Degree Ramsey and On-Line Degree Ramsey Numbers

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Parameter Ramsey Numbers

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- $R_\rho(G_1, G_2, G_3, \ldots, G_s; s)$ not yet much studied.

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Size Ramsey number is also linear in n for
cycles (Haxell–Kohayakawa–Łuczak [1995]) and
bounded-degree trees (Friedman–Pippinger [1987]),
but NOT graphs w. maxdegree 3 (Rödl–Szemerédi[2000]).

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For a family $\mathcal{G}$, let $R(\mathcal{G}; s) = \min\{n\colon K_n \xrightarrow{s} \mathcal{G}\}$.

Homomorphism = edge-preserving map $\phi\colon V(G) \to V(H)$.

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Pf. **(Idea)** Let $k = R(\mathcal{G}; s)$. Apply the bipartite Ramsey theorem repeatedly to a complete k-partite H with huge parts to get a complete k-partite subgraph H' with parts of size $|V(G)|$ where the edges joining any two parts have the same color.

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The collapsed coloring of $E(K_k)$ has a monochromatic homomorphic image of G, which expands to a monochromatic G in H'. ■

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Ex. $\chi(G) = 3 \;\Rightarrow\; 5 \le R_\chi(G) \le 6$.

Equality holds in lower bound $\iff$ $\exists$ hom. $\phi\colon G \to C_5$.

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BEL proved it for $k \le 4$.

Zhu ([1998] for $k = 5$, [2010] for all k) proved it!

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Thm. $R_\Delta(K_{1,m}; s) = \begin{cases} s(m-1) & m \text{ even} \\ s(m-1)+1 & m \text{ odd} \end{cases}$.

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Improves when m is even:
When k is odd and $r > k$, there is an r-regular graph H having no k-factor (Bollobás–Saito–Wormald [1985]).
With $k = m-1$ and $r = s(m-1)$, s-coloring $E(H)$ with no monochromatic $K_{1,m}$ requires a k-factorization.

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Put $m-1$ matchings into each color.

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Petersen's Theorem decomposes $s(m-1)$-regular supergraph H' into 2-factors. Putting $(m-1)/2$ in each color avoids degree m in one color at any vertex. ■

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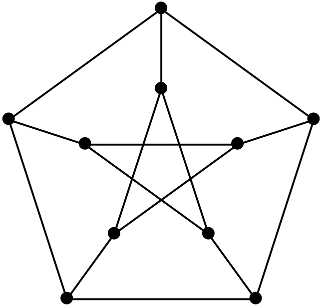

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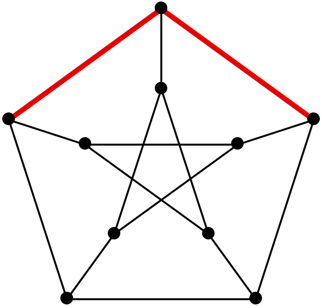

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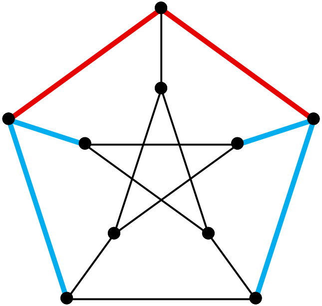

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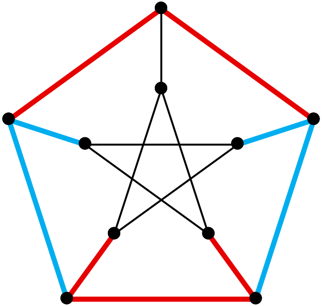

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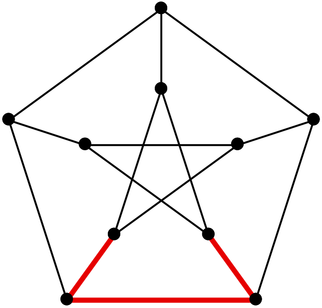

