Some Old and New Results on Degree Lists of Graphs

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Joint work with
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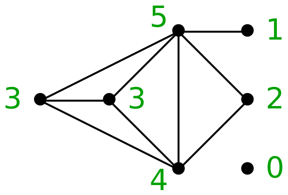

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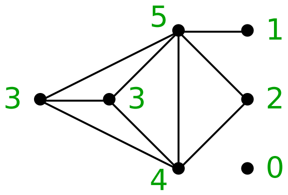

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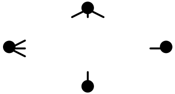
Each edge contributes to degree at two vertices, so $\sum_{v \in V(G)} d(v)$ is even.

Questions to Ask

Not all lists of nonnegative integers are degree lists.



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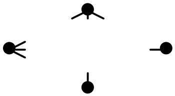

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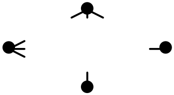

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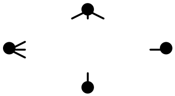

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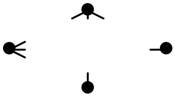

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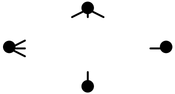

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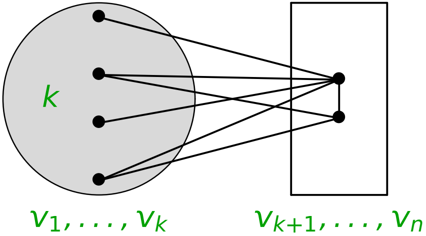
• Degrees are also studied in relation to graph coloring, connectivity, matchings, cycle lengths, etc.

Erdős–Gallai Condition

- Necessary condition



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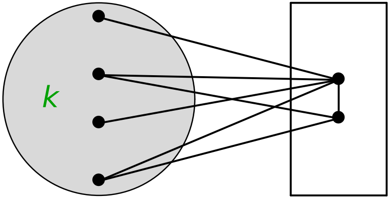

$v_1, \ldots, v_k$ $\qquad$ $v_{k+1}, \ldots, v_n$

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$

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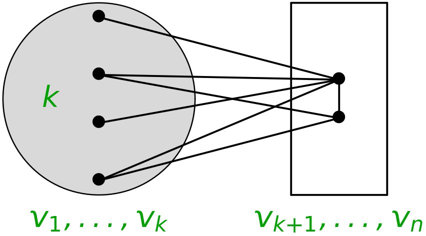

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Thm. (Erdős–Gallai [1960]) A nonincreasing nonneg. integer n-tuple d is graphic if and only if the sum is even and the Erdős–Gallai inequalities hold for all k.

Approach to Sufficiency

Idea: (Tripathi–Venugopalan–West [2010])
A subrealization of a list $d_1, \ldots, d_n$ is a graph with vertices $v_1, \ldots, v_n$ such that $d(v_i) \le d_i$ for $1 \le i \le n$.

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Write $v_i \leftrightarrow v_j$ when $v_i v_j \in E(G)$; otherwise, $v_i \not\leftrightarrow v_j$.

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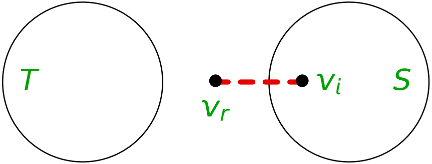
Case 0: $v_r \not\leftrightarrow v_i$ for some v_i with $d(v_i) < d_i$.



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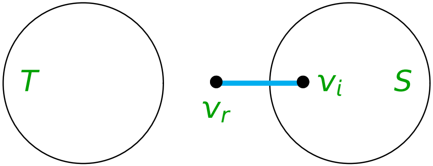
Add $v_r v_i$.



