A Guide to Discharging,

with Applications to List Coloring

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slides available on DBW preprint page

Based on a survey written with Daniel W. Cranston

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"An unavoidable set of reducible configurations"

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Def. j-vertex, j^+-vertex, j^--vertex; degree $= j$, $\geq j$, $\leq j$.
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Degree charging: give each vertex v initial charge $d(v)$.

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In particular, planar with girth $\geq 6 \Rightarrow \text{avd}(G) < 3$.

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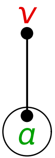

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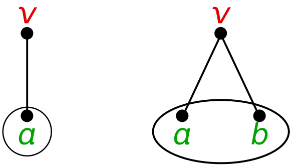

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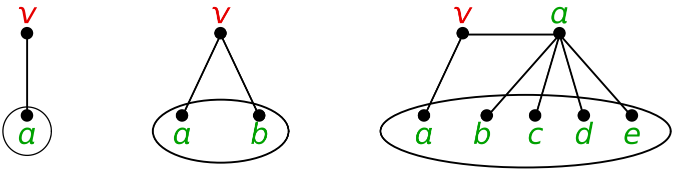
$d_G(v) = 2$ with same color on nbrs of v: avoid it and the colors on neighbors of its 5^--neighbor. ∎



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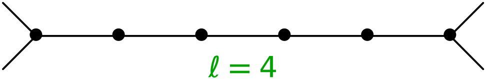

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Avoiding these configurations needs $\alpha\text{vd}(G) \geq 2 + \frac{2}{3\ell-1}$. ■

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G has no ℓ-threads, has $\ell\frac{3n}{2}$ edges, and has $n + (\ell-1)\frac{3n}{2}$ vertices, so $\text{avd}(G) = 2 + \frac{2}{3\ell-1}$.

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But, $\mathrm{Mad}(G) = k-1$ yields no better bound, due to K_k.

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($\mathrm{Mad}(G) < b$ is the inductive analogue of $\mathrm{avd}(G) < b$.)

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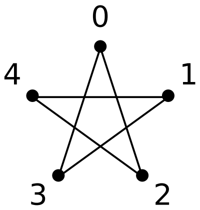

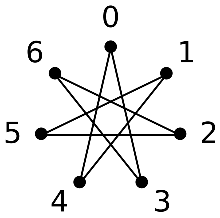

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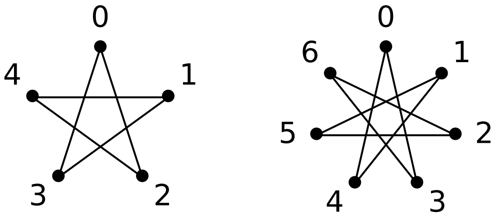

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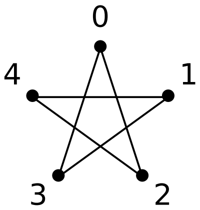

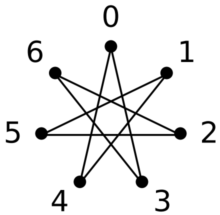

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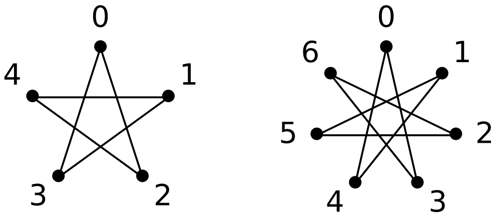

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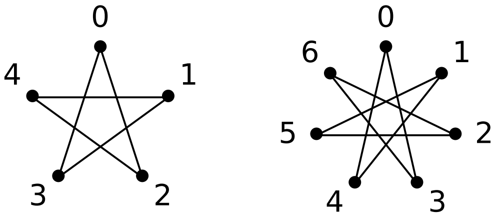

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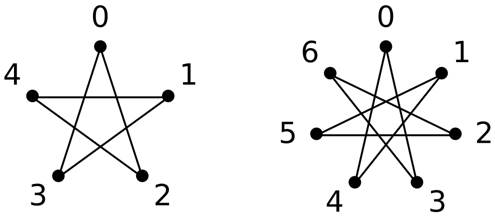

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Odd cycles don't map into longer odd cycles, so $\chi_c(G) \le 2 + \frac{1}{t}$ requires odd girth at least $2t+1$.

