Degree Ramsey and On-Line Degree Ramsey Numbers

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Joint work with
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Parameter Ramsey Numbers

**Def.** $H \rightarrow G$ means every 2-coloring of $E(H)$ gives a monochromatic $G$. Ramsey’s Theorem $\Rightarrow$ $H$ exists. 
Ramsey number $R(G) = \min\{n : K_n \rightarrow G\}$. 
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- **Extension to many colors:** $R_{\rho}(G; s) = \min\{\rho(H): \text{every } s\text{-coloring of } E(H) \text{ gives monochr. } G\}$. 
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- $R_\rho(G_1, G_2, G_3, \ldots, G_s; s)$ not yet much studied.
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For size Ramsey number, always $R_m(G) \leq \binom{R(G)}{2}$.

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**Thm.** (Beck [1983]) $R_m(P_n) \leq cn$ for some $c$.  

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Size Ramsey number is also linear in $n$ for cycles (Haxell–Kohayakawa–Łuczak [1995]) and bounded-degree trees (Friedman–Pippinger [1987]), but NOT graphs w. maxdegree 3 (Rödl–Szemerédi[2000]).
Chromatic Ramsey Number

For a family $G$, let $R(G; s) = \min\{n: K_n \rightarrow G\}$.

Homomorphism = edge-preserving map $\phi: V(G) \rightarrow V(H)$.

Ex. A proper $k$-coloring is a homomorphism into $K_k$. 
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Thm. (Burr–Erdős–Lovász [1976]) $R_{\chi}(G; s) = R(\mathcal{G}; s)$, where $\mathcal{G}$ is the family of all homomorphic images of $G$. 
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**Pf. (Idea)** Let \( k = R(G; s) \). Apply the bipartite Ramsey theorem repeatedly to a complete \( k \)-partite \( H \) with huge parts to get a complete \( k \)-partite subgraph \( H' \) with parts of size \( |V(G)| \) where the edges joining any two parts have the same color.
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The collapsed coloring of $E(K_k)$ has a monochromatic homomorphic image of $G$, which expands to a monochromatic $G$ in $H'$.
Prop. \[ R_{\chi}(G; s) > (\chi(G) - 1)^s. \]
Chromatic Ramsey Number

**Prop.** $R_{\chi}(G; s) > (\chi(G) - 1)^s$.

**Pf.** Let $k = \chi(G) - 1$.
Any $H$ with $\chi(H) = k^s$ has proper coloring $f: V(H) \to [k]^s$. 
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Any \( H \) with \( \chi(H) = k^s \) has proper coloring \( f : V(H) \rightarrow [k]^s \).

Give each edge \( uv \) in \( H \) a color \( i \) such that \( f(u)_i \neq f(v)_i \).

\( \therefore \) Color \( i \) subgraph \( H_i \) is \( k \)-colorable (by \( f_i \)).
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**Ex.** \( \chi(G) = 3 \Rightarrow 5 \leq R_{\chi}(G) \leq 6 \).

Equality holds in lower bound \( \iff \exists \) hom. \( \phi: G \rightarrow C_5 \).

**Ex.** \( R_{\chi}(C_5) = 5 \).
**Prop.** \( R_\chi(G; s) > (\chi(G) - 1)^s \).

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**Conj.** (BEL [1976]) \( \min\{R_\chi(G) : \chi(G) = k\} = (k-1)^2 + 1 \).
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BEL proved it for \( k \leq 4. \)

Zhu ([1998] for \( k = 5, [2010] \) for all \( k \)) proved it!
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Burr–Erdős–Lovász [1976]: Complete graphs and Stars
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**Obs.** $R_{\chi}(G) \leq R_{\Delta}(G) \leq R(G) - 1$; equality for $G = K_n$. 
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**Thm.** (BEL): \( R_{\Delta}(K_{1,m}) = \begin{cases} 2m - 2 & m \text{ even} \\ 2m - 1 & m \text{ odd} \end{cases} \).
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- Valid as a lower bound whenever $\Delta(G) = m$.
- We have various results for trees and cycles, some for multiple colors: $R_\Delta(G; s) = \min\{\Delta(H): H \rightarrow^s G\}$.
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**Thm.** \( R_\Delta(K_{1,m}; s) = \begin{cases} s(m - 1) & m \text{ even} \\ s(m - 1) + 1 & m \text{ odd} \end{cases} \).
\[ s(m - 1) \leq R_\Delta(K_{1,m}; s) \leq s(m - 1) + 1 \]

**Pf.** Upper Bound: \( K_{1, s(m-1)+1} \overset{s}{\rightarrow} K_{1,m} \).
\[ s(m - 1) \leq R_\Delta(K_{1,m}; s) \leq s(m - 1) + 1 \]