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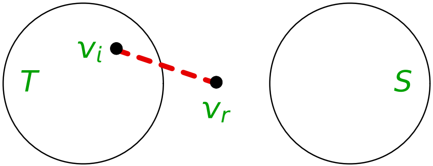

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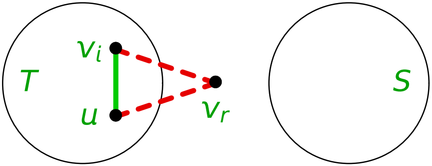

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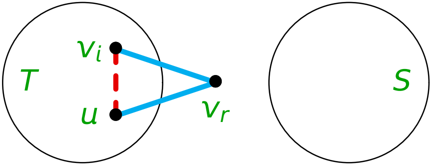

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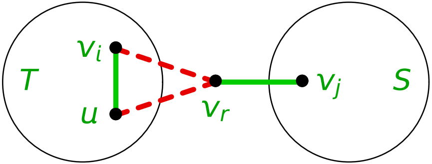
If $d_r - d(v_r) \geq 2$, then switch uv_i to uv_r, v_iv_r.
If $d_r - d(v_r) = 1$, then $\sum d_i - \sum d(v_i)$ even $\Rightarrow$ $\exists\, v_j \in S$
with $d(v_j) < d_j$. Not Case 0 $\Rightarrow$ $v_r \leftrightarrow v_j$.



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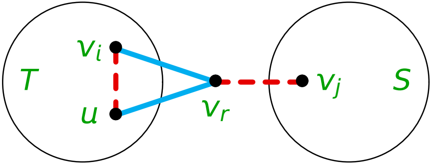
Switch v_rv_j, uv_i to uv_r, v_iv_r.



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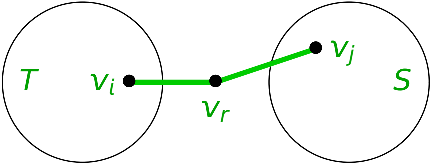

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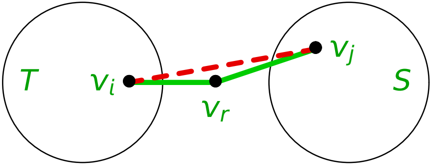

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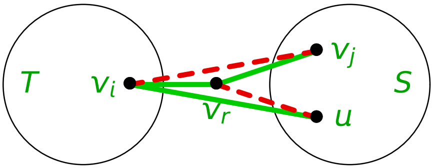

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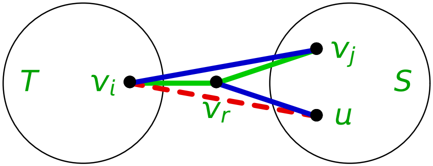

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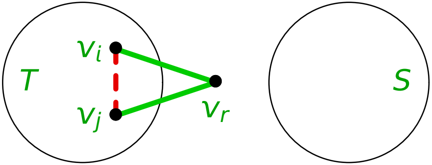

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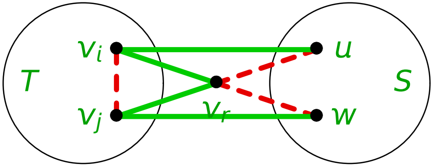

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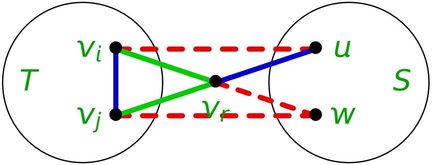
Now $v_r \leftrightarrow T \;\Rightarrow\; u, w \in S$; Switch uv_i, wv_j to uv_r, v_iv_j.



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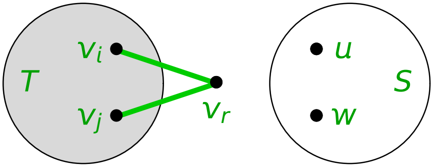

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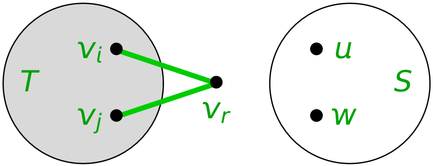

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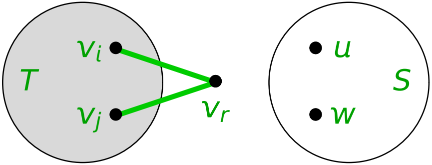
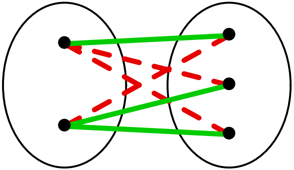
E-G inequality $\Rightarrow$ $d(v_r) = d_r$. Defect is 0, augment r.



