. This prevents the pot from supplying charge to the graph, so making all the vertices happy still contradicts the initial hypothesis on average degree.

We will also see that the configurations originally found to be reducible may not suffice.

Theorem 4.3 ([14, 58])**.** *If* $\Delta(G) \leq 6$ *and* $\mathrm{mad}(G) < \frac{5}{2}$*, then* $\chi_\ell(G^2) \leq 7$*.*

Proof. Let G be a minimal counterexample. Let $k = 6$. By Lemma 4.2(A), we may assume $\delta(G) \geq 2$. By Lemma 4.2(B), G has no 4-thread (or longer), and 3-threads have k-vertices at both ends. By Lemma 4.2(C), the union of the 3-threads is an acyclic subgraph H. Hence the number of 6-vertices is greater than the number of 3-threads.

We now seek discharging rules to prove that if $\mathrm{mad}(G) < \frac{5}{2}$ and $\delta(G) \geq 2$, then some configuration of type (B) or (C) in Lemma 4.2 must occur. This will not quite work; we will need to add more configurations to the set, but they will be reducible.

(R1) Vertices with degree 5 or 6 give $\frac{1}{2}$ to each neighbor.
(R2) A 2-vertex with one neighbor of degree 2 and one of degree 3 or 4 takes $\frac{1}{2}$ from the higher-degree neighbor.
(R3) A 2-vertex whose neighbors both have degree 3 or 4 takes $\frac{1}{4}$ from each neighbor.
(R4) Each 6-vertex contributes $\frac{1}{2}$ to the pot, and each 2-vertex at the center of a 3-thread takes $\frac{1}{2}$ from the pot.

Since there are more 6-vertices than 3-threads, the pot ends with positive charge. By the discharging rules, each 2-vertex explicitly gains charge $\frac{1}{2}$ and ends happy. A 5-vertex can afford to give $\frac{5}{2}$, and a 6-vertex can afford to give $\frac{6}{2}$ to its neighbors plus $\frac{1}{2}$ to the pot.

A 4-vertex is unhappy if it loses more than $\frac{3}{2}$ without having a 5^+-neighbor. A 3-vertex is unhappy if it loses more than $\frac{1}{2}$ without having a 5^+-neighbor. Fortunately, the configurations in which vertices can become unhappy are reducible for $\chi_\ell(G^2) \leq 7$, so their occurrence causes no difficulty. See Figure 13.

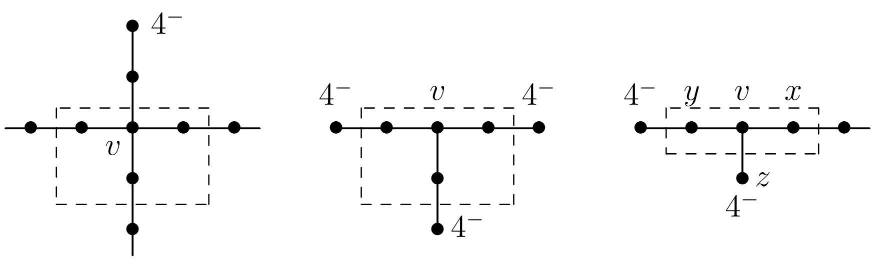

Figure 13: Additional reducible configurations for Theorem 4.3

For a 4-vertex v to lose more than $\frac{3}{2}$ all its neighbors must be 2-vertices and at least three of the incident threads must be 2-threads. We show that this configuration is reducible. Define G' from G by deleting v and its neighbors on three incident 2-threads; note that G'^2

is the subgraph of G^2 induced by $V(G')$. Since $|N_{G^2}(v) \cap V(G')| = 5$, we can extend an L-coloring of G'^2 to v. When we restore the deleted 2-neighbors of v, the numbers of vertices whose colors they must avoid are $4, 5, 6$, respectively, so at each step a color is available.

For a 3-vertex to end unhappy by losing more than $\frac{1}{2}$, it must have no 5^+-neighbor (since it gives at most $\frac{1}{2}$ to each neighbor). It may give at least $\frac{1}{4}$ to each of three neighbors or give $\frac{1}{2}$ to one neighbor and at least $\frac{1}{4}$ to another.

In the first case, let $N_G(v) = \{x_1, x_2, x_3\}$, and let $G' = G - N_G[v]$. The neighbor of x_i other than v has degree at most 4. As we restore $N_G(v)$, the number of vertices whose colors they must avoid are $4, 5, 6$, so at each step a color is available. We can then replace v; it must avoid the colors on six vertices.

In the second case, v gives charge to exactly two 2-neighbors, x and y, where x lies on a 2-thread and takes $\frac{1}{2}$ from v, and the other neighbor of y is a 4^--vertex. Let z be the third neighbor of v; note that $d(z) \le 4$. With $S = \{v, x, y\}$, let $G' = G - S$; again $G'^2 = G^2 - S$. Restore v, then y, then x. As each is restored, its color is chosen from its list to avoid the colors on at most six other vertices. □

Cranston and Škrekovski [58] proved more generally that if $\Delta(G) \ge 6$ and $\text{mad}(G) < 2 + \frac{4\Delta(G)-8}{5\Delta(G)+2}$, then $\chi_\ell(G^2) = \Delta(G) + 1$. Thus when $\text{mad}(G)$ is sufficiently small compared to $\Delta(G)$, the trivial lower bounds on $\chi(G^2)$ and $\chi_\ell(G^2)$ are tight. With a similar but shorter proof, Bonamy, Lévêque, and Pinlou [14] proved the less precise statement that for each positive ϵ, there exists k_ϵ such that $\chi_\ell(G^2) = \Delta(G)+1$ for $\Delta(G) \ge k_\epsilon$ when $\text{mad}(G) < \frac{14}{5} - \epsilon$. In [15] they extended this to $\text{mad}(G) < 3 - \epsilon$.

Even for planar graphs and ordinary coloring, $\text{mad}(G) < 4$ does not yield $\chi(G) \le \Delta(G^2) + c$ for any constant c. Note that girth 4 implies $\text{mad}(G) < 4$ when G is planar. Consider the 3-vertex multigraph in which each pair has multiplicity k; this is sometimes called the *fat triangle*. Subdividing each edge once yields a planar graph with girth 4 and maximum degree $2k$ whose square has chromatic number $3k$ (see Figure 14). Nevertheless, [14] obtained a function c such that if $\text{mad}(G) < 4 - \epsilon$, then $\chi_\ell(G^2) \le \Delta(G) + c(\epsilon)$. Yancey [130] refined this for large $\Delta(G)$, proving for $c \ge 3$ that if $\text{mad}(G) < 4 - \frac{4}{c+1}$ and $\Delta(G)$ is sufficiently large, then $\chi_\ell(G^2) \le \Delta(G) + c$.

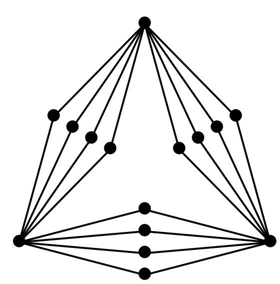

Figure 14: Construction with girth 4 and $\chi(G^2) = 3k$ (here $k = 4$)

When G is planar, larger girth restricts $\mathrm{mad}(G)$ more tightly. Motivated by the subdivided fat triangle, Wang and Lih [123] conjectured that for $g \geq 5$, there exists k_g such that $\Delta(G) \geq k_g$ implies $\chi(G^2) = \Delta(G) + 1$ when G is a planar graph with girth at least g. The conjecture is false for $g \in \{5, 6\}$; [27] and [61] both contain infinite sequences of planar graphs with girth 6, growing maximum degree, and $\chi(G^2) = \Delta(G) + 2$.

However, the Wang–Lih Conjecture holds and can be strengthened to list coloring when $g \geq 7$. Ivanova [81] proved $\chi_\ell(G^2) = \Delta(G) + 1$ for planar G having girth at least 7 and $\Delta(G) \geq 16$ (improving on $\Delta(G) \geq 30$ from [27]), and she also showed that the thresholds $10, 6, 5$ on $\Delta(G)$ are sufficient when G has girth at least $8, 10, 12$, respectively.

For girth 6, Dvořák, Král', Nejedlý, and Škrekovski [61] proved $\chi(G^2) \leq \Delta(G) + 2$ for planar G with $\Delta(G) \geq 8821$ (they also conjectured $\chi(G^2) \leq \Delta(G) + 2$ for girth 5 when $\Delta(G)$ is large enough). For girth 6, Borodin and Ivanova [31] improved $\Delta(G) \geq 8821$ to $\Delta(G) \geq 18$; in [32] and then [33] they showed that $\Delta(G) \geq 36$ and then $\Delta(G) \geq 24$ yields $\chi_\ell(G^2) \leq \Delta(G) + 2$ [32].

Bonamy, Lévêque, and Pinlou [13] proved $\chi_\ell(G^2) \leq \Delta(G) + 2$ when $\Delta(G) \geq 17$ and $\mathrm{mad}(G) < 3$, regardless of planarity. As we have noted, the hypothesis "$\mathrm{mad}(G) < 3$" in place of "planar with girth at least 6" yields a stronger result.

Now consider again the result of Cranston and Škrekovski [58]. Reducing the bound on $\mathrm{mad}(G)$ from 3 to $2 + \frac{4\Delta(G)-8}{5\Delta(G)+2}$ yields $\chi_\ell(G^2) = \Delta(G) + 1$ rather than $\chi_\ell(G^2) \leq \Delta(G) + 2$, even for the larger family where $\Delta(G) \geq 6$. Furthermore, as $\Delta(G)$ grows, the needed bound on $\mathrm{mad}(G)$ tends to $\frac{14}{5}$, which is the bound guaranteed for planar graphs with girth at least 7. Hence it seems plausible that $\chi_\ell(G^2) = \Delta(G) + 1$ for planar graphs with girth at least 7 even when $\Delta(G) \geq 6$. For fuller exploration, we suggest an open question.

Question 4.4. Among the family of graphs such that $\Delta(G) \geq k$, what is the largest value $b_{j,k}$ such that $\mathrm{mad}(G) < b_{j,k}$ implies $\chi_\ell(G^2) \leq \Delta(G) + j$?

Next we weaken the requirements. A coloring where vertices at distance 2 have distinct colors but adjacent vertices need not is an *injective coloring* (the coloring is injective on each vertex neighborhood). For motivation, consider a network of transmitters that broadcast on fixed frequencies; frequencies in a neighborhood must differ so that a receiver can know which neighbor is sending the message. The *injective chromatic number*, written $\chi^i(G)$, is the minimum number of colors needed, and the *injective choice number*, $\chi^i_\ell(G)$, is the least k such that G has an injective L-coloring when L is any k-uniform list assignment.