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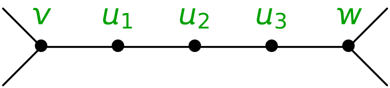

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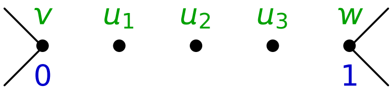

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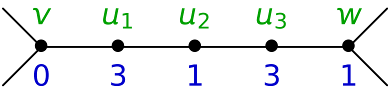

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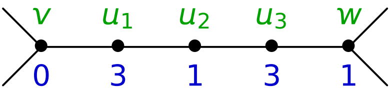

This succeeds since any two vertices of C_{2t+1} are joined by a path of even length at most $2t$. To follow a path of length $2s$ in $2t$ steps, repeat last edge $2t-2s$ times. ∎

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Lem. If $\alpha\text{vd}(G) < 2 + \frac{1}{2t-1}$ and G has no 2-regular component, then G has (1) a 1^--vertex,
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If $d(v) \geq 4$ and v has no incident ℓ-thread, then v loses at most $(\ell-1)\rho$ along each incident thread. The final charge is at least $d(v)[1-(\ell-1)\rho]$, which is at least $2+2\rho$ for $d(v) \geq 4$ when $\ell = 2t-1$. ∎

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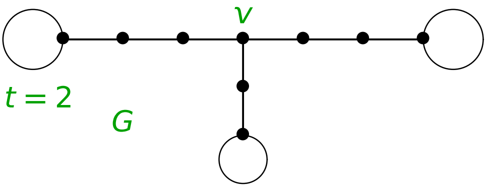

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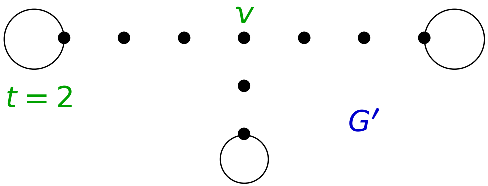

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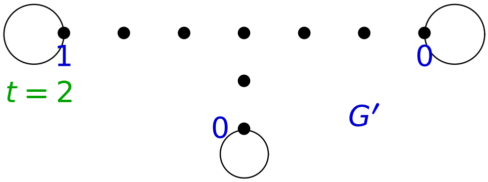

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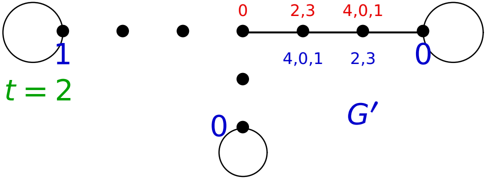

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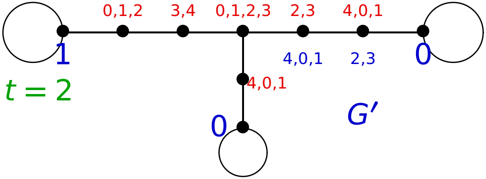

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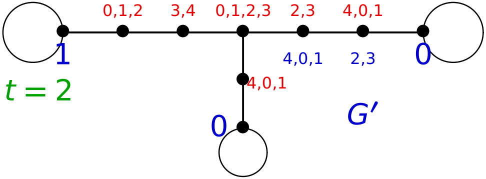

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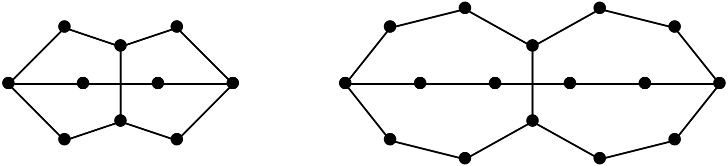
Sharpness example: Two $(2t+1)$-cycles sharing one edge, plus $(2t-2)$-thread joining the opposite vertices. $\text{Mad}(G_t) = 2+\frac{2}{3t-1}$, but $\chi_c(G_t) = 2+\frac{1}{t-1/2}$.