Koren's Condition

For any vertex partition, every cross edge is in G or $\overline{G}$.



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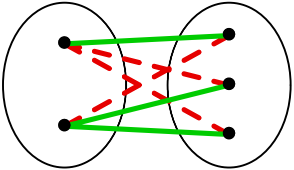

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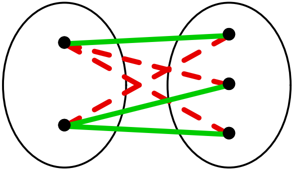

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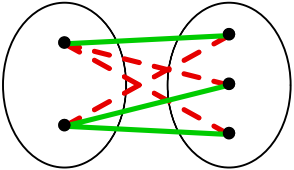

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• The result extends to loopless multigraphs with edge multiplicity $\leq p$ via $\sum_{i\in I} d_i + \sum_{j\in J}[p(n-1)-d_j] \geq p\,|I|\,|J|$.

Tools for Sufficiency

Lem. (Fulkerson–Hoffman–McAndrew [1965])
If d is a nonincreasing graphic list, and $d_j > 0$ for some j with $j > 1$, then $v_j \leftrightarrow v_1$ in some realization.

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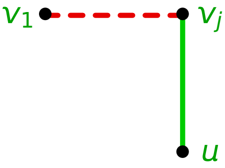

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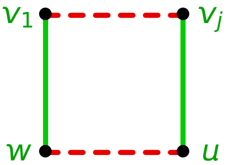

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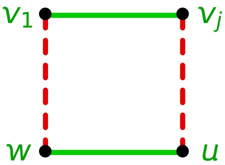

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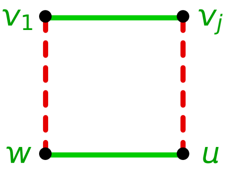

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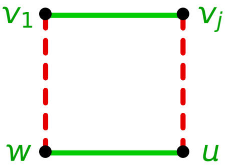

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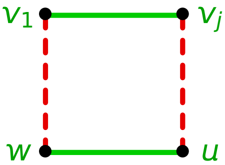

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Basis: $n + \sum d_i = 1$. One vertex, no edges.

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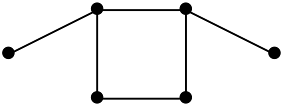
Other reductions leave only $d = (n-2, \ldots, n-2, 1)$, which has odd sum. ■

Well-Behaved Lists are Graphic

Although $(3, 3, 1, 1)$ is not graphic, $(3, 3, 2, 2, 1, 1)$ is.



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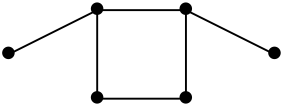

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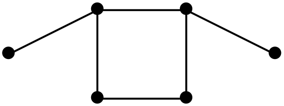

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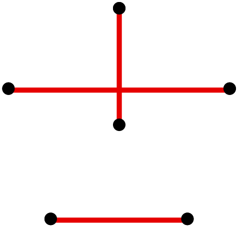

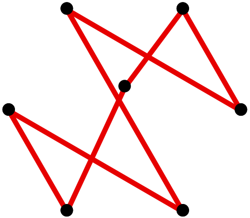

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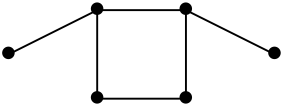

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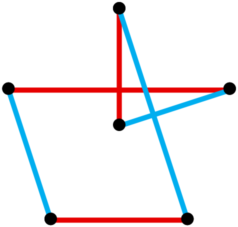

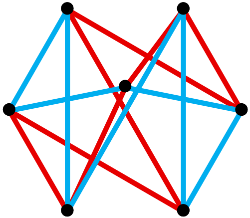

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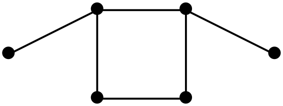