From the definition, always $\chi^i(G) \leq \chi(G^2)$ and $\chi^i_\ell(G) \leq \chi_\ell(G^2)$. The trivial lower bound on $\chi^i(G)$ is $\Delta(G)$ rather than $\Delta(G) + 1$. We seek results like those above, with a bound on $\chi^i(G)$ or $\chi^i_\ell(G)$ that is one less than the corresponding bound for $\chi(G^2)$ or $\chi_\ell(G^2)$. Again when $\mathrm{mad}(G)$ is small relative to $\Delta(G)$, the value is close to the lower bound. In [13], for example, it is noted that the proof there also yields $\chi^i_\ell(G) \leq \Delta(G) + 1$ when $\Delta(G) \geq 17$ and $\mathrm{mad}(G) < 3$ ([15] and [58] also translate to injective coloring).

Nevertheless, the analogue of Problem 4.4 for injective coloring remains largely open. When $j = 0$, rather tight bounds on $\mathrm{mad}(G)$ suffice. Cranston, Kim, and Yu [56] proved that $\chi^i(G) = \Delta(G)$ when $\mathrm{mad}(G) < \frac{42}{19}$ and $\Delta(G) \geq 3$. Sharpness is not known, even for $\Delta(G) = 3$. Subdividing one edge of K_4 yields a graph H such that $\chi'(H) > \Delta(H)$, and then subdividing every edge of H yields a bipartite graph G such that $\chi^i(G) > \Delta(G)$ and $\mathrm{mad}(G) = \frac{7}{3}$. The largest b such that $\mathrm{mad}(G) < b$ implies $\chi^i(G) = \Delta(G)$ when $\Delta(G) = 3$ is not known; it is at least $\frac{42}{19}$ and at most $\frac{7}{3}$.

To yield $\chi^i_\ell(G) \leq \Delta(G) + 1$, it suffices to have $\mathrm{mad}(G) \leq \frac{5}{2}$ when $\Delta(G) \geq 3$ [56]. For $\Delta(G) \geq 4$ this is fairly easy (it uses Exercise 2.5); for $\Delta(G) \geq 6$ it follows from [58].

To yield $\chi^i_\ell(G) \leq \Delta(G) + 2$, it suffices to have $\mathrm{mad}(G) < \frac{36}{13}$ when $\Delta(G) = 3$ [57]; we will see that this is sharp. For $\Delta(G) \geq 4$, it suffices to have $\mathrm{mad}(G) < \frac{14}{5}$ [57]; the cases $\Delta(G) \in \{4, 5\}$ are difficult, and sharpness is not known. Note that when $\Delta(G) \geq 4$ the allowed values of $\mathrm{mad}(G)$ are larger than when $\Delta(G) = 3$; the loosest condition on $\mathrm{mad}(G)$ that suffices for a given bound on $\chi^i(G) - \Delta(G)$ should grow (somewhat) as $\Delta(G)$ grows.

We use one of these results to further explore how discharging arguments are found. In the discharging process, charge may travel distance 2.

Theorem 4.5. ([57]) *If* $\Delta(G) \leq 3$ *and* $\mathrm{mad}(G) < \frac{36}{13}$, *then* $\chi^i_\ell(G) \leq 5$.

Proof. We present the discharging argument and leave the reducibility of the configurations in the resulting unavoidable set to Exercise 4.7. We claim that every graph G with $\Delta(G) = 3$ and $\overline{d}(G) < \frac{36}{13}$ contains one of the following configurations: a 1^--vertex, adjacent 2-vertices, a 3-vertex with two 2-neighbors, or adjacent 3-vertices each having a 2-neighbor.

If none of these configurations occurs, then $\delta(G) \geq 2$. With initial charge equal to degree, only 2-vertices need charge; all other vertices are 3-vertices. A way to allow 2-vertices to reach charge $\frac{36}{13}$ without taking too much from 3-vertices is as follows:

(R1) Every 2-vertex takes $\frac{3}{13}$ from each neighbor.
(R2) Every 2-vertex takes $\frac{1}{13}$ via each path of length 2 from a 3-vertex.

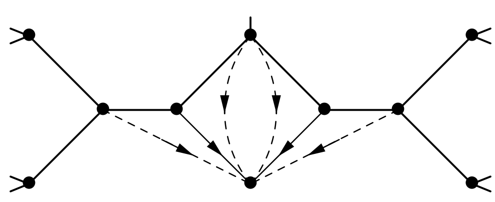

Figure 15: Discharging rules for Theorem 4.5; dashes move $\frac{1}{13}$

Each 3-vertex v having a 2-neighbor gives it $\frac{3}{13}$. Since no two 2-vertices are adjacent, and adjacent 3-vertices cannot both have 2-neighbors, v loses no other charge. Each 3-vertex w having no 2-neighbor loses at most $\frac{1}{13}$ along each incident edge, because its 3-neighbors do not have two 2-neighbors. (Under (R2), a 3-vertex opposite a 2-vertex x on a 4-cycle gives $\frac{2}{13}$ to x.) Thus every 3-vertex ends with charge at least $\frac{36}{13}$.

A 2-vertex gains $\frac{3}{13}$ from each neighbor, and it also gains $\frac{1}{13}$ along each of the two other edges incident to each neighbor (see Figure 15). Hence it gains $\frac{10}{13}$ and reaches charge $\frac{36}{13}$. (With no adjacent 3-vertices having 2-neighbors, no 2-vertex lies on a triangle.)

We have shown that $\overline{d}(G) \geq \frac{36}{13}$ when the specified configurations do not occur. □

Remark 4.6. The proof of Theorem 4.5 allows every vertex to end with charge exactly $\frac{36}{13}$. This can happen, making the structure theorem sharp. In fact, here also the coloring result is sharp. Deleting one vertex from the Heawood graph (the incidence graph of the Fano plane) yields a graph H with $\overline{d}(H) = \frac{36}{13}$, $\Delta(H) = 3$, and $\chi^i(G) = 6$.

The discharging rules in Theorem 4.5 follow naturally from the bound on $\mathrm{mad}(G)$ and the forbidden configurations, but how are those found? To discover the structure theorem, first study the coloring problem to find reducible configurations. A 1^--vertex and two adjacent 2-vertices are easy to show reducible. With a bit more thought, a 3-vertex with two 2-neighbors is reducible. These configurations form an unavoidable set for $\mathrm{mad}(G) < \frac{8}{3}$, using the discharging rule that each 2-vertex takes $\frac{1}{3}$ from each neighbor. That yields the desired conclusion when $\mathrm{mad}(G) < \frac{8}{3}$, but we can do better.

After adding the reducible configuration consisting of two adjacent 3-vertices having 2-neighbors, we seek the loosest bound on $\mathrm{mad}(G)$ under which this larger set is unavoidable. It will exceed $\frac{8}{3}$. The 2-vertices can take charge only from 3-vertices, but when $\mathrm{mad}(G) > \frac{8}{3}$ their neighbors cannot afford to give enough to satisfy them. When two adjacent 3-vertices with 2-neighbors are forbidden, the 2-vertices can also gain charge along paths of length 2.

Now we have the "avenues" of discharging. Let each 2-vertex take a from each neighbor and b along each path of length 2. Now 2-vertices end with $2 + 2a + 4b$, 3-vertices having 2-neighbors end with $3 - a$, and 3-vertices without 2-neighbors end with as little as $3 - 3b$. We seek a and b to maximize the minimum of $\{2 + 2a + 4b, 3 - a, 3 - 3b\}$. If $3 - a$ and $3 - 3b$ are not equal, then the value can be improved, so take $a = 3b$. Now $\min\{2 + 10b, 3 - 3b\}$ is maximized when $2 + 10b = 3 - 3b$, or $b = \frac{1}{13}$. Hence the proof works when $\mathrm{mad}(G) < \frac{36}{13}$ and fails for any larger bound (as also implied by the sharpness example).

We also apply Lemma 3.5 to the problem of coloring the square of a planar graph, where there is another well-known conjecture (the original conjecture was more general).

Conjecture 4.7 (Wegner's Conjecture [125])**.** *If G is planar, then $\chi(G^2) \leq \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor + 1$ for $\Delta(G) \geq 8$; also $\chi(G^2) \leq \Delta(G) + 5$ for $4 \leq \Delta(G) \leq 7$ and $\chi(G^2) \leq 7$ for $\Delta(G) \leq 3$.*

The case $\Delta(G) = 3$ was recently proved by Hartke, Jahanbekam, and Thomas [75] using discharging and computerized checking of reducibility. Wegner gave sharpness constructions; fixing $\Delta(G)$, these are planar graphs of diameter 2 (so $\chi(G^2) = |V(G)|$) with the most vertices. The general situation uses graphs studied by Erdős and Rényi [65], shown on the left in Figure 16. The other graphs there with maximum degree k have diameter 2 with $k+5$ vertices for $4 \le k \le 7$, taken from [78] (the half-edges in the graph for $k = 6$ meet at the eleventh vertex). Wegner found the three leftmost graphs.

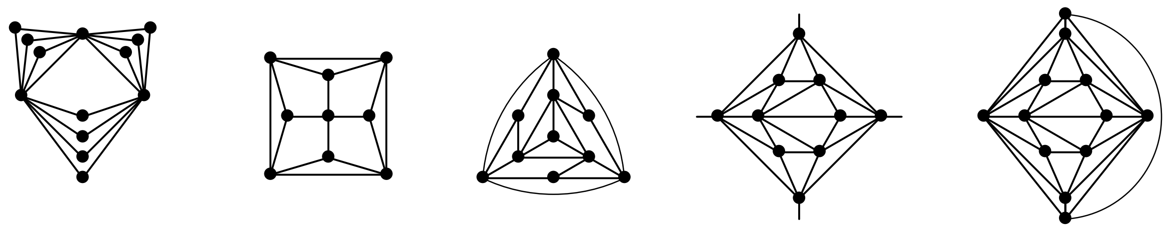

Figure 16: Large graphs with diameter 2 and fixed maximum degree

For upper bounds, van den Heuvel and McGuinness [79] proved $\chi(G^2) \le 2\Delta(G)+25$, but their argument also yields $\chi_\ell(G^2) \le 2\Delta(G) + 25$ (we list tighter bounds later). We present a weaker version of this, showing that $\chi_\ell(G^2) \le 2\Delta(G) + 34$ when G is planar. To keep our additive constant small, we use an enhanced version of Lemma 3.2. Instead of letting each 5^--vertex v take $\frac{6-d(v)}{d(v)}$ from each 7^+-neighbor, change the rule for 5-vertices to take $\frac{1}{4}$ from each 7^+-neighbor. Now a 5-vertex becomes happy when it has at least four 7^+-neighbors. With 4^--neighbors forbidden, a 7-vertex lost at most $\frac{3}{5}$ before, now at most $\frac{3}{4}$, so again it remains happy. Hence we conclude the following.