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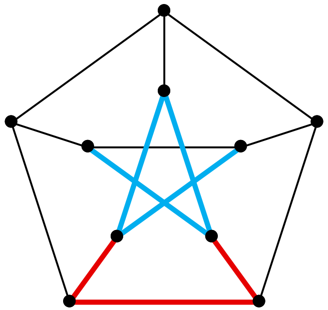

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s colors on sm edges puts $\geq m$ in some color class.
Since $|V(H)| = m$, this subgraph has a cycle.
Since girth$(H) \geq n$, this color class contains P_n. ∎

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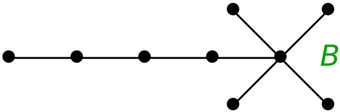

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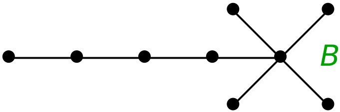

- $R_\Delta(K_{1,m};s)$ grows with m, but $R_\Delta(P_l;s) \leq 2s$.

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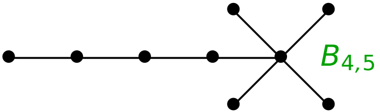

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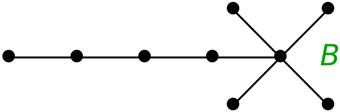

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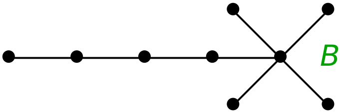

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Upper Bound: Let H be a regular graph with the specified degree and girth at least $l+2$. For even m, also require H to have no $(m-1)$-factor (H exists by a variation on Bollobás–Saito–Wormald [1985])

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Odd m: each color class has at most $(m-1)/2$ edges, but H has $[s(m-1)+1]n/2$ edges.

Even m: no $(m-1)$-factor $\Rightarrow$ each color class has less than $(m-1)n/2$ edges, but $|E(H)| = s(m-1)n/2$. ■

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- The bound is sharp for paths and is twice the true value for brooms.

- Surprisingly, the bound is asymptotically sharp for all $\Delta(G)$, using double-stars.

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For b even and $a < b$, we need a stronger upper bound for $S_{b-1,b}$.

Improved upper bound (b even and $a < b$)

Claim:
$H \to S_{b-1,b}$
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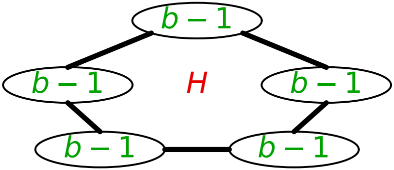

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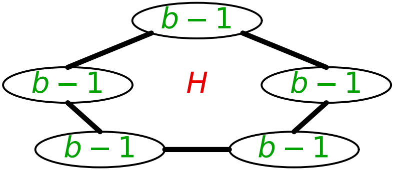

H is $(2b-2)$-regular

Pf. Vertices are majority red or majority blue or tied.
Not all are tied (would be odd regular of odd order).

No $S_{b-1,b}$ $\Rightarrow$ all nbrs (via red) of maj red are maj blue.

A maj red vertex forces a maj blue in each direction;
after 5 steps, one set has a maj red and a maj blue.

Now its neighboring sets together need b maj blue and b maj red vertices, but they have only $2b-2$ total. ■

Double-Stars, s colors

Thm. If $s \geq 2$, then $R_\Delta(S_{a,b}; s) \leq 2(s-1)(b-1)+1$. If $b \geq 2a-1$, then $R_\Delta(S_{a,b}; s) \leq s(b-1)+1$ ($= R_\Delta(K_{1,b}; s)$).