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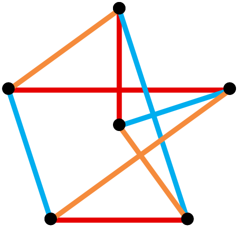

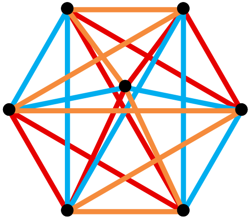

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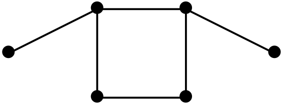

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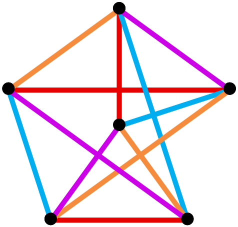

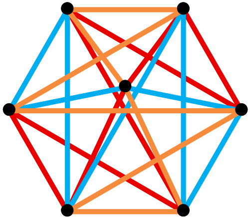

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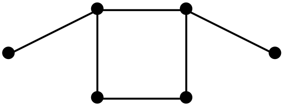

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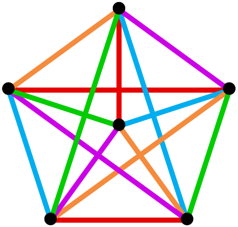

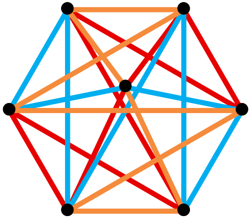

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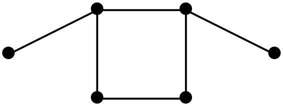

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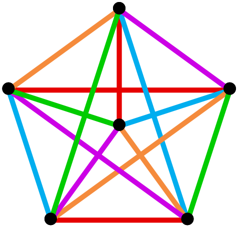

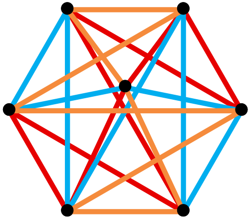

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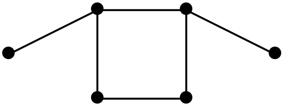

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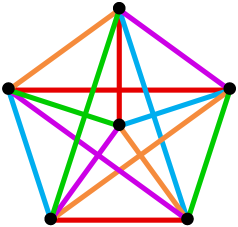

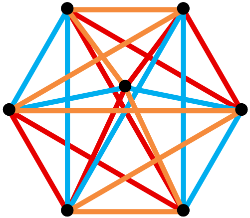

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Thm. (B–H–J–W) When consecutive entries in d differ by at most g, let $q = (r-s)/g$. The list d (with even sum) is graphic if $n \ge \frac{r+s+q}{2} + \frac{(r+s)(r-s-q+2)+\epsilon}{4s}$, and this is sharp. (Here $\epsilon \in \{0, g, g+1\}$.)

Long Lists are Graphic

Def. A nonincreasing integer list is gap-free if all values between the largest and smallest are present. Consider lists $(r, \dots, s)$ with length n.

Thm. (Barrus–Hartke–Jao–West [2010+])
A gap-free even-sum list d is graphic if $n \ge r + \frac{r+s+1}{2s}$.
With no restriction on gaps, d is graphic if $n \ge \frac{(r+s+1)^2}{4s}$.
Both thresholds are sharp.

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Set $g = 1 = \epsilon - 1$ and $g = r - s = \epsilon$ for the special cases.

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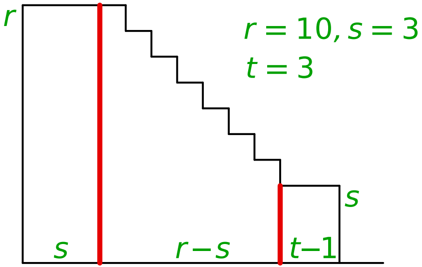