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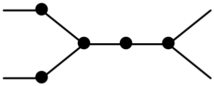

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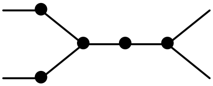

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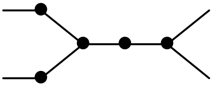

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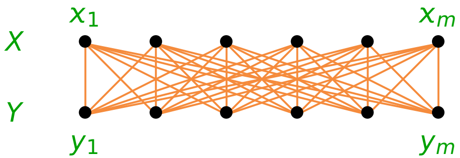

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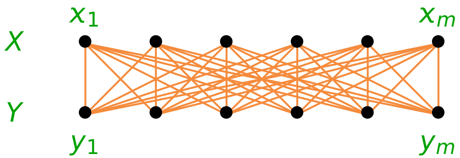

Choosing colors from all lists requires using k colors in each part, but then some color is used in both parts. ∎

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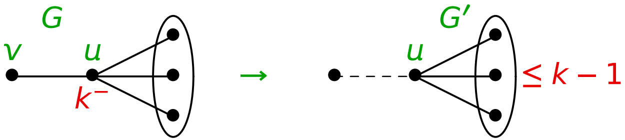

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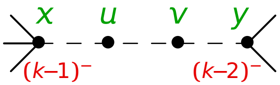

$G' = G - \{u, v\}$

Note that $G'^2 = G^2 - \{u, v\}$.

Extend L-coloring of G'^2:
(1) pick color on u (avoid at most k on $N(x) \cup \{x, y\}$).
(2) pick color on v (avoid at most k on $N(y) \cup \{x, y, u\}$).

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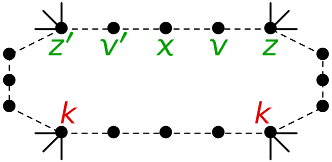

Form G' by deleting the 2-vertices; $G'^2 = G^2 -$these.

v must avoid colors on z and $N(z)$: at most $k-1$.
Type-v all have ≥ 2 colors left; even cycle 2-choosable!

Type-x verts must avoid ≤ 4 colors; okay since $k \ge 4$. ■

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Pf. A smallest counterexample G has no (A), (B), (C).

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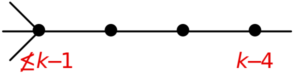

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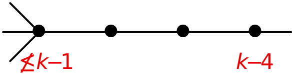

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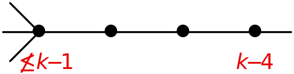

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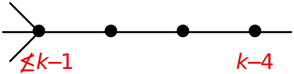

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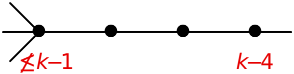

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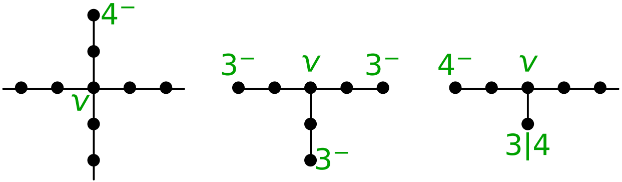

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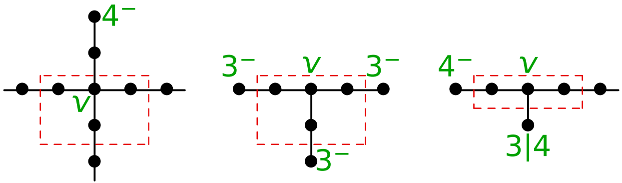

The configs forced by unhappy vertices are reducible! ■

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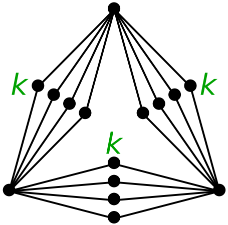

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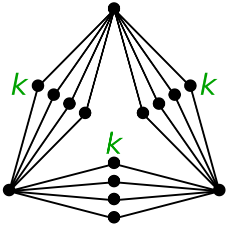

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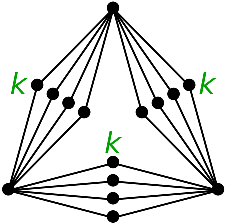

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Ques. Does $\text{Mad}(G) < 4$ imply $\chi_\ell(G^2) \leq \frac{3}{2}\Delta(G)$?