**Pf.** Upper Bound: \( K_{1,s(m-1)+1} \xrightarrow{s} K_{1,m} \).

**Improves when** \( m \) **is even:**
When \( k \) is odd and \( r > k \), there is an \( r \)-regular graph \( H \) having no \( k \)-factor (Bollobás–Saito–Wormald [1985]).
With \( k = m - 1 \) and \( r = s(m - 1) \), \( s \)-coloring \( E(H) \) with no monochromatic \( K_{1,m} \) requires a \( k \)-factorization.
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**Lower bound:** When \( \Delta(H) \leq s(m - 1) - 1 \), Vizing’s Theorem \( \Rightarrow \ H \) is \( s(m - 1) \)-edge-colorable.
Put \( m - 1 \) matchings into each color.
\[ s(m - 1) \leq R_{\Delta}(K_{1,m}; s) \leq s(m - 1) + 1 \]

**Pf.** Upper Bound: \( K_{1,s(m-1)+1} \xrightarrow{S} K_{1,m} \).

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**Improves** when \( m \) is odd. When \( \Delta(H) \leq s(m - 1) \), Petersen’s Theorem decomposes \( s(m - 1) \)-regular supergraph \( H' \) into 2-factors. Putting \( (m - 1)/2 \) in each color avoids degree \( m \) in one color at any vertex.  
\[ \blacksquare \]
Paths

Thm. \( R_\Delta(P_n) = \begin{cases} 
3 & n \in \{4, 5\} \\
3 \text{ or } 4 & n = 6 \quad \leftarrow \text{Open} \\
4 & n \geq 7
\end{cases} \).
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\[ R_\Delta(P_4) \leq 3: \text{ Petersen } \rightarrow P_4 \]
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**Thm.** (Alon–Ding–Oporowski–Vertigan [2003])
\[ R_\Delta(P_n; s) \leq 2s \text{ always.} \quad R_\Delta(P_n; s) = 2s \text{ for } n > n_0(s). \]
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Lower bound: \( \exists \ n_0(s) \) such that every graph $H$ with $\Delta(H) = 2s - 1$ has an $s$-edge-coloring where all monochr. components have at most $n_0(s)$ edges.
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Upper bound:

Let $H$ be $2s$-regular with girth $\geq n$, and $m = |V(H)|$. 
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Upper bound:
Let $H$ be 2s-regular with girth $\geq n$, and $m = |V(H)|$.
$s$ colors on $sm$ edges puts $\geq m$ in some color class. Since $|V(H)| = m$, this subgraph has a cycle. Since girth$(H) \geq n$, this color class contains $P_n$.  ■
Brooms

**Def.** Broom $B_{l,m} = \text{tree with } l + m \text{ vertices formed by replacing an edge of } K_{1,m} \text{ with a path of length } l.$
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- $R_\Delta(K_{1,m}; s)$ grows with $m$, but $R_\Delta(P_l; s) \leq 2s$. 

![Diagram of B4,5 tree](image-url)
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\[ B_{4,5} \]

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Upper Bound: Let $H$ be a regular graph with the specified degree and girth at least $l + 2$. For even $m$, also require $H$ to have no $(m - 1)$-factor ($H$ exists by a variation on Bollobás–Saito–Wormald [1985]).
In an $s$-edge-coloring, let $H_i$ be the spanning subgraph of $H$ using color $i$. Split $H_i$ into two subgraphs:
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A cycle $C$ in $A$ has length $\geq l + 2$, but then a path in $A$ from a high-degree vertex to $C$ yields $B_{l,m} \subseteq A$). Hence

$$|E(H_i)| \leq |V(A)| + \frac{m - 1}{2} |V(B)| \leq \frac{m - 1}{2} n;$$

equality only if $H_i$ is $(m - 1)$-regular. (Here $n = |V(H)|$.)
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Odd $m$: each color class has at most $(m-1)/2$ edges, but $H$ has $[s(m-1)+1]n/2$ edges.

Even $m$: no $(m-1)$-factor $\Rightarrow$ each color class has less than $(m-1)n/2$ edges, but $|E(H)| = s(m-1)n/2$.  

Prop. If $G$ is a tree, then $R_\Delta(G; s) \leq 2s(\Delta(G) - 1)$. 

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Prop. If $G$ is a tree, then $R_{\Delta}(G; s) \leq 2s(\Delta(G) - 1)$.