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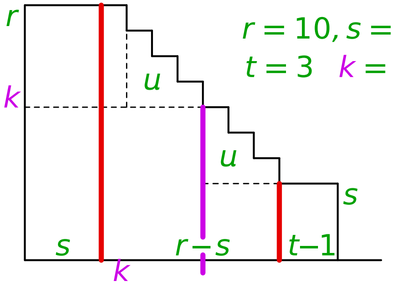

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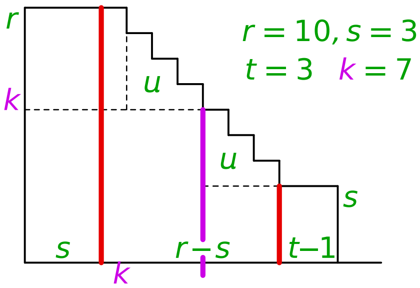

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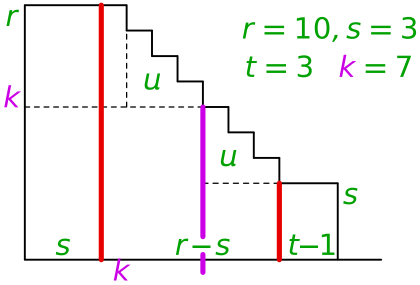

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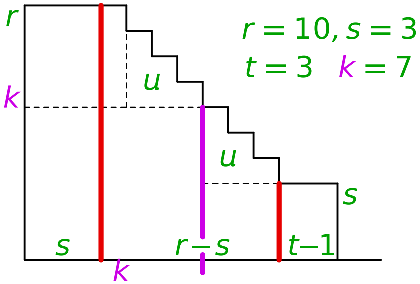

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Compare $s+r-k+1 = \left\lfloor \frac{r+s}{2} \right\rfloor + 1$ with $s(t-1) = s\left\lfloor \frac{r-s+1}{2s} \right\rfloor$.

Without Gap Constraint

Sufficiency threshold $= h = \left\lceil \frac{(r+s+1)^2}{4s} \right\rceil$

Let $x = \left\lfloor \frac{r+s}{2} \right\rfloor$. Let $y = h - x - 2$.

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• The constructions may produce lists with odd sum.
We actually obtain the sharp threshold such that every "(r, s, g)-list" satisfies all the Erdős–Gallai inequalities.

The Aigner–Triesch Method

Idea: (Aigner–Triesch [1994]) To prove that a condition Q on lists is sufficient for a property R:

1: Define a partial order P on the lists and show that if d satisfies R and $d' < d$ in P, then d' satisfies R.

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Def. A list d dominates a list d' if $\sum_{i=1}^{k} d_i \ge \sum_{i=1}^{k} d'_i$ for all k (trailing 0s added as needed).

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In any $P_{m,r,s,g,n}$, all or none satisfy the length threshold.

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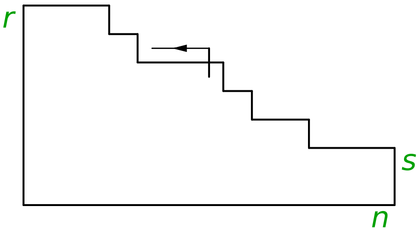

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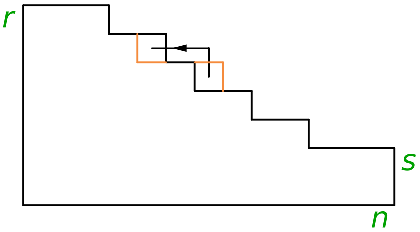

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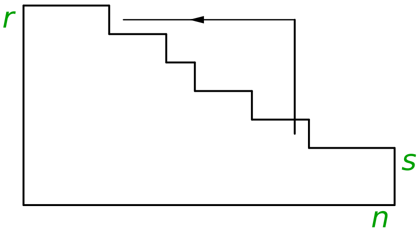

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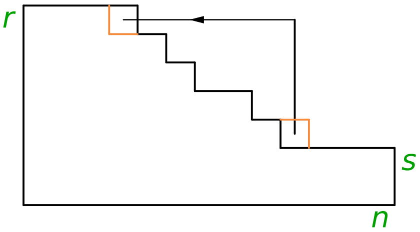