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To avoid $S_{b,b}$, $\exists$ j-minor endpt for each edge of color j.
Each vertex is j-major for some j if $d > s(b-1)$. Hence

$$\frac{nd}{2} = |E(H)| \leq \sum_v \sum_{j \in M(v)} d_j(v) \leq n(s-1)(b-1),$$

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To study $S_{a,b}$, vertices not j-major or j-minor are j-medium. With more careful counting, using $b \geq 2a-1$, the upper bound on d to avoid $S_{a,b}$ becomes $s(b-1)$. ■

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The same idea constructs an s-edge-coloring of $K_{n,n}$ that avoids $S_{b,b}$, where $n = \left\lfloor 2\frac{s-1}{s+1}s(b-1) \right\rfloor$. This gives a lower bound for the "bipartite Ramsey number" of $S_{b,b}$.

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Cor. $R_\Delta(C_3) = 5$.

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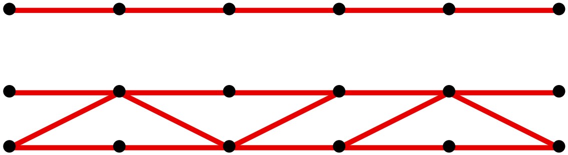
Thm. (Jiang–Milans–West)
$R_\Delta(C_{2k}) \le 96$ and $R_\Delta(C_{2k+1}) \le 3458$.

Trick to Force Cycles

Idea: Force a long even cycle by forcing a blowup of a long monochromatic path.



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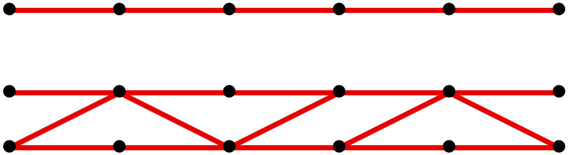

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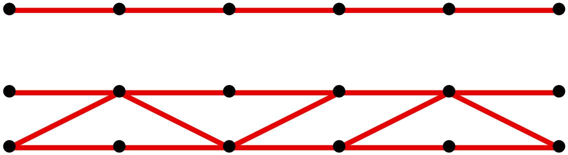

Lem. $K_{3,3} \to P_4$.

Pf. Each vertex of X has a majority in some color.
Two vertices have majority in the same color, say red.
Since $|Y| = 3$, they have a common neighbor in Y. ∎

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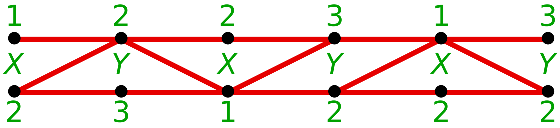
This class contains a cycle; length is $\ge k$; contains P_k.

Since the edges have the same Type, in H they yield pasted copies of P_4 in the same color c.
This yields a monochromatic C_{2k}. ∎

Comments

Same type $\Rightarrow$ can paste X-side onto same vertices, similarly for Y



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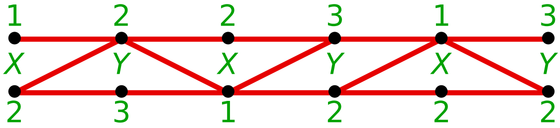

Cor. If F is bipartite and $\Delta(F) = 2$, then $R_\Delta(F) \leq 108$.

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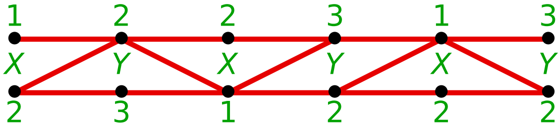

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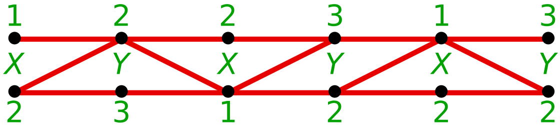

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• Reduction to $R_\Delta(F) \leq 96$ uses that only 8 of the 9 edges in $K_{3,3}$ are needed to arrow P_4.