Pf. Let $H$ be $2s(\Delta(G) - 1)$-regular with girth $> \text{diam}(G)$. $s$-coloring $E(H)$ $\Rightarrow$ avgdeg $\geq 2(\Delta(G) - 1)$ in some color $i$. Color $i$ has a subgraph with minimum degree $\geq \Delta(G)$. 
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- The bound is sharp for paths and is twice the true value for brooms.
- Surprisingly, the bound is asymptotically sharp for all $\Delta(G)$, using double-stars.
Double-Stars

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**Pf.** Lower bound: $R_\Delta(S_{a,b}) \geq R_\Delta(K_{1,b})$ suffices, except for $S_{b,b}$ with $b$ even. Alternating red and blue along an Eulerian circuit of a $(2b - 2)$-regular graph avoids $S_{b,b}$. 
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For $b$ even and $a < b$, we need a stronger upper bound for $S_{b-1,b}$.
Improved upper bound \((b \text{ even and } a < b)\)

Claim: \(H \rightarrow S_{b-1,b}\) for \(b\) even

\(H\) is \((2b-2)\)-regular
Improved upper bound (\(b\) even and \(a < b\))

Claim:
\[H \rightarrow S_{b-1,b}\]
for \(b\) even

\(H\) is \((2b - 2)\)-regular

\textbf{Pf.} Vertices are majority \textcolor{red}{red} or majority \textcolor{blue}{blue} or tied. Not all are tied (would be odd regular of odd order).

No \(S_{b-1,b}\) \Rightarrow all nbrs (via \textcolor{red}{red}) of maj \textcolor{red}{red} are maj \textcolor{blue}{blue}.

A maj \textcolor{red}{red} vertex forces a maj \textcolor{blue}{blue} in each direction; after 5 steps, one set has a maj \textcolor{red}{red} and a maj \textcolor{blue}{blue}.

Now its neighboring sets together need \(b\) maj \textcolor{blue}{blue} and \(b\) maj \textcolor{red}{red} vertices, but they have only \(2b - 2\) total. ■
Double-Stars, $s$ colors

**Thm.** If $s \geq 2$, then $R_\Delta(S_{a,b}; s) \leq 2(s - 1)(b - 1) + 1$. If $b \geq 2a - 1$, then $R_\Delta(S_{a,b}; s) \leq s(b - 1) + 1 \ (= R_\Delta(K_{1,b}; s))$. 
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**Pf.** Let $H$ be a $d$-regular triangle-free $n$-vertex graph. Given a coloring, let $d_j(\nu) =$\#edges with color $j$ at $\nu$. $\nu$ is $j$-major when $d_j(\nu) \geq b$ and $j$-minor when $d_j(\nu) < a$. 
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To avoid $S_{b,b}$, exists $j$-minor endpt for each edge of color $j$. Each vertex is $j$-major for some $j$ if $d > s(b - 1)$. Hence

$$\frac{nd}{2} = |E(H)| \leq \sum_{\nu} \sum_{j \in M(\nu)} d_j(\nu) \leq n(s - 1)(b - 1),$$

so $d \leq 2(s - 1)(b - 1)$. 
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To study $S_{a,b}$, vertices not $j$-major or $j$-minor are $j$-medium. With more careful counting, using $b \geq 2a - 1$, the upper bound on $d$ to avoid $S_{a,b}$ becomes $s(b - 1)$. ■
Asymptotic Sharpness

**Thm.** $R_{\Delta}(S_{b,b},s) > (2-\epsilon)s(b-1)$ for sufficiently large $b$. 
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**Pf. (Idea)** Consider $G$ with $\Delta(G) = (2 - \epsilon)s(b - 1)$. Choose for each vertex a random color. When adjacent vertices have the same color, give the edge some other color, at random. When adjacent vertices have different colors, give the edge one of those two colors.
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The same idea constructs an $s$-edge-coloring of $K_{n,n}$ that avoids $S_{b,b}$, where $n = \left\lceil 2 \frac{s - 1}{s + 1} s(b - 1) \right\rceil$. This gives a lower bound for the “bipartite Ramsey number” of $S_{b,b}$. 

Fixed Cycles, s colors

Cycles are much more difficult to analyze than paths.
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**Pf.** If \( \Delta(H) < 2s \), then \( H \) decomposes into \( s \) forests, by Nash-Williams’ Arboricity Formula.
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**Thm.** \( R_\Delta(C_{2k+1}; s) \geq 2^s + 1 \).
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**Pf.** $R_{\Delta}(G; s) \geq R_{\chi}(G; s) \geq (\chi(G) - 1)^s + 1$.  


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**Cor.** $R_{\Delta}(C_3) = 5$. 