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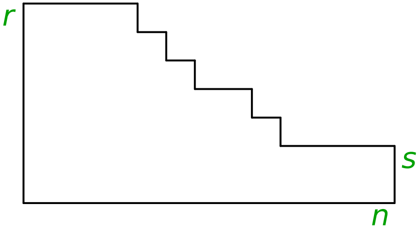

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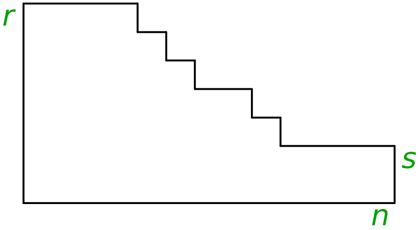

The unique maximal list has at most one repeated entry not in $\{r, s\}$.

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Pf. From $k-1$ to k, left side adds d_k, right adds $2k-2-d_k$ (given that $\min\{k, d_k\} = d_k$). ∎

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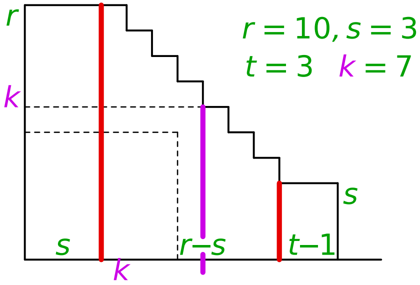
Pf. Reducing k, left side loses more than right side. ∎



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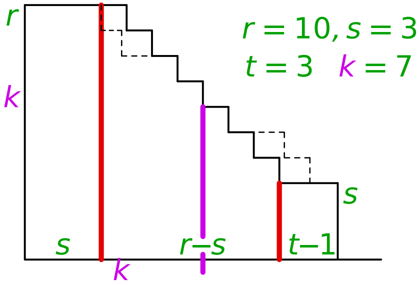

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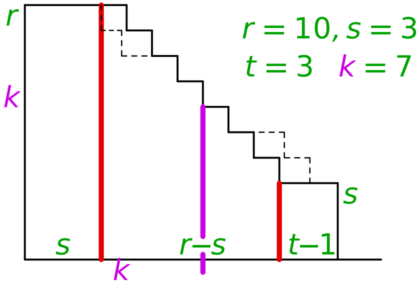

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Ex. Let $d = (3, 1, 2, 2, 0, 0)$ and $d' = (1, 3, 0, 0, 2, 2)$. The sum $(4, 4, 2, 2, 2, 2)$ is realized by $K_{2,4}$, and d and d' are graphic. In every realization of d or d', the vertex of degree 3 is adjacent to all other nonisolated vertices. Thus v_1 and v_2 are adjacent in every realization of d or d'; the lists do not pack.

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- Sauer–Spencer implies that G and G' pack when $\Delta(G) + \Delta(G') < \sqrt{2n}$, but that allows reordering vertices, so the result on packing graphic lists is stronger.

Sharpness Construction

For $m > 1$, let $n = 2m^2$. Let

$$d = (m, m, (2m)^{(m-1)}, 0^{(m-1)}, 1^{(m^2-m)}, 0^{(m^2-m)}),$$

$$d' = (m, m, 0^{(m-1)}, (2m)^{(m-1)}, 0^{(m^2-m)}, 1^{(m^2-m)}).$$

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In a realization G, let S be the $m + 1$ vertices with degree > 1. Their degree-sum is $2m^2$, which equals $2\binom{m+1}{2} + (m^2 - m)$. To reach this, S must be a clique, and all other edges must join S to leaves. Thus $v_1 \leftrightarrow v_2$ in every realization of d or d', so they don't pack.