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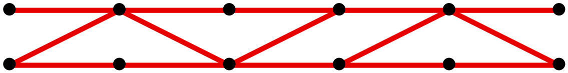

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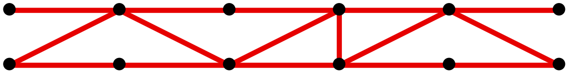

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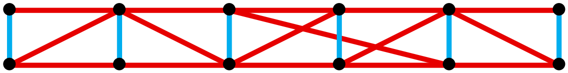

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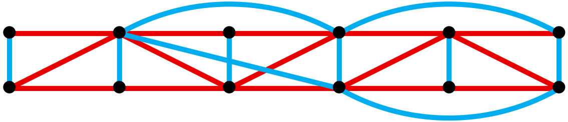

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The Big Question

Ques. Does there exist a function f such that every graph G satisfies $R_\Delta(G) \leq f(\Delta(G))$?

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And Now For Something

Sort Of

Completely Different

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Can Builder playing on $\mathcal{H}$ force a monochromatic G?

On-Line Ramsey Parameters

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- Lower bound for $\mathring{R}_\Delta(C_n) \ge 4$ comes from charzn of $\mathring{R}_\Delta(G) \le 3$, which uses greedy linear-forest Painter .

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- Prove upper bounds on $\mathring{R}_\Delta$ for trees and cycles by algorithms for Builder to defeat a consistent Painter .

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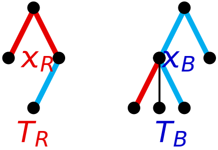
An active vertex becomes
satisfied if it has k children via its own color.
dangerous if it has k incident edges of the other color.

Builder Strategy

Builder plays pendant edges at active vertices (in T_R or T_B) until Painter makes one satisfied or dangerous.



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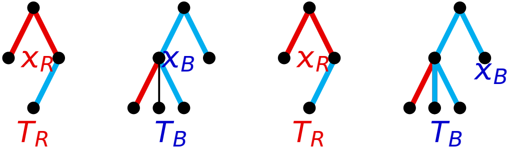

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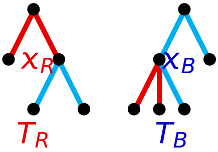

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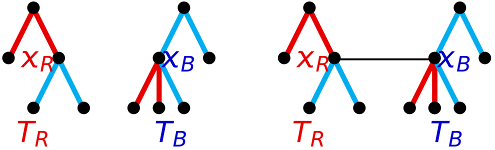

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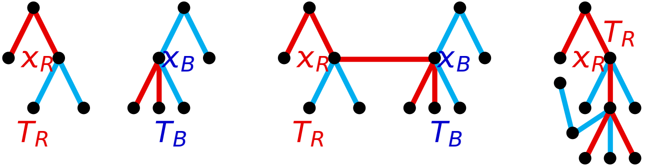

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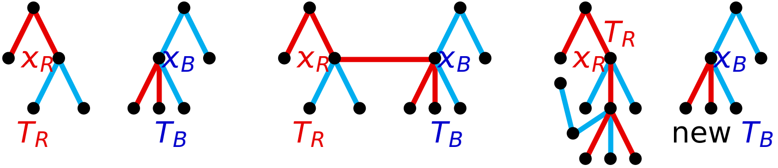
Then Builder regenerates the other tree.



Even Cycles

Assume Builder plays on $\mathcal{S}_k$ and Painter is consistent.
(Weight = bound on total red + blue at a vertex.)

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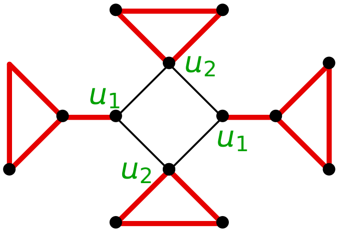
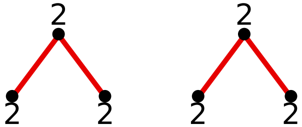
Pf. Builder forces $q/2$ copies of F_1 and F_2 and then adds a cycle alternating between the copies of u_1 and u_2. ■



Trees for Even Cycles

Consistent Painter makes the same monochr. P_3 (with weights 2) in any isolated triangle; we may assume it is red. Painter wants to avoid a monochromatic C_q.