Lemma 4.8. *Every normal plane map G has a 3-vertex with a 10^--neighbor, or a 4-vertex with a 7^--neighbor, or a 5-vertex with two 6^--neighbors.*

Lemma 4.8 strengthens an early result of Franklin [69]: if G is planar with minimum degree 5, then G has a 5-vertex with a 5-neighbor or two 6-neighbors.

Our first bound on $\chi_\ell(G^2)$ helps when $\Delta(G)$ is small. It slightly refines an idea from [85].

Theorem 4.9. *If G is a planar graph, then* $\chi_\ell(G^2) \le \begin{cases} \Delta(G)^2 + 1 & \text{when } \Delta(G) \le 5, \\ 7\Delta(G) - 7 & \text{when } \Delta(G) \ge 6. \end{cases}$

Proof. For $\Delta(G) \le 5$, the claim is just the trivial upper bound from $\Delta(G^2)$, so we may assume $\Delta(G) \ge 6$.

Index the vertices from v_n to v_1 as follows. Having chosen $v_n, \ldots, v_{i+1}$, let $G_i = G - \{v_n, \ldots, v_{i+1}\}$. If $\delta(G_i) \le 3$, then let v_i be a vertex of minimum degree; otherwise let v be a vertex as guaranteed by Lemma 4.8. Let $S_i = \{v_1, \ldots, v_i\}$. We choose colors for vertices in the order $v_1, \ldots, v_n$ so that the coloring of S_i satisfies all the constraints in the full graph G^2 from pairs of vertices in S_i.

Let $j = |N(v_i) \cap S_i|$; by the choice of the ordering, $j \le 5$. The other neighbors of v_i occur later and are not yet colored. However, v_i must avoid the colors on the neighbors in S_i of these vertices; there may be up to $4(\Delta(G)-j)$ such colors. For $u \in N(v_i)\cap S_i$, all neighbors of u may lie in S_i, so there may be as many as $d(u)$ colors that v_i must avoid due to u.

If $j \le 3$, then the number of colors v_i must avoid is at most $4(\Delta(G)-j)+j\Delta(G)$. If $j=4$, then the bound is $7\Delta(G)-9$, since v_i has a 7^--neighbor in S_i. If $j=5$, then it is $7\Delta(G)-8$, since v_i has two 6^--neighbors in S_i. Hence always the bound is at most $7\Delta(G)-8$. □

Using Lemma 3.5 instead of Lemma 4.8, a vertex with j earlier neighbors must avoid at most $6\Delta(G)+7j-22$ colors. Hence $\chi_\ell(G^2) \le 6\Delta(G)+14$. We strengthen this bound.

Theorem 4.10. *If G is a planar graph, then $\chi_\ell(G^2) \le 2\Delta(G)+34$.*

Proof. Theorem 4.9 provides upper bounds that are at most $2\Delta(G)+34$ when $\Delta(G) \le 8$. Hence we may assume $\Delta(G) \ge 9$.

Let G be a minimal counterexample, with list assignment L from which no such coloring can be chosen. We will form G' by contracting an edge of G incident to a 5^--vertex v, viewed as absorbing v into the other endpoint u; the new vertex retains the list assigned to u. All the constraints forcing vertices of G to have distinct colors are present also in G', so any proper coloring of the square of G' can also be used on $V(G')$ in G. If $\Delta(G')$ is small enough, then the induction hypothesis applies to properly color the square of G' from lists of the desired size, and the task is only to show that few enough other vertices are within distance 2 of v in G, leaving a color available for v to complete an L-coloring of G^2.

If $\delta(G) \le 2$, then let v be a vertex of minimum degree. The contracted vertex has degree at most $\Delta(G)$. At most $2\Delta(G)$ vertices are within distance 2 of v, which leaves a color available for v. Hence we may assume $\delta(G) \ge 3$, so Lemmas 4.8 and 3.5 apply.

Case 1: $9 \le \Delta(G) \le 13$. In this case we prove $\chi_\ell(G^2) \le 52 \le 2\Delta(G)+34$. Let v be a vertex as guaranteed by Lemma 4.8: a 3-vertex with a 10^--neighbor u, a 4-vertex with a 7^--neighbor u, or a 5-vertex with 6^--neighbors u and u'. Contract the edge uv into a new vertex; it has degree at most 11. Thus $\Delta(G') \le 13$, and the induction hypothesis yields a proper L-coloring of the square of G' from lists of size 52 (if $\Delta(G')=8$, then $\chi_\ell(G'^2) \le 2\Delta(G')+34 \le 52$). The bound on the number of other vertices within distance 2 of v is $2\Delta(G)+10$ for $d(v)=3$, $3\Delta(G)+7$ for $d(v)=4$, and $3\Delta(G)+12$ for $d(v)=5$. Since $\Delta(G) \le 13$, the value in each case is at most 51.

Case 2: $\Delta(G) \ge 14$. Let v be a vertex as guaranteed by Lemma 3.5; note that $d(v) \le 5$. Since $d(v) \ge 3$ and v has at most two 12^+-neighbors, v has an 11^--neighbor u; contract the edge uv. The contracted vertex has degree at most 14, so the induction hypothesis applies. Also, the number of vertices within distance 2 of v in G is bounded by $2\Delta(G)+33$. □

The improved upper bound of $2\Delta(G)+25$ in [79] uses a slightly stronger version of Lemma 3.5 to improve the argument for large $\Delta(G)$ (see Exercise 4.9). The main additional work was proving a second lemma specifically for graphs with small maximum degree.

Havet, van den Heuvel, McDiarmid, and Reed [76] proved for planar graphs that $\chi_\ell(G^2) \le (\frac{3}{2} + o(1))\Delta(G)$ as $\Delta(G) \to \infty$, by probabilistic methods. Hence we also seek bounds below $2\Delta(G)$ when $\Delta(G)$ is "small". Borodin et al. [24] proved for planar graphs that $\chi_\ell(G^2) \le 59$ when $\Delta(G) \le 20$ and $\chi_\ell(G^2) \le \max\{\Delta(G) + 39, \lceil \frac{9}{5}\Delta(G) \rceil + 1\}$ when $\Delta(G) > 20$. In particular, if $\Delta(G) \ge 47$, then $\chi_\ell(G^2) \le \lceil \frac{9}{5}\Delta(G) \rceil + 1$ ([1] proved this bound for $\Delta(G) \ge 750$). Also, [24] proved that G^2 is k-degenerate, where $k = \max\{\Delta(G) + 38, \lceil \frac{9}{5}\Delta(G) \rceil\}$. For the coloring problem alone, Molloy and Salavatipour [98] proved $\chi(G^2) \le \frac{5}{3}\Delta(G) + 78$ for all planar G. We explore results in terms of $\mathrm{mad}(G)$ (without planarity) in the exercises.

Exercise 4.1. (Cranston–Kim [55]) Apply Exercise 2.8 to prove that if $\Delta(G) \le 3$ and $\mathrm{mad}(G) \le \frac{14}{5}$, then $\chi_\ell(G^2) \le 7$.

Exercise 4.2. (Kim–Park [89]) Prove that if $\delta(G) \ge 2$ and $\overline{d}(G) < \frac{4k}{k+2}$ with $k \ge 4$, then G has a 3^--vertex with a $(k-1)^-$-neighbor. Guarantee a 2-vertex with a $(k-1)^-$-neighbor when $k \le 6$. Conclude that if $\mathrm{mad}(G) < \frac{4k}{k+2}$ with $k \ge 4$ (and no components are 5-cycles if $k = 4$), then from any lists of size at least k a proper coloring of G can be chosen so that every vertex with degree at least 2 has neighbors with distinct colors. Show also that this is sharp: there exists G with $\mathrm{mad}(G) = \frac{4k}{k+2}$ and an assignment of k-lists from which no such coloring can be chosen.

Exercise 4.3. In Problem 4.4, prove that $b_{1,k} \ge 2$. Show that equality holds when $k \in \{2, 3\}$.

Exercise 4.4. (Cranston–Erman–Škrekovski [53]) Prove that a cycle of length divisible by 3 with vertices whose degrees cycle repeatedly through $2, 2, 3$ is reducible for 5-choosability of G^2. Use discharging to conclude that if $\Delta(G) \le 4$ and $\mathrm{mad}(G) < 16/7$. then $\chi_\ell(G^2) \le 5$.

Exercise 4.5. (Cranston–Erman–Škrekovski [53]) Prove that if $\Delta(G) \le 4$ and $\overline{d}(G) < \frac{18}{7}$, then G contains one of: (C1) a 1^--vertex, (C2) two adjacent 2-vertices, (C3) a 3-vertex with three 2-neighbors, or (C4) a four-vertex path alternating between 2-vertices and 3-vertices. Conclude that if $\Delta(G) \le 4$ and $\mathrm{mad}(G) < \frac{18}{7}$, then $\chi_\ell(G^2) \le 7$.

Exercise 4.6. (Cranston–Erman–Škrekovski [53]) Prove that if $\Delta(G) \le 4$ and $\overline{d}(G) \le \frac{10}{3}$, then G contains one of: (C1) a 1^--vertex, (C2) a 2-vertex with a 3^--neighbor, (C3) a 3-vertex with two 3-neighbors, or (C4) a 4-vertex with a 2-neighbor and a 3^--neighbor. Construct infinitely many graphs with average degree $\frac{10}{3}$ and maximum degree 4 that contain no such configuration. Prove that if $\Delta(G) \le 4$ and $\mathrm{mad}(G) < \frac{10}{3}$, then $\chi_l(G^2) \le 12$.

Exercise 4.7. (Cranston–Kim–Yu [57]) Complete the proof of Theorem 4.5 by showing that those configurations are reducible for $\chi^i(G) \le 5$ in the family of graphs with $\Delta(G) \le 3$.

Exercise 4.8. (Cranston–Kim–Yu [57]) Prove that if $\overline{d}(G) < \frac{14}{5}$ and $\Delta(G) \ge 6$, then G contains one of the following configurations: (C1) a 1^--vertex, (C2) adjacent 2-vertices, (C3) a 3-vertex with neighbors of degrees $2, a, b$, where $a + b \le \Delta(G) + 2$, or (C4) a 4-vertex having four 2-neighbors, one of which has other neighbor of degree less than $\Delta(G)$. Argue that none of these configurations can appear in a minimal graph G such that $\Delta(G) \ge 6$ and $\chi^i(G) > \Delta(G) + 2$. Reducibility of the first two configurations and part of (C3) is already requested in Exercise 4.7.