Fixed Cycles, 2 colors

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**Ques.** Is $R_\Delta(C_n)$ bounded?
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**Thm.** (Jiang–Milans–West)

$R_\Delta(C_{2k}) \leq 96$ and $R_\Delta(C_{2k+1}) \leq 3458$. 
**Trick to Force Cycles**

**Idea:** Force a long even cycle by forcing a blowup of a long monochromatic path.
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\[ \text{Lem. } K_{3,3} \rightarrow P_4. \]
Trick to Force Cycles

**Idea:** Force a long even cycle by forcing a blowup of a long monochromatic path.

![Graph showing the idea](image)

**Lem.** $K_{3,3} \rightarrow P_4$.

**Pf.** Each vertex of $X$ has a majority in some color. Two vertices have majority in the same color, say red. Since $|Y| = 3$, they have a common neighbor in $Y$. ![End of proof]

Even Cycles

**Thm.** \( R_\Delta(C_{2k}) \leq 108. \)
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**Pf.** Let \( G \) be a 36-regular \( X,Y \)-bigraph with girth \( \geq k \). Let \( H = G[\overline{K}_3] \); this is 108-regular. Claim: \( H \rightarrow C_{2k} \).
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**Pf.** Let $G$ be a 36-regular $X,Y$-bigraph with girth $\geq k$. Let $H = G\overline{K_3}$; this is 108-regular. Claim: $H \rightarrow C_{2k}$.

Consider a 2-coloring $f$ of $E(H)$. Each edge $xy \in E(G)$ becomes $K_{3,3}$ with parts $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$. 
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Consider a 2-coloring $f$ of $E(H)$. Each edge $xy \in E(G)$ becomes $K_{3,3}$ with parts $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$. Say that $xy$ has \textbf{Type} $(c; i, j)$ if the resulting $P_4$ in the $xy$-copy of $K_{3,3}$ has color $c$ and omits $x_i$ and $y_j$. 
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The **18 Types** yield an **18-coloring** of \( E(G) \). Some color class has average degree at least **2**.
**Even Cycles**

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This class contains a cycle; length is \( \geq k \); contains \( P_k \).
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Consider a 2-coloring $f$ of $E(H)$. Each edge $xy \in E(G)$ becomes $K_{3,3}$ with parts $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$.

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The 18 Types yield an 18-coloring of $E(G)$. Some color class has average degree at least 2.

This class contains a cycle; length is $\geq k$; contains $P_k$.

Since the edges have the same Type, in $H$ they yield pasted copies of $P_4$ in the same color $c$.
This yields a monochromatic $C_{2k}$. ■
Comments

Same type ⇒ can paste $X$-side onto same vertices, similarly for $Y$
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Cor. If $F$ is bipartite and $\Delta(F) = 2$, then $R_\Delta(F) \leq 108$. 
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- Bipartite $G$ helps pasting but gives only even cycles. In fact, the resulting $H$ was also bipartite.
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Cor. If $F$ is bipartite and $\Delta(F) = 2$, then $R_{\Delta}(F) \leq 108$.

- Bipartite $G$ helps pasting but gives only even cycles. In fact, the resulting $H$ was also bipartite.

- Reduction to $R_{\Delta}(F) \leq 96$ uses that only 8 of the 9 edges in $K_{3,3}$ are needed to arrow $P_4$. 
Odd Cycles

**Thm.** $R_{\Delta}(C_{2k-1}) \leq 3890$. 
Odd Cycles

**Thm.** \( R_\Delta(C_{2k-1}) \leq 3890 \).

**Pf.** Let \( G \) be a 36-regular \( X,Y \)-bigraph with girth \( > 2k \). Let \( H = G^2[K_3] \); this is 3890-regular \((3 \times 36^2 + 2)\). Claim: \( H \to C_{2k-1} \).
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Claim: $H \to C_{2k-1}$.

Consider 2-coloring $f$ of $E(H)$. Again make 18-coloring of $E(G)$ (not $G^2$); it has monochromatic $P_{2k}$, say red.
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Pf. Let $G$ be a 36-regular $X,Y$-bigraph with girth $> 2k$. Let $H = G^2[K_3]$; this is 3890-regular $(3 \times 36^2 + 2)$. Claim: $H \rightarrow C_{2k-1}$.

Consider 2-coloring $f$ of $E(H)$. Again make 18-coloring of $E(G)$ (not $G^2$); it has monochromatic $P_{2k}$, say red.

Any red edge "inside" $\Rightarrow$ red $C_{2k-1}$. 
Odd Cycles

**Thm.** $R_\Delta(C_{2k-1}) \leq 3890$.

**Pf.** Let $G$ be a 36-regular $X,Y$-bigraph with girth $> 2k$. Let $H = G^2[K_3]$; this is 3890-regular ($3 \times 36^2 + 2$). Claim: $H \rightarrow C_{2k-1}$.