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Color $E(G-e)$ from the lists of size k; then e. ■

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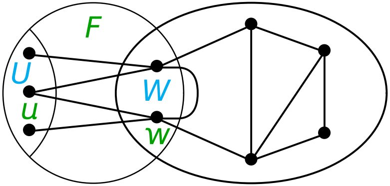

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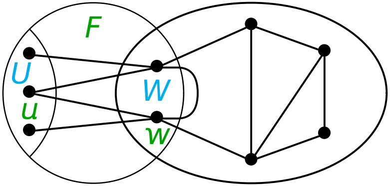

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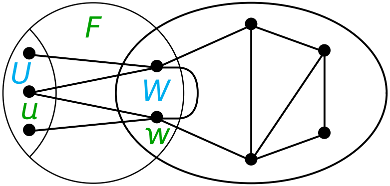

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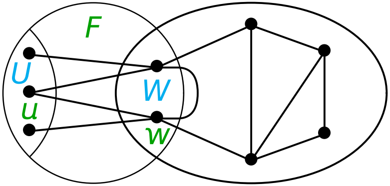

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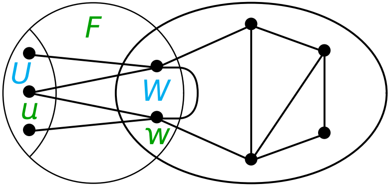

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BKW is stronger than Galvin's $\chi_\ell'(G) = \Delta(G)$ for bipartite.

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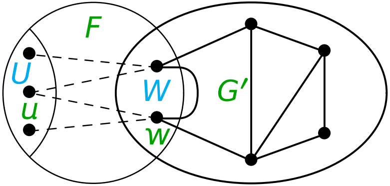

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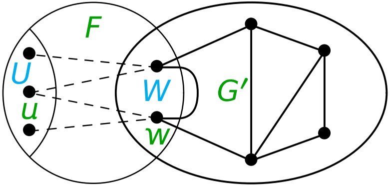

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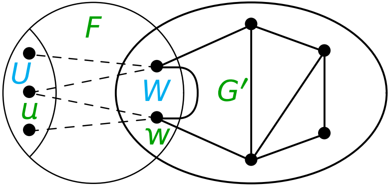

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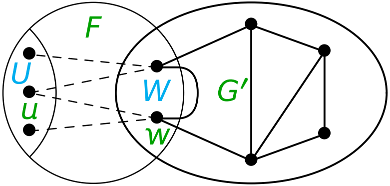

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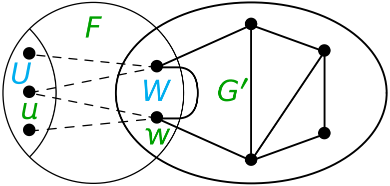

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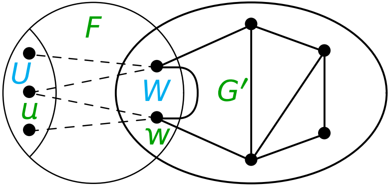

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Now BKW colors $E(F)$ from what remains of the lists. ■

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Now delete $\{w\} \cup (N(w) \cap U)$ from F. What remains is not i-alternating. Iterate til each $u \in U$ gains charge 1.

continued...

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Allowing $\mathrm{Mad}(G) > \sqrt{2\Delta(G)}$ in the list version is harder!

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Prop. In a plane graph G, with $d(F)$ denoting the length of a face F,

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Pf. Multiply Euler's Formula by -6 or -4 and split the contribution from the number of edges:

$$-6n + 2m + 4m - 6f = -12; \quad -4n + 2m + 2m - 4f = -8.$$

Substitute the degree-sum in G or G^* for $2m$. ■

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Results for planar graphs with girth g may extend to $\text{Mad}(G) < \frac{2g}{g-2}$ (when faces are not used), but general graphs may need tighter $\text{Mad}(G)$.

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Recall: For $\delta(G) = 2$, forcing a 2-vertex with a j^--nbr. When $\mathrm{Mad}(G) < 2 + 2\rho$, each 2-vert takes ρ from each nbr. If no 2-vert has j^--nbr, then all end happy if $(j+1)[1-\rho] \geq 2+2\rho$. Works if $\rho \leq \frac{2j-2}{j+3}$, which is sharp.

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Pf. Use balanced charging.

(R1) Each 2-vertex takes 1 from each incident face.
(R2) Each 3-vertex takes 1/3 from each incident face.