Exercise 4.9. (van den Heuvel–McGuinness [79]) Prove that every planar graph G with $\delta(G) \geq 3$ has a 5^--vertex v with at most two 12^+-neighbors such that v has a 7^--neighbor if $d(v) \in \{4, 5\}$, and v has an additional 6^--neighbor if $d(v) = 5$. Use this to prove that $\chi(G^2) \leq 2\Delta(G) + 25$ when $\Delta(G) \geq 12$. (Hint: Extend the proof of Lemma 3.5 by allowing 5^--vertices to take some charge from their 11^--neighbors.)

5 Edge-coloring and List Edge-coloring

We have mentioned the famous result of Vizing [116, 118] and Gupta [74] known as *Vizing's Theorem*. It gives an upper bound for $\chi'(G)$ when G is a multigraph (allowing multiedges) and specializes to $\chi'(G) \leq \Delta(G) + 1$ when G is a graph. Deciding whether $\chi'(G)$ equals $\Delta(G)$ or $\Delta(G) + 1$ is NP-complete [80], so we seek sufficient conditions for equality.

Conjecture 5.1 (Vizing's Planar Graph Conjecture [117, 119])**.** *If G is a planar graph and $\Delta(G) \geq 6$, then $\chi'(G) = \Delta(G)$.*

Both conditions in Vizing's Conjecture are needed. The complete graph K_7 is 6-regular but not planar. Each color can be used on at most three edges, so $\chi'(K_7) \geq \frac{21}{3} = 7$. Similarly, obtain G from a 5-regular planar graph with $2k$ vertices by subdividing one edge. Since G has $5k + 1$ edges, and at most k edges can receive the same color, $\chi'(G) \geq 6$. This difficulty does not arise for $\Delta(G) \geq 6$, because regular planar graphs have degree at most 5.

Vizing [117] proved Conjecture 5.1 for $\Delta(G) \geq 8$, using Vizing's Adjacency Lemma (VAL). It is common to say that G is *Class 1* if $\chi'(G) = \Delta(G)$, *Class 2* otherwise. An *edge-critical graph* G is then a Class 2 graph such that $\chi'(G - e) = \Delta(G)$ for all $e \in E(G)$. In fact, VAL implies that every edge-critical graph has at least three vertices of maximum degree, so $\Delta(G) = \Delta(G - e)$. Note also that every Class 2 graph contains an edge-critical graph with the same maximum degree.

Theorem 5.2 (Vizing's Adjacency Lemma [117])**.** *If x and y are adjacent in an edge-critical graph G, then at least* $\max\{1 + \Delta(G) - d(y), 2\}$ *neighbors of x have degree* $\Delta(G)$*.*

Using VAL, Vizing proved the conjecture for $\Delta(G) \geq 8$ via counting arguments about vertices of various degrees. The proof is clearer in the language of discharging, which was not then in use. Luo and Zhang [96] used VAL and discharging to prove $\chi'(G) = \Delta(G)$ for the larger family of graphs G with $\text{mad}(G) \leq 6$ and $\Delta(G) \geq 8$. We present a slightly simpler proof of a slightly weaker result, requiring $\text{mad}(G) < 6$. In fact, Miao and Sun [97] proved $\chi'(G) = \Delta(G)$ also when $\Delta(G) \geq 8$ and $\text{mad}(G) < \frac{13}{2}$. Their result (and that of [96]) uses additional adjacency lemmas. Here VAL is used instead of reducibility arguments.

Theorem 5.3 ([96])**.** *If G is a graph with* $\text{mad}(G) < 6$ *and* $\Delta(G) \geq 8$*, then* $\chi'(G) = \Delta(G)$*.*

Proof. Let G be a minimal counterexample, and let $k = \Delta(G)$. Since $\chi'(G) > k$ requires an edge-critical subgraph with the same maximum degree, we may assume that G is edge-critical. By VAL, each vertex has at least two k-neighbors, so $\delta(G) \geq 2$. We use discharging with initial charge $d(v)$; it suffices to show that each vertex ends with charge at least 6.

(R1) If $d(v) \leq 4$, then v takes $\frac{6-d(v)}{d(v)}$ from each neighbor.
(R2) If $d(v) \in \{5, 6\}$, then v takes $\frac{1}{4}$ from each 6^+-neighbor.

For $v \in V(G)$, let j be the least degree among vertices in $N_G(v)$. If $j < k$, then v has at least $k + 1 - j$ neighbors of degree k, by VAL. Hence $k + 1 - j \leq d(v) - 1$, which yields $j \geq 10 - d(v)$ since $k \geq 8$. Note that 7^+-vertices take no charge.

If $d(v) \leq 4$, then $j \geq 6$, so v loses no charge, and (R1) sends enough to make v happy.

If $d(v) = 5$, then $j \geq 5$. Furthermore, $j = 5$ yields $k - 4$ neighbors with degree k. Since $k \geq 8$, charge at least $4(\frac{1}{4})$ comes to v, no charge is given away, and v is happy.

The remaining cases are all similar. We show representative cases in Figure 17. Note that v has at most $j + d(v) - 9$ neighbors with degree less than k.

If $d(v) = 6$, then $j \geq 4$. At most $j - 3$ neighbors have degree less than k. For $j \in \{4, 5, 6\}$, v gives at most $\frac{2}{4}, \frac{2}{4}, \frac{3}{4}$ and receives at least $\frac{5}{4}, \frac{4}{4}, \frac{6}{4}$, respectively, ending happy.

If $d(v) = 7$, then $j \geq 3$. At most $j - 2$ neighbors have degree less than k. For $j \in \{3, 4, 5, 6\}$, v gives at most $\frac{3}{3}, \frac{4}{4}, \frac{3}{4}, \frac{4}{4}$, respectively, and remains happy.

If $d(v) \geq 8$, then $j \geq 2$. At most $j - 1$ neighbors have degree less than k. For $j \in \{2, 3, 4, 5, 6\}$, v gives at most $\frac{4}{2}, \frac{6}{3}, \frac{6}{4}, \frac{4}{4}, \frac{5}{4}$, respectively, and remains happy. □

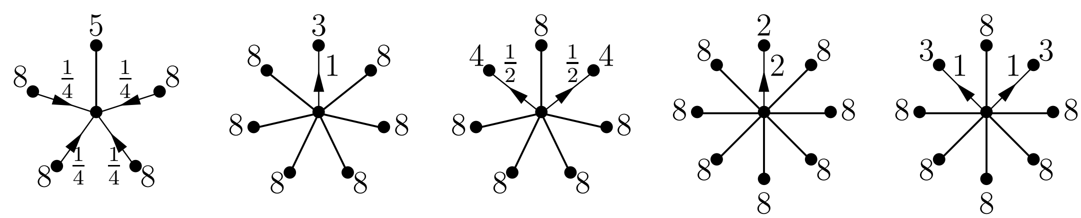

Figure 17: Some cases in Theorem 5.3 ending with charge 6

Sanders and Zhao [106] and Zhang [131] proved Conjecture 5.1 for planar graphs with $\Delta(G) = 7$, extended in [108] to graphs with maximum degree at least 7 that embed in a surface of nonnegative Euler characteristic. Since the proof above uses only $\text{mad}(G) < 6$, it holds also for graphs in the projective plane. Graphs on the torus (or Klein bottle) also satisfy $\text{mad}(G) < 6$ unless they triangulate the surface, in which case $\overline{d}(G) = 6$.

Although Conjecture 5.1 remains open when $\Delta(G) = 6$, it has been proved for various classes of planar graphs with certain subgraphs forbidden, such as short cycles with chords (see [46, 122, 124]). Note that $\text{mad}(G) < 6$ is not sufficient when $\Delta(G) = 6$; planarity really is needed. Although K_7 is forbidden by $\text{mad}(G) < 6$, consider the graph G obtained from

K_7 by subdividing one edge with a new 2-vertex v; we have $\Delta(G) = 6$ and $\mathrm{mad}(G) < 6$. In a proper edge-coloring of G, only two colors can appear four times (using edges at v); hence six colors can cover only 20 edges, but G has 22 edges.

Proper edge-coloring of G is equivalent to proper coloring of the line graph $L(G)$. Since the line graph has a clique of size $\Delta(G)$, Vizing's Theorem states that the optimization problem of proper coloring behaves much better when restricted to line graphs. The same phenomenon seems to occur with the list version of the problem.

Definition 5.4. An *edge-list assignment* L assigns lists of available colors to the edges of a graph G. Given an edge-list assignment L, an *L-edge-coloring* of G is a proper edge-coloring ϕ such that $\phi(e) \in L(e)$ for all $e \in E(G)$. A graph G is *k-edge-choosable* if G is L-edge-colorable whenever each list has size at least k. The *list edge-chromatic number* of G, written $\chi'_\ell(G)$, is the least k such that G is k-edge-choosable.

Conjecture 5.5 (List Coloring Conjecture)**.** *$\chi'_\ell(G) = \chi'(G)$ for every graph G.*

This conjecture was posed independently by many researchers. It was first published by Bollobás and Harris [11], but it was independently formulated earlier by Albertson and Collins in 1981 and by Vizing as early as 1975 (both unpublished). Kahn [87] proved the conjecture asymptotically: $\chi'_\ell(G) \le (1 + o(1))\chi'(G)$.

We first consider Vizing's weaker conjecture that always $\chi'_\ell(G) \le \Delta(G)+1$. Borodin [19] proved it for planar G with $\Delta(G) \ge 9$. Bonamy [12] extended this to $\Delta(G) = 8$, by a much longer proof with 11 reducible configurations. It was also proved for planar graphs with $\Delta(G) \ge 6$ having no two 3-faces sharing an edge [52]. It was proved in [86] for $\Delta(G) \le 4$ (including nonplanar graphs), and for $\Delta(G) = 5$ it is known for planar graphs with no 3-cycle [132], no 4-cycle [52], or no 5-cycle [123]. The proofs for $\Delta(G) = 5$ use discharging.

We present a recent use of balanced charging [50] to prove the result of Borodin [19]. Balanced charging is natural when neither the graphs nor their duals are triangulations. Again we use a pot of charge (see Theorem 4.3). In this proof, the pot facilitates moving charge from maximum-degree vertices to 3-vertices; we need not name specific recipients.