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Any red edge "inside" $\Rightarrow$ red $C_{2k-1}$.

If all are blue, consider the added edges of $G^2$ joining alternate pairs along the path. Any red $\Rightarrow$ red $C_{2k-1}$. 

![Graph diagram](image-url)
Odd Cycles

**Thm.** \( R_\Delta(C_{2k-1}) \leq 3890. \)

**Pf.** Let \( G \) be a 36-regular \( X,Y \)-bipartite graph with girth > 2k. Let \( H = G^2[K_3] \); this is 3890-regular \((3 \times 36^2 + 2)\).

Claim: \( H \rightarrow C_{2k-1} \).

Consider 2-coloring \( f \) of \( E(H) \). Again make 18-coloring of \( E(G) \) (not \( G^2 \)); it has monochromatic \( P_{2k} \), say red.

Any red edge "inside" \( \Rightarrow \) red \( C_{2k-1} \).

If all are blue, consider the added edges of \( G^2 \) joining alternate pairs along the path. Any red \( \Rightarrow \) red \( C_{2k-1} \).

If all are blue, then we have a blue \( C_{2k-1} \). \( \blacksquare \)
The Big Question

**Ques.** Does there exist a function $f$ such that every graph $G$ satisfies $R_{\Delta(G)} \leq f(\Delta(G))$?

The answer is yes for $\Delta(G) = 2$, but maybe it is unbounded for $\Delta(G) = 3$. 
The Big Question

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And Now For Something Sort Of Completely Different
The On-line Ramsey Problem

Graph Ramsey theory = a game
Builder presents a graph; Painter 2-colors the edges. Builder wins if a monochromatic $G$ is produced.
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“Arrow” $H \rightarrow G \iff$ Builder wins by playing $H$. 
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Builder presents one edge at a time; Painter colors it. Builder wins if a monochromatic $G$ is produced.
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Idea: Restrict Builder to a hereditary family $\mathcal{H}$.
After every move, the graph presented so far lies in $\mathcal{H}$.
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Builder presents one edge at a time; Painter colors it.
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Idea: Restrict Builder to a hereditary family $\mathcal{H}$.
After every move, the graph presented so far lies in $\mathcal{H}$.
This defines the on-line Ramsey game $(G, \mathcal{H})$.
Can Builder playing on $\mathcal{H}$ force a monochromatic $G$?
Def. For a monotone graph parameter $\rho$, the on-line $\rho$-Ramsey number $\hat{R}_\rho(G)$ of $G$ is $\min\{k: \text{Builder wins } (G, F_k)\}$, where $F_k = \{H: \rho(H) \leq k\}$. 

**On-Line Ramsey Parameters**
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Grytczuk–Hałuszczak–Kierstead [2004] $\hat{R}_\chi(G) = \chi(G)$. 
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** Conj.** (GHK) When $\mathcal{H} = \{\text{planar}\}$, Builder wins $(G, \mathcal{H})$ if and only if $G$ is outerplanar. Disproved by Petříčková.
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Beck [1993] - introduced on-line size Ramsey number

Grytczuk–Kierstead–Prałat [2008] For $P_n$ at most $4n - 7$, but for trees it can be quadratic.

**Def.** on-line degree Ramsey number $\hat{R}_\Delta(G) = \min \{ k : \text{Builder wins } (G, \mathcal{S}_k) \}, \text{ where } \mathcal{S}_k = \{ H : \Delta(H) \leq k \}.$
On-line Degree Ramsey Results

Obs. $\hat{R}_\Delta(G) \leq R_\Delta(G)$ for all $G$ (always $\hat{R}_\rho(G) \leq R_\rho(G)$).
On-line Degree Ramsey Results

**Obs.** \( \hat{R}_\Delta(G) \leq R_\Delta(G) \) for all \( G \) (always \( \hat{R}_\rho(G) \leq R_\rho(G) \)).

**Thm.** \( \hat{R}_\Delta(G) \leq 3 \iff \) each component of \( G \) is a path or each component is a subgraph of \( K_{1,3} \).
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**Thm.** $\hat{R}_\Delta(G) \leq 2\Delta(G) - 1$ when $G$ is a tree, sharp when $\exists$ adjacent maxdeg vertices. (Also $\hat{R}_\Delta(S_{a,b}) = a+b-1$.)
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Kinnersley: $\hat{R}_\Delta(T_1, \ldots, T_s; s) \leq 1 + \sum_{i=1}^{s}(\Delta(T_i) - 1)$. 
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**Thm.** $4 \leq \hat{R}_\Delta(C_n) \leq 5$, equal to 4 if $n$ is even or large.
On-line Degree Ramsey Results

**Obs.** $\hat{R}_\Delta(G) \leq R_\Delta(G)$ for all $G$ (always $\hat{R}_\rho(G) \leq R_\rho(G)$).

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D. Rolnick proved $\hat{R}_\Delta(C_n) = 4$ for all $n$. 
On-line Degree Ramsey Results

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**Ques.** Is $\hat{R}_\Delta(G)$ bounded by a function of $\Delta(G)$?
On-line Degree Ramsey Results

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**Ques.** Is $\hat{R}_\Delta(G)$ bounded by a function of $\Delta(G)$?