Vertices end happy! If nbrs of 2-verts are 4^+-verts, then a j-face loses $\leq j/2$ when $j \geq 8$, and ≤ 3 when $j = 7$. ■

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Appl: Every planar graph decomposes into three forests, one having maximum degree at most 8.
(Balogh–Kochol–Pluhár–Yu [2005])

Using a Pot of Charge - Preparation

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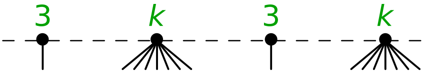

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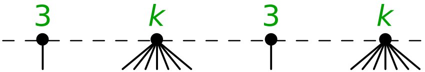

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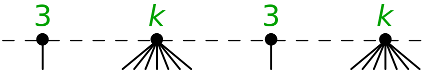

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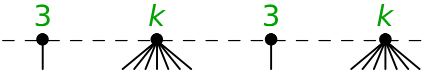

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Thus the bipartite subgraph H of edges incident to 3-vertices is acyclic, with $n_3 + n_k$ vertices and fewer edges. It also has $3n_3$ edges, so $3n_3 < n_3 + n_k$.

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For faces, length ≥ 4 loses no charge and stays happy. For a 3-face T, let j be the least incident vertex degree. Since no edges are light and $k \geq 9$, the other two are $(12 - j)^+$-vertices. In each case, T gains at least 1. ■

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We prove the last to illustrate face charging, yielding an easier proof than the original balanced charging.

An Application of Face Charging

Lem. (Borodin [1996]) Every plane graph G with $\delta(G) \geq 3$ has two adjacent 3-faces, or a j-face with $4 \leq j \leq 9$, or a 10-face with ten 3-vertices.

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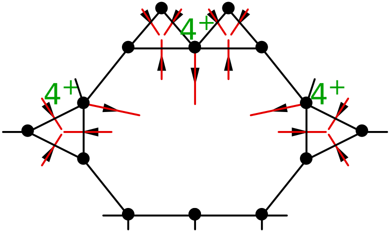

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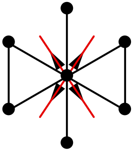

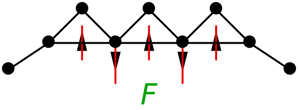

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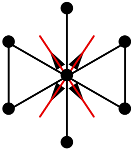

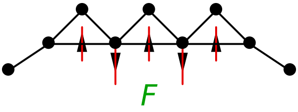

A j-face F with $j \ge 10$ loses 1 for each bounding path where the faces neighboring F are triangles and the endpoints have degree 3. Hence F loses $\le \lfloor j/2 \rfloor$. Only a 10-face with all 3^+-verts can become negative. ∎

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Every forest is star 3-colorable: In each tree T, choose a root r and color each vertex v with $d_T(v, r) \mod 3$.

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$\therefore$ G is star 4-colorable if $V(G)$ has an I, F-partition (into sets I and F inducing a 3-distant set and a forest).

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A 1^--vertex v can be added to F in any I, F-partition of $G - v$.

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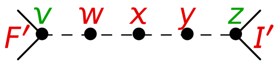

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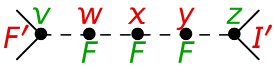

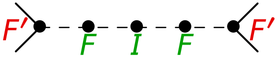

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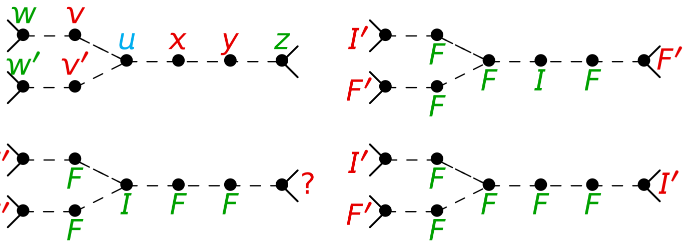

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After finding reducible configurations for the property "$\tau(G) \leq 2\nu(G)$", the discharging is easy.

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Let r =#good vertices. Since $(a,b,c) = (2, \frac{10}{3}, \frac{24}{7})$,
$r > \frac{c-b}{c-a}n = \frac{72-70}{72-42}n = \frac{n}{15}$. ∎

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By the Lemma, they save at least $n-2$ vertices with probability $> \frac{1}{15}$ and otherwise save at least 2. ∎