Theorem 5.6 ([19])**.** *If G is a planar graph and $\Delta(G) \ge 9$, then $\chi'_\ell(G) \le \Delta(G) + 1$.*

Proof. (Cohen and Havet [50]) Let G be a minimal counterexample, with an edge-list assignment L such that each list has size $\Delta(G) + 1$ and G has no L-edge-coloring. An edge with weight at most $\Delta(G) + 2$ is reducible. Hence we may assume that $\delta(G) \ge 3$ and that every neighbor of a j-vertex has degree at least $\Delta(G) + 3 - j$. Let $k = \Delta(G)$; since $k \ge 9$, the degree-sum of any two adjacent vertices is at least 12.

We use balanced charging, with initial charge equal to degree or length minus 4. Initially, the pot of charge is empty. The discharging rules must make each vertex and face happy and keep the charge in the pot nonnegative to contradict the assumption of a counterexample.

(R1) Every 3-vertex takes 1 from the pot, and every k-vertex gives $\frac{1}{2}$ to the pot.
(R2) Each 3-face takes $\frac{1}{2}$ from each incident 8^+-vertex and $\frac{j-4}{j}$ from each incident j-vertex with $j \in \{5, 6, 7\}$.

To ensure positive charge in the pot, we prove $n_k > 2n_3$, where n_j is the number of j-vertices in G. The edges incident to 3-vertices form a bipartite graph H; its parts are the 3-vertices and the k-vertices. If H has a cycle C, then C has even length, since H is bipartite. By the minimality of the counterexample, $G - E(C)$ has an L-edge-coloring. Each edge of C is incident to $\Delta(G) - 1$ edges that have now been colored, so there remain at least two available colors on each edge (see Figure 18). Since even cycles are 2-edge-choosable (by Lemma 4.1 and cycles being isomorphic to their line graphs), the L-edge-coloring extends to G. Since G is a counterexample, we thus may assume that H is acyclic and therefore has fewer than $n_3 + n_k$ edges. Since it also has $3n_3$ edges, we have $3n_3 < n_3 + n_k$, as desired.

For vertices, (R1) immediately makes 3-vertices happy. A j-vertex v with $j \in \{4, 5, 6, 7\}$ loses altogether at most $j - 4$, its initial charge. An 8-vertex loses at most 4, since $k \geq 9$. For $j \geq 9$, possibly sending $\frac{1}{2}$ to the pot, a j-vertex loses at most $\frac{j+1}{2}$ and is happy.

For faces, the 4^+-faces lose no charge and remain happy; we must show that each 3-face f gains at least 1. Let j be the least degree among vertices incident to f. If $j \leq 4$, then two incident 8^+-vertices give $\frac{1}{2}$ each. If $j = 5$, then two incident 7^+-vertices give at least $\frac{3}{7}$ each, plus $\frac{1}{5}$ for the 5-vertex. If $j \geq 6$, then each vertex incident to f gives at least $\frac{1}{3}$ to f. □

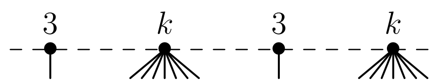

Figure 18: Excluded cycles in Theorem 5.6

This proof fits the model of discharging to produce an unavoidable set of reducible configurations. The reducible configurations are light edges (degree-sum at most $\Delta(G) + 2$) and cycles alternating between 3-vertices and $\Delta(G)$-vertices. The first use of arbitrarily large reducible configurations (cycles alternating between 2-vertices and $\Delta(G)$-vertices) was in Borodin [17]. Notions analogous to the pot of charge for long-distance transfer of charge appear in [77] and [30]; a general term for such methods is "global discharging".

Now we return to the full List Coloring Conjecture $\chi'_\ell(G) = \chi'(G)$. This was proved for bipartite multigraphs by Galvin [70], where always $\chi'(G) = \Delta(G)$. With Vizing conjecturing $\chi'(G) = \Delta(G)$ when G is planar and $\Delta(G) \geq 6$ (Conjecture 5.1), we also seek $\chi'_\ell(G) = \Delta(G)$ for such graphs. Borodin [19] proved it for $\Delta(G) \geq 14$. This later was strengthened to $\Delta(G) \geq 12$ by Borodin, Kostochka, and Woodall [41]. We present an alternative proof of the result of Borodin [19]. The result in [41] uses similar discharging, but it requires more reducible configurations and more detailed analysis.

A *t-alternating* cycle alternates between t-vertices and vertices of higher degree (introduced in Borodin [17]). We used 3-alternating cycles in Theorem 5.6.

Lemma 5.7 ([19]). *If G is a simple plane graph with $\delta(G) \geq 2$, then G contains*
(C1) an edge uv with $d(u) + d(v) \leq 15$, or
(C2) a 2-alternating cycle C.

Proof. In a counterexample G, we have $d(u) + d(v) \geq 16$ for every edge uv. Both neighbors of any 2-vertex are 14^+-vertices. Since G is simple, every 2-vertex lies on a 4^+-face.

To obtain a contradiction, we use face charging, with initial charge $2d(v) - 6$ at each vertex v and $\ell(f) - 6$ at each face f. We also keep a central pot of charge (initially empty) and use the following discharging rules (see Figure 19).

(R1) Each 14^+-vertex gives charge 1 to the pot, and each 2-vertex takes 1 from the pot.
(R2) Each 4^+-vertex distributes its charge remaining after (R1) equally to its incident faces.
(R3) Each 4^+-face gives charge 1 to each incident 2-vertex.

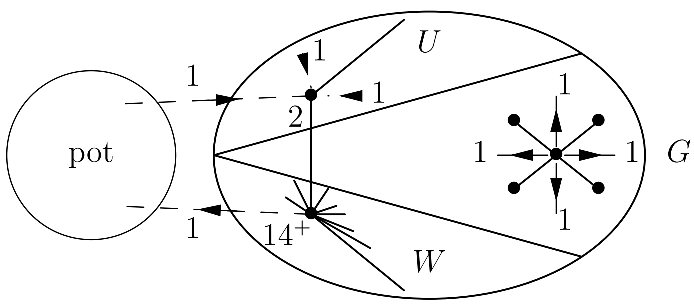

Figure 19: Discharging for Lemma 5.7

To keep the charge in the pot nonnegative, we need $|U| \leq |W|$, where U and W denote the sets of 2-vertices and 14^+-vertices, respectively. Let H be the bipartite subgraph of G with vertex set $U \cup W$ and edge set consisting of all edges with endpoints in both U and W. Since (C2) does not occur in G, the components of H are trees. Also (C1) does not occur, so $2|U| = |E(H)| < |U| + |W|$. Thus $|U| < |W|$.

A 2-vertex takes 1 from the pot and 1 from an incident 4^+-face (since G is simple) and ends happy. A 3-vertex starts and ends with no charge. By (R2), a 4^+-vertex also ends with charge 0. Hence all vertices are happy.

Faces give charge to 2-vertices and take charge from 4^+-vertices. Under (R2), a face takes charge $\frac{2j-6}{j}$ or $\frac{2j-7}{j}$ from an incident j-vertex when $j \geq 4$, the latter when $j \geq 14$. Thus the value is at least $\frac{1}{2}$ when $j \geq 4$, at least 1 when $j \geq 6$, and at least $\frac{3}{2}$ when $j \geq 12$.

If a face f has no incident 3^--vertices, then it receives at least $\frac{1}{2}\ell(f)$; its final charge is at least $\frac{3}{2}\ell(f) - 6$, which is nonnegative when $\ell(f) \geq 4$. When f is a 3-face or a face incident to some 3^--vertex, let k be the least degree among the vertices incident to f. Prohibiting (C1) gives f two incident $(16-k)^+$-vertices.

A 3-face needs to receive charge at least 3. When $k \geq 6$, it receives at least 1 from each incident vertex. When $k = 2$, the other incident vertices have degree at least 14, and each provides $\frac{3}{2}$. When $3 \leq k \leq 5$, it receives at least $\frac{2k-6}{k} + 2 \cdot \frac{26-2k}{16-k}$, which is at least 3.

A 4^+-face f needs 2 (or maybe less) to become happy. If $k \geq 3$ or f has exactly one incident 2-vertex, then f receives at least 3 and gives away at most 1. If f has at least two incident 2-vertices, then each is followed on f (in a consistent direction) by a 14^+-vertex, which contributes at least $\frac{3}{2}$. These pairs net at least $\frac{1}{2}$ each for f. If G has no 2-alternating cycle, then f has another incident 14^+-vertex that has not been counted, which provides more than enough charge to f. □

Theorem 5.8 ([19]). *If G is a plane graph with $\Delta(G) \geq 14$, then $\chi'_\ell(G) = \Delta(G)$.*

Proof. Let G be a minimal counterexample, having no L-edge-coloring from edge-list assignment L. If G has a 1-vertex with incident edge e, then $G - e$ has an L-edge-coloring, and it extends to e. Thus $\delta(G) \geq 2$. By Lemma 5.7, G has an edge uv with $d(u) + d(v) \leq 15$ or a 2-alternating cycle C. In the first case, we can extend an L-edge-coloring of $G - uv$, since $|L(uv)| \geq 14$ and at most 13 colors are restricted from use on uv. In the other case, by minimality $G - E(C)$ has an L-edge-coloring. Since each list has size at least $\Delta(G)$, each edge of C has at least two colors remaining available, and the 2-edge-choosability of even cycles allows us to extend the edge-coloring. □

Finally, we come full circle and return to the role of bounding the maximum average degree. Vizing [119] conjectured that an n-vertex edge-critical graph G must have at least $\frac{1}{2}[n(\Delta(G)-1)+3]$ edges, which in our language translates to "$\chi'(G) = \Delta(G)$ when $\mathrm{mad}(G) \leq \Delta(G)-1$". Based on the List Coloring Conjecture, Woodall [127] conjectured that $\mathrm{mad}(G) < \Delta(G)-1$ also implies $\chi'_\ell(G) = \Delta(G)$. In this direction, it is known that $\chi'_\ell(G) = \Delta(G)$ when $\mathrm{mad}(G) < \sqrt{2\Delta(G)}$. The result is implicit in [41], using the following tool.

Theorem 5.9 (Borodin–Kostochka–Woodall [41]). *If lists on the edges of a bipartite multigraph G satisfy $|L(uv)| \geq \max\{d_G(u), d_G(v)\}$ for $uv \in E(G)$, then G has an L-edge-coloring.*

Woodall [127] rephrased the argument using discharging, introducing an exciting new way of moving charge in successive stages. When the average degree is large, vertices with very small degree need a lot of charge. It may be too hard to specify exactly where it all comes from. Hence he allows charge to move in phases, which we call *iterated discharging*. Besides light edges, we will need another reducible configuration (see Figure 20).