**Thm.** $\hat{R}_\Delta(G) \leq 6$ if $\Delta(G) \leq 2$. 
Lower Bounds – Greedy Painter

**Def.** The greedy $\mathcal{F}$-Painter colors each edge red if the resulting red graph lies in $\mathcal{F}$; otherwise blue.
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**Thm.** $\hat{\Delta}(G) \geq \Delta(G) - 1 + \max_{uv \in E(G)} \min\{d(u), d(v)\}$.
Lower Bounds – Greedy Painter

**Def.** The greedy \( \mathcal{F} \)-Painter colors each edge red if the resulting red graph lies in \( \mathcal{F} \); otherwise blue.

**Thm.** \( \hat{\chi}_\Delta(G) \geq \Delta(G) - 1 + \max_{uv \in E(G)} \min\{d(u), d(v)\} \).

**Pf.** Let \( m = \Delta(G) \). \( S_{m-1} \)-Painter never makes red \( G \). An edge gets blue \iff\ an endpt already has \( m - 1 \) red.
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**Pf.** Let $m = \Delta(G)$. $S_{m-1}$-Painter never makes red $G$. An edge gets blue $\iff$ an endpt already has $m - 1$ red. Let $xy$ be an edge with maxmin degree in $G$. A blue $G$ has an edge for $xy$; it has $m - 1$ red at one endpt and at least $\min \{d_G(x), d_G(y)\}$ blue there. ■
Def. The greedy \( F \)-Painter colors each edge red if the resulting red graph lies in \( F \); otherwise blue.

Thm. \( \hat{\Delta}(G) \geq \Delta(G) - 1 + \max_{uv \in E(G)} \min \{d(u), d(v)\}. \)

Pf. Let \( m = \Delta(G) \). \( S_{m-1} \)-Painter never makes red \( G \). An edge gets blue \( \iff \) an endpt already has \( m - 1 \) red.

Let \( xy \) be an edge with maxmin degree in \( G \). A blue \( G \) has an edge for \( xy \); it has \( m - 1 \) red at one endpt and at least \( \min \{d_G(x), d_G(y)\} \) blue there.

Cor. \( \hat{\Delta}(P_n) \geq 3; \hat{\Delta}(K_{1,m}) \geq m; \hat{\Delta}(S_{a,b}) \geq a + b - 1; \)
\( \hat{\Delta}(G) \geq 2\Delta(G) - 1 \) if \( \exists \) adjacent maxdegree vertices.
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**Cor.** $\hat{R}_\Delta(P_n) \geq 3$; $\hat{R}_\Delta(K_{1,m}) \geq m$; $\hat{R}_\Delta(S_{a,b}) \geq a + b - 1$; $\hat{R}_\Delta(G) \geq 2\Delta(G) - 1$ if $\exists$ adjacent maxdegree vertices.

- Lower bound for $\hat{R}_\Delta(C_n) \geq 4$ comes from charzn of $\hat{R}_\Delta(G) \leq 3$, which uses greedy linear-forest Painter.
Upper Bounds – Consistent Painter

**Def.** Painter follows a consistent strategy if the color used on a new edge depends only on the current 2-colored component(s) containing its endpoints (regardless of what else has been played).
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**Thm.** If $\mathcal{H}$ is an additive family (closed under disjoint unions), and $\mathcal{A}$ is a Painter strategy on $\mathcal{H}$, then there is a consistent Painter strategy $\mathcal{A}'$ on $\mathcal{H}$ such that for any list $E'$ presented by Builder, there is another list $E$ such that $\mathcal{A}'(E') \subseteq \mathcal{A}(E)$ (as 2-colored graphs).
Def. Painter follows a consistent strategy if the color used on a new edge depends only on the current 2-colored component(s) containing its endpoints (regardless of what else has been played).

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Cor. To prove that $\hat{R}_\Delta(G) \leq k$, it suffices to show that Builder can win against any consistent Painter on $S_k$. 
Def. Painter follows a consistent strategy if the color used on a new edge depends only on the current 2-colored component(s) containing its endpoints (regardless of what else has been played).