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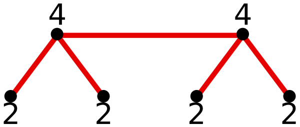

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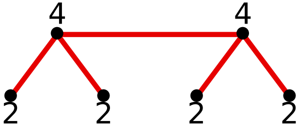
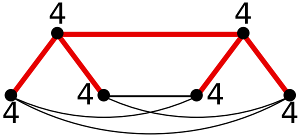


$q = 4$



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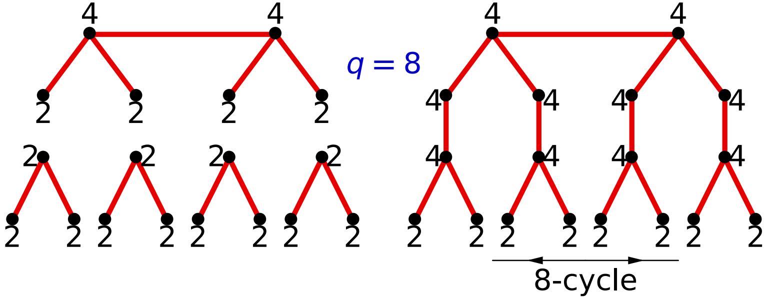

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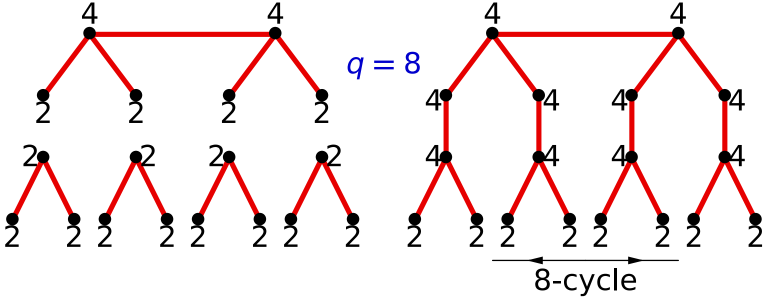
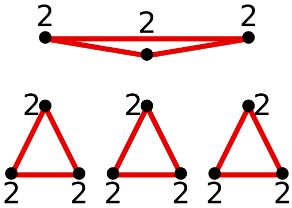


Further extensions of the tree force any even cycle C_q (just extend one half if $q \equiv 2 \mod 4$), but C_6 and C_{10} are special.

Special Case: C_6 (C_{10} is similar)

Consistent Painter makes consistent triangles.

Case 1: monochromatic



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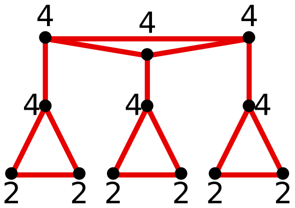

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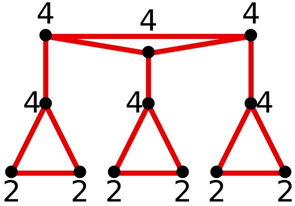

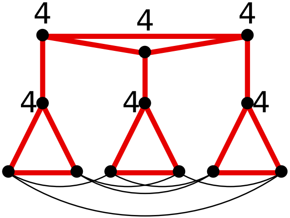

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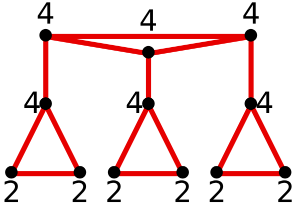

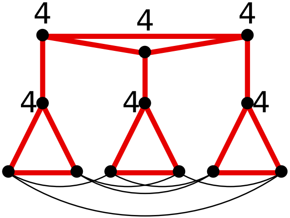

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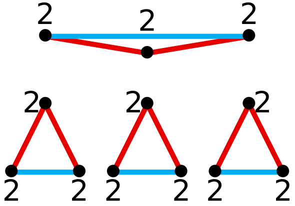