Definition 5.10. In a multigraph G, an *i-alternating subgraph* is a bipartite submultigraph F with parts U and W such that $d_F(u) = d_G(u) \leq i$ when $u \in U$ and $d_G(w) - d_F(w) \leq \Delta(G) - i$ when $w \in W$. Note that cycles in F alternate between W and i^--vertices in U.

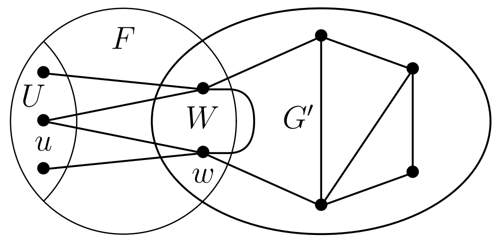

Figure 20: A 2-alternating subgraph F

Lemma 5.11 ([41, 127])**.** *i-alternating subgraphs are reducible for the property that edge-choosability equals maximum degree.*

Proof. Let L be a $\Delta(G)$-uniform edge-list assignment for such a multigraph G. Let F be an i-alternating subgraph of G, and let $G' = G - E(F)$ (see Figure 20). Choose an L-edge-coloring of G', and delete the chosen colors from the lists of their incident edges in F. We claim that the lists remain large enough to apply Theorem 5.9 to F.

For $uw \in E(F)$, no colors have been lost to edges incident at u, since all edges incident to u lie in F. The number of colors lost to edges incident to w, by definition, is at most $d_G(w) - d_F(w)$. Since $d_G(w) \le \Delta(G)$, the list on uw retains at least $d_F(w)$ colors. Also $d_F(u) \le i \le \Delta(G) - (d_G(w) - d_F(w))$, so the list on uw also retains at least $d_F(u)$ colors. Now Theorem 5.9 applies to complete the L-edge-coloring of G. □

To avoid technicalities, we make the bound on $\text{mad}(G)$ slightly tighter than needed.

Theorem 5.12 ([41, 127])**.** *If* $\text{mad}(G) \le \sqrt{2\Delta(G)} - 1$*, then* $\chi'_\ell(G) = \Delta(G)$*.*

Proof. Let $b = \sqrt{2\Delta(G)} - 1$. It suffices to show that every graph G with average degree at most b contains a edge with weight at most $\Delta(G) + 1$ or an i-alternating subgraph with $i \le b$. Suppose that G contains neither. An edge incident to a 1-vertex would be light, so we may assume $\delta(G) \ge 2$.

Use degree charging. In phase i of discharging, for $2 \le i \le \lfloor b \rfloor$, each i^--vertex receives charge 1 from a neighbor. We want every vertex to end with charge at least $\lceil b \rceil$.

To begin phase i, let $U = \{v \colon d_G(v) \le i\}$, and let W be the set of all vertices having neighbors in U. Note that U is independent (no light edge). Let F be the subgraph with vertex set $U \cup W$ containing all edges incident to U. Since G has no i-alternating subgraph, there exists $w \in W$ such that $d_F(w) \le d_G(w) + i - \Delta(G) - 1 \le i - 1$. Move charge 1 from this vertex w to each of its neighbors in U.

Now delete $\{w\} \cup (N(w) \cap U)$ from F. Each deleted vertex in U has received charge 1, and w lost at most $i - 1$. Iterate. What remains of U and W at each step cannot form an i-alternating subgraph, so we continue to find the desired vertex until U is empty.

Since each vertex with degree at most i receives a unit of charge in phase i, vertices with degree less than $\left\lfloor\sqrt{2\Delta(G)}\right\rfloor$ have their charge increased to at least $\left\lfloor\sqrt{2\Delta(G)}\right\rfloor$ (and they never lose charge). Since there is no light edge, vertices with larger degree j lose charge only on rounds i with $i \geq \Delta(G)+2-j$. Hence such a vertex loses charge at most $\sum_{i=\Delta(G)+2-j}^{\lfloor b\rfloor}(i-1)$. With each reduction of 1 in j, the amount of lost charge declines by more than 1, so it suffices to show that vertices with degree $\Delta(G)$ keep sufficient charge. Their lost charge is bounded by $\frac{1}{2}b(b-1)$, so they keep charge at least $\frac{3}{2}b$, which is more than enough. □

This use of discharging in [127] replaced extensive manipulations of finite sums in [41]; it is another illustration of the notion of "amortized counting" we mentioned earlier. Woodall also gave an example to show that the discharging argument is essentially sharp, meaning that more reducible configurations will be needed to weaken the hypothesis on $\mathrm{mad}(G)$.

For the Vizing conjecture saying approximately that $\mathrm{mad}(G) \leq \Delta(G)-1$ implies $\chi'(G) = \Delta(G)$, Fiorini [68] made the first major step, proving that $\mathrm{mad}(G) < \frac{1}{2}(\Delta(G)+1)$ suffices (by counting edges in edge-critical graphs). After a number of papers using similar manipulations of finite sums, Sanders and Zhao [107] greatly simplified the proof by using discharging and improved the result; they showed that $\mathrm{mad}(G) < \frac{1}{2}(\Delta(G) + \sqrt{2\Delta(G)-1})$ suffices. Woodall [126] then proved that $\mathrm{mad}(G) < \frac{2}{3}(\Delta(G)+)$ suffices. The list version seems to be much harder, and the more restrictive requirement of $\mathrm{mad}(G) < \sqrt{2\Delta(G)}$ in Theorem 5.12 is a first step.

Exercise 5.1. Let G be a graph with maximum degree at least 8 that embeds on the torus. By a closer examination of the proof of Theorem 5.3, prove that $\chi'(G) = \Delta(G)$ except possibly when G is obtained from a 6-regular triangulation H of the torus by inserting vertices of degree 3 into one-third of the faces in H, chosen so that each vertex in H lies on exactly two of the chosen faces, and making each new vertex adjacent to the vertices of H on its face. It suffices to show that otherwise every vertex ends with charge at least 6 and some vertex ends with larger charge.

Exercise 5.2. Prove that if $\Delta(G) \leq 6$ and $\overline{d}(G) < \frac{7}{2}$, then G contains an isolated vertex, an edge with weight at most 7, or a cycle alternating between 2-vertices and 6-vertices. Conclude that if $\Delta(G) \leq 6$ and $\mathrm{mad}(G) < \frac{7}{2}$, then G is 6-edge-choosable.

Exercise 5.3. (Borodin [17], Borodin–Kostochka–Woodall [41]) A *total coloring* assigns colors to both edges and vertices, so that elements get distinct colors if they are either incident or adjacent. Adapt the proofs of Theorems 5.8 and 5.12 to prove analogous versions for choosing total colorings from lists. In each case, the bound for the size of lists to permit choosing a total coloring is larger by 1 than that for choosing a proper edge-coloring.

References

[1] G. Agnarsson and M.M. Halldórsson, Coloring powers of planar graphs, *SIAM J. Discrete Math.* **16** (2003), no. 4, 651–662.

[2] J. Akiyama, G. Exoo, and F. Harary, Covering and packing in graphs III: Cyclic and acyclic invariants, *Math Slovaca* **30** (1980), 405–417.

[3] J. Akiyama, G. Exoo, and F. Harary, Covering and packing in graphs. IV. Linear arboricity, *Networks* **11** (1981), no. 1, 69–72.

[4] M.O. Albertson, A lower bound for the independence number of a planar graph, *J. Combin. Theory Ser. B* **20** (1976), no. 1, 84–93.

[5] M.O. Albertson, G.G. Chappell, H.A. Kierstead, A. Kündgen, and R. Ramamurthi, Coloring with no 2-colored P_4's, *Electron. J. Combin.* **11** (2004), no. 1, Research Paper 26, 13 pp.

[6] N. Alon, The linear arboricity of graphs, *Israel J. Math.* **62** (1988), 311–325.

[7] K. Appel and W. Haken, Every planar map is four colorable. I. Discharging, *Illinois J. Math.* **21** (1977), no. 3, 429–490.

[8] K. Appel and W. Haken, Every planar map is four colorable, with the collaboration of J. Koch, *Contemporary Mathematics* **98**, American Mathematical Society, Providence, RI, 1989. xvi+741 pp.

[9] K. Appel, W. Haken, and J. Koch, Every planar map is four colorable. II. Reducibility, *Illinois J. Math.* **21** (1977), no. 3, 491–567.

[10] J. Balogh, M. Kochol, A. Pluhár, and X. Yu, Covering planar graphs with forests, *J. Combin. Theory Ser. B* **94** (2005), no. 1, 147–158.

[11] B. Bollobás and A.J. Harris, List-colourings of graphs, *Graphs Combin.* **1** (1985), 115–127.

[12] M. Bonamy, Planar graphs with $\Delta \geq 8$ are $(\Delta+1)$-edge-choosable, *SIAM J. Discrete Math.* **29** (2015), no. 3, 1735–1763.

[13] M. Bonamy, B. Lévêque, and A. Pinlou, Graphs with maximum degree D at least 17 and maximum average degree less than 3 are list 2-distance $(D+2)$-colorable, *Discrete Math.* **317** (2014), 19–32.

[14] M. Bonamy, B. Lévêque, and A. Pinlou, 2-distance coloring of sparse graphs, *J. Graph Theory* **77** (2014), no. 3, 190–218.

[15] M. Bonamy, B. Lévêque, and A. Pinlou, List coloring the square of sparse graphs with large degree, *European J. Combin.* **41** (2014), 128–137.

[16] O.V. Borodin, On acyclic colorings of planar graphs, *Discrete Math.* **25** (1979), no. 3, 211–236.

[17] O.V. Borodin, On the total coloring of planar graphs, *J. Reine Angew. Math.* **394** (1989), 180–185.

[18] O.V. Borodin, Solving the Kotzig and Grünbaum problems on the separability of a cycle in planar graphs, *Mat. Zametki* **46** (1989), no. 5, 9–12, 103 [Russian]; translation in *Math. Notes* **46** (1989), no. 5-6, 835–837 (1990).

[19] O.V. Borodin, A generalization of Kotzig's theorem and prescribed edge coloring of planar graphs, *Mat. Zametki* **48** (1990), no. 6, 22–28, 160 [Russian]; translation in *Math. Notes* **48** (1990), no. 5-6, 1186–1190 (1991).

[20] O.V. Borodin, A new proof of the 6 color theorem, *J. Graph Theory* **19** (1995), no. 4, 507–521.

[21] O.V. Borodin, Structural properties of plane graphs without adjacent triangles and an application to 3-colorings, *J. Graph Theory* **21** (1996), no. 2, 183–186.