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Cor. To prove that $\mathring{\mathcal{R}}_\Delta(G) \leq k$, it suffices to show that Builder can win against any consistent Painter on $S_k$.

• Prove upper bounds on $\mathring{\mathcal{R}}_\Delta$ for trees and cycles by algorithms for Builder to defeat a consistent Painter.
Thm. If $G$ is a tree, then $\hat{R}_\Delta(G) \leq 2\Delta(G) - 1$. 
Trees

**Thm.** If $G$ is a tree, then $\mathcal{R}_\Delta(G) \leq 2\Delta(G) - 1$.

**Pf.** Idea: Builder forces a large monochromatic complete $k$-ary tree, where $k = \Delta(G) - 1$. 
**Thm.**  If $G$ is a tree, then $\hat{R}_\Delta(G) \leq 2\Delta(G) - 1$.

**Pf.**  Idea: Builder forces a large monochromatic complete $k$-ary tree, where $k = \Delta(G) - 1$.

Candidate tree $T_R$ or $T_B$ has an active vertex $x_R$ or $x_B$ - a vertex of least depth w/o $k$ children via its own color.
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Candidate tree $T_R$ or $T_B$ has an active vertex $x_R$ or $x_B$ - a vertex of least depth w/o $k$ children via its own color.

Invariant: In $T_R$, each vertex other than $x_R$ either 1) is a leaf in $T_R$ with no other incident edge, or 2) has $k$ red children and at most $k$ blue incident edges. (Symmetrically for $T_B$).
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Pf. Idea: Builder forces a large monochromatic complete $k$-ary tree, where $k = \Delta(G) - 1$.

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Invariant: In $T_R$, each vertex other than $x_R$ either
1) is a leaf in $T_R$ with no other incident edge, or
2) has $k$ red children and at most $k$ blue incident edges.

(Symmetrically for $T_B$).

An active vertex becomes satisfied if it has $k$ children via its own color.
dangerous if it has $k$ incident edges of the other color.
Builder Strategy

**Builder** plays pendant edges at active vertices (in $T_R$ or $T_B$) until **Painter** makes one satisfied or dangerous.
Builder Strategy

**Builder** plays pendant edges at active vertices (in $T_R$ or $T_B$) until **Painter** makes one satisfied or dangerous.

When an active vertex is satisfied, **Builder** rechooses it (closest to root w/o $k$ children via its own color).
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Builder plays pendant edges at active vertices (in $T_R$ or $T_B$) until Painter makes one satisfied or dangerous.

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If $x_R$ and $x_B$ are both dangerous,
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If $x_R$ and $x_B$ are both dangerous, Builder plays $x_Rx_B$. 
Builder Strategy

Builder plays pendant edges at active vertices (in $T_R$ or $T_B$) until Painter makes one satisfied or dangerous.

When an active vertex is satisfied, Builder rechooses it (closest to root w/o $k$ children via its own color).

If $x_R$ and $x_B$ are both dangerous, Builder plays $x_R x_B$.

This edge enters the tree for its color, dragging the other tree with it.
Builder Strategy

Builder plays pendant edges at active vertices (in $T_R$ or $T_B$) until Painter makes one satisfied or dangerous.

When an active vertex is satisfied, Builder rechooses it (closest to root w/o $k$ children via its own color).

If $x_R$ and $x_B$ are both dangerous, Builder plays $x_R x_B$.

This edge enters the tree for its color, dragging the other tree with it.

Then Builder regenerates the other tree.
Even Cycles

Assume Builder plays on $S_k$ and Painter is consistent. (Weight = bound on total red + blue at a vertex.)

**Lem.** Let $F_1, F_2$ be weighted graphs Builder can force in red, with vertices $u_1, u_2$. Form $F$ from $F_1 + F_2$ by adding $u_1u_2$ and increasing weights on $u_1$ and $u_2$ by 2. If $q$ is even, then Builder can force a red $F$ or a blue $C_q$. 
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**Pf.** Builder forces $q/2$ copies of $F_1$ and $F_2$ and then adds a cycle alternating between the copies of $u_1$ and $u_2$. ■
Trees for Even Cycles

Consistent Painter makes the same monochromatic $P_3$ (with weights 2) in any isolated triangle; we may assume it is red. Painter wants to avoid a monochromatic $C_q$. 

![Diagram showing two trees with weights 2, one red and one black.](image)
Trees for Even Cycles

Consistent Painter makes the same monochromatic $P_3$ (with weights 2) in any isolated triangle; we may assume it is red. Painter wants to avoid a monochromatic $C_q$. 

![Diagram of a tree with even cycles]

- Node labeled 4 is connected to two nodes labeled 2.
- Node labeled 4 is connected to two nodes labeled 2.