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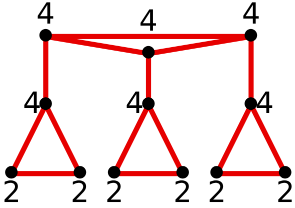

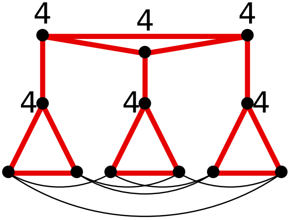

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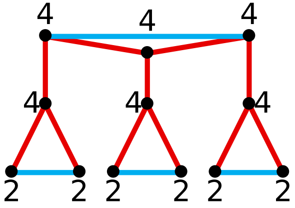

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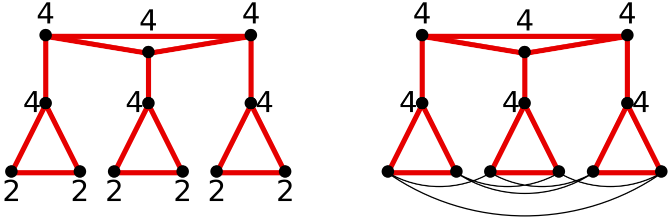

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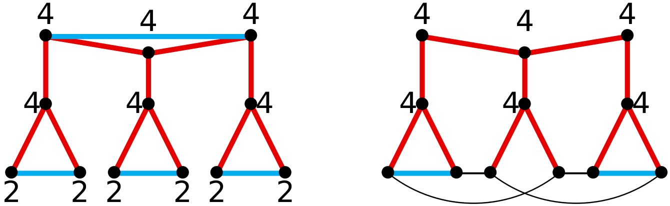

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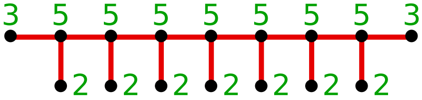

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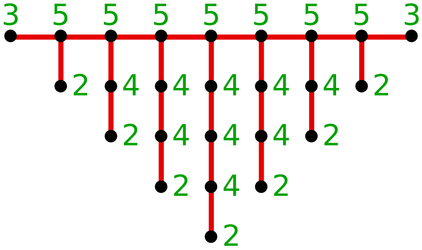

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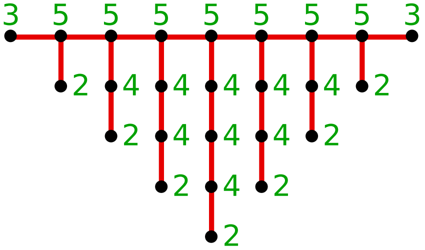

Leaf distances $q-1$ (opposite halves or to middle).
Cycle through the leaves is all blue or some red.

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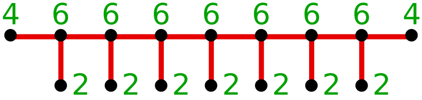

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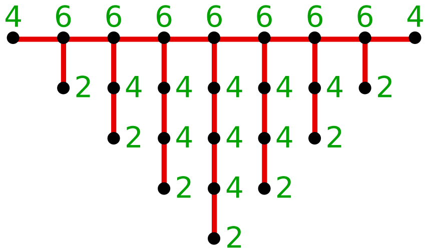

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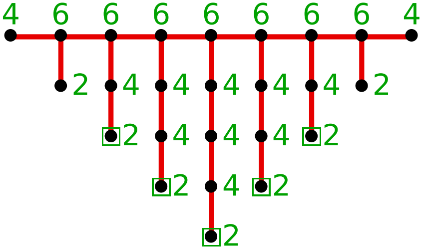

Finally, Builder plays a b-cycle on the leaves.