[22] O.V. Borodin, Structural theorem on plane graphs with application to the entire coloring number, *J. Graph Theory* **23** (1996), no. 3, 233–239.

[23] O.V. Borodin, Colorings of plane graphs: A survey, *Discrete Math.* **313** (2013), no. 4, 517–539.

[24] O.V. Borodin, H. J. Broersma, A. Glebov, and J. van den Heuvel, Stars and bunches in planar graphs, Part II: General planar graphs and colourings, *CDAM Research Report Series* **2002–05** (2002). Original Russian version. *Diskretn. Anal. Issled. Oper. Ser. 1* **8** (2001), no. 4, 9–33, available at: `http://www.cdam.lse.ac.uk/Reports/Abstracts/cdam-2002-05.html`.

[25] O.V. Borodin, M. Chen, A.O. Ivanova, and A. Raspaud, Acyclic 3-choosability of sparse graphs with girth at least 7, *Discrete Math.* **310** (2010), no. 17–18, 2426–2434.

[26] O.V. Borodin, D.G. Fon-Der Flaass, A.V. Kostochka, A. Raspaud, and E. Sopena, Acyclic list 7-coloring of planar graphs, *J. Graph Theory* **40** (2002), no. 2, 83–90.

[27] O.V. Borodin, A.N. Glebov, A.O. Ivanova, T.K. Neustroeva, and V.A. Tashkinov, Sufficient conditions for planar graphs to be 2-distance $(\Delta + 1)$-colorable, *Sib. Élektron. Mat. Izv.* **1** (2004), 129–141 [Russian; English summary].

[28] O.V. Borodin, A.N. Glebov, A. Raspaud, and M.R. Salavatipour, Planar graphs without cycles of length from 4 to 7 are 3-colorable, *J. Combin. Theory Ser. B* **93** (2005), no. 2, 303–311.

[29] O.V. Borodin, S.G. Hartke, A.O. Ivanova, Kostochka, A. V., and D.B. West, Circular (5,2)-coloring of sparse graphs, *Sib. Èlektron. Mat. Izv.* **5** (2008), 417–426.

[30] O.V. Borodin and A.O. Ivanova, Almost proper vertex 2-colorings of sparse graphs, *Diskretn. Anal. Issled. Oper.* **16** (2009), no. 2, 16–20, 98; translation in *J. Appl. Ind. Math.* **4** (2010), no. 1, 21–23.

[31] O.V. Borodin and A.O. Ivanova, 2-distance $(\Delta + 2)$-coloring of planar graphs with girth six and $\Delta \geq 18$, *Discrete Math.* **309** (2009), no. 23–24, 6496–6502.

[32] O.V. Borodin and A.O. Ivanova, List 2-distance $(\Delta + 2)$-coloring of planar graphs with girth six, *European J. Combin.* **30** (2009), no. 5, 1257–1262.

[33] O.V. Borodin and A.O. Ivanova, List 2-distance $(\Delta+2)$-coloring of planar graphs with girth six and $\Delta \geq 24$, *Sibirsk. Mat. Zh.* **50** (2009), no. 6, 1216–1224 (Russian); translation in *Sib. Math. J.* **50** (2009), no. 6, 958–964.

[34] O.V. Borodin and A.O. Ivanova, Acyclic 5-choosability of planar graphs without adjacent short cycles, *J. Graph Theory* **68** (2011), no. 2, 169–176.

[35] O.V. Borodin and A.O. Ivanova, Acyclic 5-choosability of planar graphs without 4-cycles, *Sibirsk. Mat. Zh.* **52** (2011), no. 3, 522–541 (Russian); translation in *Sib. Math. J.* **52** (2011), no. 3, 411–425.

[36] O.V. Borodin and A.O. Ivanova, Acyclic 4-choosability of planar graphs with no 4- and 5-cycles, *J. Graph Theory* **72** (2013), 374–397.

[37] O.V. Borodin and A.O. Ivanova, Describing $(d-2)$-stars at d-vertices, $d \leq 5$, in normal plane maps, *Discrete Math.* **313** (2013), no. 17, 1700–1709.

[38] O.V. Borodin and A.O. Ivanova, Describing 3-faces in normal plane maps with minimum degree 4, *Discrete Math.* **313** (2013), no. 23, 2841–2847.

[39] O.V. Borodin, A.O. Ivanova, and T. K. Neustroeva, Sufficient conditions for planar graphs with girth 6 to be 2-distance colourable, *Sib. Elektron. Mat. Izv.* **3** (2006), 441–450 [Russian], available at: `http://semr.math.nsc.ru/v3.html`.

[40] O.V. Borodin, S.-J. Kim, A.V. Kostochka, and D.B. West, Homomorphisms from sparse graphs with large girth, *J. Combin. Theory Ser. B* **90** (2004), no. 1, 147–159.

[41] O.V. Borodin, A.V. Kostochka, and D.R. Woodall, List edge and list total colourings of multigraphs, *J. Combin. Theory Ser. B* **71** (1997), no. 2, 184–204.

[42] O.V. Borodin and D. Sanders, On light edges and triangles in planar graphs of minimum degree five, *Math. Nachr.* **170** (1994), 19–24.

[43] A. Brandt, M. Ferrara, M. Kumbhat, S. Loeb, D. Stolee, and M. Yancey, I,F-partitions of Sparse Graphs *European J. Combinatorics* **57** (2016), 1–12.

[44] R.L. Brooks, On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.* **37** (1941), 194–197.

[45] Y. Bu, D. Cranston, M. Montassier, A. Raspaud, and W. Wang, Star coloring of sparse graphs, *J. Graph Theory* **62** (2009), no. 3, 201–219.

[46] Y. Bu and W. Wang, Some sufficient conditions for a planar graph of maximum degree six to be class 1, *Discrete Math.* **306** (2006), no. 13, 1440–1445.

[47] M. Chen and A. Raspaud, Planar graphs without 4- and 5-cycles are acyclically 4-choosable, *Discrete Appl. Math.* **161** (2013), no. 7-8, 921–931.

[48] M. Chen, A. Raspaud, and W. Wang, 8-star-choosability of a graph with maximum average degree less than 3, *Discrete Math. Theor. Comput. Sci.* **13** (2011), no. 3, 97–110.

[49] M. Chen, A. Raspaud, and W. Wang, 6-star-coloring of subcubic graphs, *J. Graph Theory* **72** (2013), no. 2, 128–145.

[50] N. Cohen and F. Havet, Planar graphs with maximum degree $\Delta \geq 9$ are $(\Delta + 1)$-edge-choosable—a short proof, *Discrete Math.* **310** (2010), no. 21, 3049–3051.

[51] V. Cohen-Addad, M. Hebdige, D. Král', Z. Li, and E. Salgado, Steinberg's Conjecture is false, *J. Combin. Theory Ser. (B)* (2016), http://dx.doi.org/10.1016/j.jctb.2016.07.006.

[52] D.W. Cranston, Edge-choosability and total-choosability of planar graphs with no adjacent 3-cycles, *Discuss. Math. Graph Theory* **29** (2009), no. 1, 163–178.

[53] D.W. Cranston, R. Erman, and R. Škrekovski, Choosability of the square of a planar graph with maximum degree four, *Australas. J. Combin.* **59** (2014), 86–97.

[54] D.W. Cranston, S. Jahanbekam, and D.B. West, The 1,2,3-Conjecture and 1,2-Conjecture for sparse graphs, *Discuss. Math. Graph Theory* 34 (2014), no. 4, 769–799.

[55] D.W. Cranston and S.-J. Kim, List-coloring the square of a subcubic graph, *J. Graph Theory* **57** (2008), no. 1, 65–87.

[56] D.W. Cranston, S.-J. Kim, and G. Yu, Injective colorings of sparse graphs, *Discrete Math.* **310** (2010), no. 21, 2965–2973.

[57] D.W. Cranston, S.-J. Kim, and G. Yu, Injective colorings of graphs with low average degree, *Algorithmica* **60** (2011), no. 3, 553–568.

[58] D.W. Cranston and R. Škrekovski, Sufficient sparseness conditions for G^2 to be $(\Delta + 1)$-choosable, when $\Delta \geq 5$, *Discrete Appl. Math.* **162** (2014), 167–176.

[59] D.W. Cranston and G. Yu, Linear choosability of sparse graphs, *Discrete Math.* **311** (2011), no. 17, 1910–1917.

[60] M. Cygan, Ł. Kowalik, and B. Lužar, A planar linear arboricity conjecture, in *Algorithms and complexity, Lecture Notes in Comput. Sci.* **6078**, (Springer, 2010), 204–216.

[61] Z. Dvořák, D. Král', P. Nejedlý, and R. Škrekovski, Coloring squares of planar graphs with girth six, *European J. Combin.* **29** (2008), no. 4, 838–849.

[62] Z. Dvořák, K. Kawarabayashi, and R. Thomas, Three-coloring triangle-free planar graphs in linear time, *Proc. 20th Annual ACM-SIAM Symposium on Discrete Algorithms (extended abstract)* 1176–1182, SIAM, Philadelphia, PA, 2009, and *ACM Trans. Algorithms* **7** (2011), no. 4, Art. 41, 14 pp.

[63] H. Enomoto and B. Péroche, The linear arboricity of some regular graphs, *J. Graph Theory* **8** (1984), 309–324.

[64] P. Erdős, A.L. Rubin, and H. Taylor, Choosability in graphs, *Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979), Congress. Numer.* **26** (Utilitas Math., 1980), 125–157.

[65] P. Erdős and A. Rényi, On a problem in the theory of graphs, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962), 623–641 [Hungarian].

[66] L. Esperet, J. van den Heuvel, F. Maffray, and F. Sipma, Fire containment in planar graphs, *J. Graph Theory* **73** (2013), no. 3, 267–279.

[67] G. Fertin, A. Raspaud, and B. Reed, *Star coloring of graphs*, J. Graph Theory **47** (2004), no. 3, 163–182.

[68] S. Fiorini, Some remarks on a paper by Vizing on critical graphs, *Math. Proc. Camb. Phil. Soc.* **77** (1975), 475–483.

[69] P. Franklin, The Four Color Problem, *Amer. J. Math.* **44** (1922), no. 3, 225–236.

[70] F. Galvin, The List Chromatic Index of a Bipartite Multigraph, *J. Combin. Theory Ser. B* **63** (1995), no. 1, 153–158.

[71] H. Grötzsch, Ein dreifarbensatz für dreikreisfreie netze auf der kugel, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* **8** (1959), 109–120 [German].