Weights: 2, 2, 2, 2.
Trees for Even Cycles

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Trees for Even Cycles

Consistent Painter makes the same monochromatic $P_3$ (with weights 2) in any isolated triangle; we may assume it is red. Painter wants to avoid a monochromatic $C_q$.

Further extensions of the tree force any even cycle $C_q$ (just extend one half if $q \equiv 2 \pmod{4}$), but $C_6$ and $C_{10}$ are special.
Special Case: $C_6$ ($C_{10}$ is similar)

Consistent Painter makes consistent triangles.

Case 1: monochromatic
Special Case: $C_6$ ($C_{10}$ is similar)

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Case 1: monochromatic
Special Case: $C_6$ ($C_{10}$ is similar)

Consistent Painter makes consistent triangles.

**Case 1**: monochromatic

**Case 2**: not monochromatic
Special Case: $C_6$ ($C_{10}$ is similar)

Consistent Painter makes consistent triangles.

Case 1: monochromatic

Case 2: not monochromatic
Special Case: $C_6$ ($C_{10}$ is similar)

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Case 1: monochromatic

Case 2: not monochromatic
**Lem.** Against consistent Painter, if Builder can force red $F$ or monochr. $C_q$ ($q$ odd), then Builder can force red $F+uv$ or monochr. $C_q$, with wt on $u$ and $v$ up by 2.
Odd Cycles

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**Thm.** $\hat{R}_\Delta(C_q) \leq 5$ when $q$ is odd.
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**Pf.** Force monochr. $P_q$ (say red) with weights 3. Grow pendant paths.

```
3 5 5 5 5 5 5 5 3
2 2 2 2 2 2 2
```
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```
  3  5  5  5  5  5  5  5  3
  2  4  4  4  4  4  4  2
  2  4  4  4  4  2
  2  2
```

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Leaf distances $q - 1$ (opposite halves or to middle). Cycle through the leaves is all blue or some red.
Idea for Max Degree 2

\[ \Delta(G) = 2; \text{ we may assume each component is a cycle.} \]
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$\Delta(G) = 2$; we may assume each component is a cycle.

If Builder can’t force $G$ against a consistent Painter, then $\exists \; r$ and $b$ where Builder can’t force red $C_r$ or blue $C_b$. One Case: both are odd ($b \leq r$ by symmetry).
\( \Delta(G) = 2 \); we may assume each component is a cycle.

If Builder can’t force \( G \) against a consistent Painter, then \( \exists \ r \) and \( b \) where Builder can’t force red \( C_r \) or blue \( C_b \). One Case: both are odd (\( b \leq r \) by symmetry).

Builder forces red 2-weighted \( P_2 \) by playing a \( b \)-cycle.
Δ(G) = 2; we may assume each component is a cycle.

If Builder can’t force G against a consistent Painter, then ∃ r and b where Builder can’t force red C_r or blue C_b. One Case: both are odd (b ≤ r by symmetry).

Builder forces red 2-weighted P_2 by playing a b-cycle. By earlier lemmas, Builder can force red 4-weighted P_r.
Idea for Max Degree 2

$\Delta(G) = 2$; we may assume each component is a cycle.

If Builder can’t force $G$ against a consistent Painter, then $\exists$ $r$ and $b$ where Builder can’t force red $C_r$ or blue $C_b$. One Case: both are odd $(b \leq r$ by symmetry).

Builder forces red 2-weighted $P_2$ by playing a $b$-cycle.

By earlier lemmas, Builder can force red 4-weighted $P_r$. Similarly, Builder can force the red 6-weighted tree.
Idea for Max Degree 2

\( \Delta(G) = 2 \); we may assume each component is a cycle.

If Builder can’t force \( G \) against a consistent Painter, then \( \exists \ r \) and \( b \) where Builder can’t force red \( C_r \) or blue \( C_b \). One Case: both are odd (\( b \leq r \) by symmetry).

Builder forces red 2-weighted \( P_2 \) by playing a \( b \)-cycle.

By earlier lemmas, Builder can force red 4-weighted \( P_r \). Similarly, Builder can force the red 6-weighted tree.
\[ \Delta(G) = 2; \text{ we may assume each component is a cycle.} \]

If Builder can’t force \( G \) against a consistent Painter, then \( \exists r \) and \( b \) where Builder can’t force red \( C_r \) or blue \( C_b \). One Case: both are odd \((b \leq r \text{ by symmetry})\).

Builder forces red 2-weighted \( P_2 \) by playing a \( b \)-cycle.

By earlier lemmas, Builder can force red 4-weighted \( P_r \). Similarly, Builder can force the red 6-weighted tree.

Finally, Builder plays a \( b \)-cycle on the leaves.