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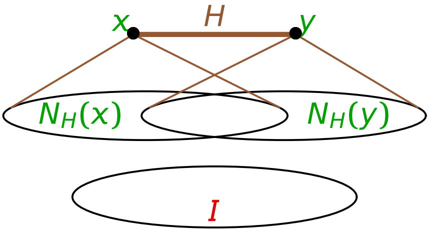

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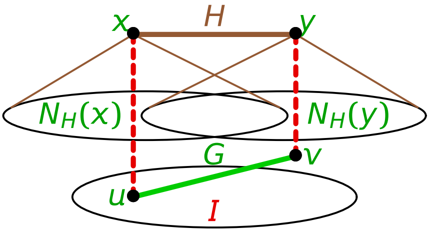

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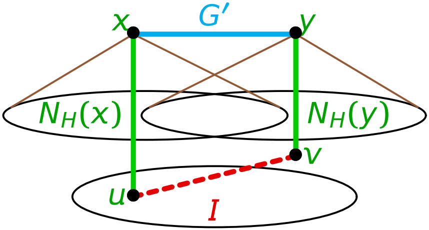
If G has an edge uv with $u \in I$ and $v \notin N_H(y)$, then switch xy, uv to xu, yv in G: fewer shared edges.



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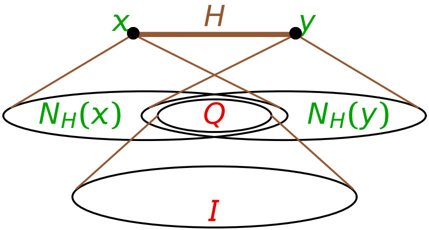
Pf. Realize d by G and d' by G' on the same vertices using the fewest common edges. If not a packing, . . .

Fix $xy \in E(G) \cap E(G')$. Let $H = G \cup G'$ and $I = \overline{N}_H(x) \cap \overline{N}_H(y)$.

If G has an edge uv with $u \in I$ and $v \notin N_H(y)$, then switch xy, uv to xu, yv in G: fewer shared edges.

Thus $Q = N_H(I) \subseteq N_G(x) \cap N_G(y)$.



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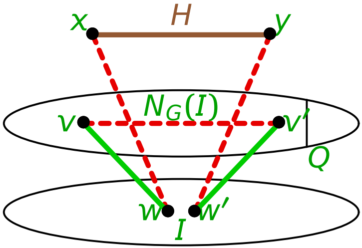

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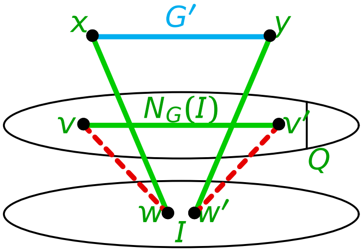

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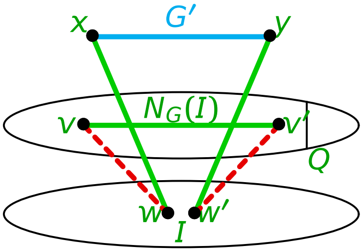

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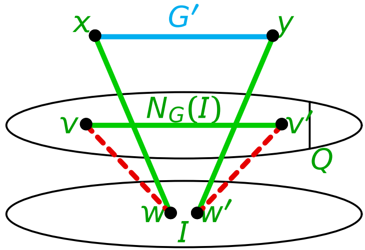

$\therefore$ Q is covered by two cliques in H. With $r = |E(H[Q])|$,

$$r \geq \binom{|Q|}{2} - \frac{|Q|^2}{4} = \frac{|Q|(|Q|-2)}{4}, \qquad (1)$$

since the complement $\overline{H}[Q]$ is bipartite.

Counting Argument

Since xy is a shared edge, $|N_H(x)|, |N_H(y)| \leq D - 1$.
Since also $Q \subseteq N_H(x) \cap N_H(y))$,

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Conj. Graphic n-tuples d and d' pack if always $d_i d'_i < n/2$. This stronger statement would be a more direct analogue of the Sauer–Spencer Theorem.

References

These slides can be found at
http://www.math.uiuc.edu/~west/pubs/degreet.pdf.

The proofs and results are from the papers below, found at http://www.math.uiuc.edu/~west/pubs/publink.html.

A. Tripathi, S. Venugopalan, and D.B. West, A short proof of the Erdős-Gallai characterization of degree lists, *Discrete Math.* 310 (2010), 843–844.

G. Isaak and D.B. West, The edge-count criterion for degree lists, (submitted).

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