[72] B. Grünbaum, Acyclic colorings of planar graphs, *Israel J. Math.* **14** (1973), 390–408.

[73] F. Guldan, The linear arboricity of 10-regular graphs, *Math Slovaca* **36** (1986), 225–228.

[74] R.P. Gupta, The chromatic index and the degree of a graph, *Notices Amer. Math. Soc.* **13** (1966), abstract 66T-429.

[75] S.G. Hartke, S. Jahanbekam, and B. Thomas, The chromatic number of the square of subcubic planar graphs, http://arxiv.org/abs/1604.06504.

[76] F. Havet, J. van den Heuvel, C. McDiarmid, and B. Reed, List colouring squares of planar graphs, preprint, available at: `http://arxiv.org/abs/0807.3233`.

[77] F. Havet and J.-S. Sereni, Improper choosability of graphs and maximum average degree, *J. Graph Theory* **52** (2006), no. 3, 181–199.

[78] P. Hell and K. Seyffarth, Largest planar graphs of diameter two and fixed maximum degree, *Discrete Math.* **111** (1993), no. 1–3, 313–322.

[79] J. van den Heuvel and S. McGuinness, Coloring the square of a planar graph, *J. Graph Theory* **42** (2003), no. 2, 110–124.

[80] I. Holyer, The NP-completeness of edge-coloring, *SIAM J. Comput.* **10** (1981), no. 4, 718–720.

[81] A.O. Ivanova, List 2-distance $(\Delta+1)$-coloring of sparse planar graphs with girth at least 7, *Diskretn. Anal. Issled. Oper.* **17** (2010), no. 5, 22–36, 94 [Russian]; *J. Appl. Indust. Math.* **5** (2011), 221–230.

[82] F. Jaeger, Nowhere-zero flow problems, in *Selected Topics in Graph Theory, Vol. 3* (L. Beineke et al., eds), Academic Press (1988), 91–95.

[83] S. Jendrol', A short proof of Kotzig's theorem on minimal edge weights of convex 3-polytopes, *Proc. Intl. Scient. Conf. Math.* (V. Balint Ed.) Žilina (1998) EDIS Univ. Press, Žilina (1999), 35–38.

[84] S. Jendrol' and H.-J. Voss, Light subgraphs of graphs embedded in the plane–A survey, *Discrete Math.* **313** (2013), no. 4, 406–421.

[85] T.K. Jonas, Graph coloring analogues with a condition at distance two: L(2,1)-labellings and list lambda-labellings, *Ph.D. Thesis*, University of South Carolina. 1993. 95 pp.

[86] M. Juvan, B. Mohar, and R. Škrekovski, Graphs of degree 4 are 5-edge-choosable, *J. Graph Theory* **32** (1999), no. 3, 250–264.

[87] J. Kahn, Asymptotics of the list chromatic index for multigraphs, *Random Structures & Algorithms* **17** (2000), no. 2, 117–156.

[88] S.-J. Kim and B. Park, Counterexamples to the list square coloring conjecture, *J. Graph Theory* **78** (2015), no. 4, 239–247.

[89] S.-J. Kim and W.-J. Park, List dynamic coloring of sparse graphs, *Combinatorial optimization and applications, Lecture Notes in Comput. Sci.* **6831**, (Springer, Heidelberg, 2011), 156–162.

[90] A.V. Kostochka and L.S. Mel'nikov, Note to the paper of Grünbaum on acyclic colorings, *Discrete Math.* **14** (1976), no. 4, 403–406.

[91] A.V. Kostochka and D.R. Woodall, Choosability conjectures and multicircuits, *Discrete Math.* **240** (2001), no. 1–3, 123–143.

[92] A.V. Kostochka and M. Yancey, Ore's conjecture for $k = 4$ and Grötzsch's theorem, *Combinatorica* **34** (2014), no. 3, 323–329.

[93] A. Kotzig, Contribution to the theory of Eulerian polyhedra, *Mat. Čas SAV (Math. Slovaca)* **5** (1955), 101–113, [Slovak].

[94] H. Lebesgue, Quelques conséquences simples de la formule dEuler, *J. Math. Pures Appl.* **19** (1940), 2743 [French].

[95] L. Lovász, C. Thomassen, Y. Wu, and C.-Q. Zhang, Nowhere-zero 3-flows and modulo k-orientations, *J. Combin. Theory Ser. B* **103** (2013), 587–598.

[96] R. Luo and C.-Q. Zhang, Edge coloring of graphs with small average degrees, *Discrete Math.* **275** (2004), 207–218.

[97] L. Miao and Q. Sun, On the size of critical graphs with maximum degree 8, *Discrete Math.* **310** (2010), 2215–2218.

[98] M. Molloy and M.R. Salavatipour, A bound on the chromatic number of the square of a planar graph, *J. Combin. Theory Ser. B* **94** (2005), no. 2, 189–213.

[99] M. Montassier, Acyclic 4-choosability of planar graphs with girth at least 5, *Graph theory in Paris, Trends Math.*, (Birkhäuser, 2007), 299–310.

[100] M. Montassier, A. Raspaud, and W. Wang, Acyclic 4-choosability of planar graphs without cycles of specific lengths, in *Topics in Discrete Mathematics*, Algorithms Combin., **26** (Springer, 2006), 473–491.

[101] C.St.J.A. Nash-Williams, Decomposition of finite graphs into forests, *J. London Math. Soc.* **39** (1964), 12.

[102] J. Przybyło and M. Woźniak, On a 1,2-conjecture, *Discrete Math. Theor. Comput. Sci.* **12** (2010), 101–108.

[103] N. Robertson, D. Sanders, P. Seymour, and R. Thomas, The four-colour theorem, *J. Combin. Theory Ser. B* **70** (1997), no. 1, 2–44.

[104] N. Robertson, P. Seymour, and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* **13** (1993), 279–361.

[105] D.P. Sanders and Y. Zhao, A note on the three color problem, *Graphs Combin.* **11** (1995), no. 1, 91–94.

[106] D.P. Sanders and Y. Zhao, Planar graphs of maximum degree seven are class I, *J. Combin. Theory Ser. B* **83** (2001), no. 2, 201–212.

[107] D.P. Sanders and Y. Zhao, On the size of edge chromatic critical graphs, *J. Combin. Theory Ser. B* **86** (2002), no. 2, 408–412.

[108] D.P. Sanders and Y. Zhao, Coloring edges of graphs embedded in a surface of characteristic zero, *J. Combin. Theory Ser. B* **87** (2003), no. 2, 254–263.

[109] R. Steinberg, The state of the three color problem, in *Quo Vadis, Graph Theory?* (J. Gimbel, J.W. Kennedy, L.V. Quintas, eds.), Annals of Discrete Mathematics **55** (1993), 211–248.

[110] L. Stockmeyer, Planar 3-colorability is polynomial complete, *ACM SIGACT News* **5** (1973), 19–25.

[111] C. Thomassen, Every planar graph is 5-choosable, *J. Combin. Theory Ser. B* **62** (1994), no. 1, 180–181.

[112] C. Thomassen, Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane, *J. Combin. Theory Ser. B* **62** (1994), no. 2, 268–279.

[113] C. Thomassen, 3-list-coloring planar graphs of girth 5, *J. Combin. Theory Ser. B* **64** (1995), no. 1, 101–107.

[114] C. Thomassen, A short list color proof of Grötzsch's theorem, *J. Combin. Theory Ser. B* **88** (2003), no. 1, 189–192.

[115] C. Timmons, Star coloring high girth planar graphs, *Electron. J. Combin.* **15** (2008), no. 1, Research Paper 124, 17 pp.

[116] V.G. Vizing, On an estimate of the chromatic class of a p-graph, *Metody Diskret. Analiz.* **3** (1964), 25–30 [Russian].

[117] V.G. Vizing. Critical graphs with given chromatic class, *Metody Diskr. Anal.* **5** (1965), 9–17 [Russian].

[118] V.G. Vizing, The chromatic class of a multigraph, *Kibernetika (Kiev)*, no. **3** (1965), 29–39. [Russian]; English translation in *Cybernetics* **1**, 32–41.

[119] V.G. Vizing, Some unsolved problems in graph theory, *Uspekhi Mat. Nauk* **23** (1968), 117–134 [Russian]; English translation in *Russian Math. Surveys* **23**, 125–141.

[120] V.G. Vizing, Coloring the vertices of a graph in prescribed colors, *Diskret. Analiz No.* **29** Metody Diskret. Anal. v Teorii Kodov i Shem (1976), 3–10, 101 [Russian].

[121] M. Voigt, List colourings of planar graphs, *Discrete Math.* **120** (1993), no. 1-3, 215–219.

[122] W. Wang and Y. Chen, A sufficient condition for a planar graph to be class 1, *Theoret. Comput. Sci.* **385** (2007), no. 1-3, 71–77.

[123] W. Wang and K.-W. Lih, Choosability and edge choosability of planar graphs without five cycles, *Appl. Math. Lett.* **15** (2002), no. 5, 561–565.

[124] Y. Wang, and L. Xu, A sufficient condition for a plane graph with maximum degree 6 to be class 1, *Discrete Appl. Math.* **161** (2013), no. 1-2, 307–310.

[125] G. Wegner, Graphs with given diameter and a colouring problem, preprint, University of Dortmund (1977).

[126] D.R. Woodall, The average degree of an edge-chromatic critical graph II, *J. Graph Theory* **56** (2007), no. 3, 194–218.

[127] D.R. Woodall, The average degree of a multigraph critical with respect to edge or total choosability, *Discrete Math.* **310** (2010), no. 6-7, 1167–1171.

[128] J.-L. Wu, On the linear arboricity of planar graphs, *J. Graph Theory* **31** (1999), no. 2, 129–134.

[129] J.-L. Wu and Y.-W. Wu, The linear arboricity of planar graphs of maximum degree seven is four, *J. Graph Theory* **58** (2008), no. 3, 210–220.

[130] M. Yancey, Coloring the square of a sparse graph G with almost $\Delta(G)$ colors, preprint, available at: `http://arxiv.org/abs/1502.03132`.

[131] L. Zhang, Every planar graph with maximum degree 7 is of class 1, *Graphs Combin.* **16** (2000), no. 4, 467–495.

[132] L. Zhang and B. Wu, Edge choosability of planar graphs without small cycles, *Discrete Math.* **283** (2004), no. 1–3, 289–293.

[133] X. Zhu, Circular chromatic number: a survey, *Discrete Math.* **229** (2001), no. 1-3, 371–410.

[134] X. Zhu, Recent developments in circular colouring of graphs, in "Topics in discrete mathematics", Algorithms Combin. **26** (Springer, 2006), 497–550.