Discrete Mathematics 73 (1989) 309–310
North-Holland

NOTE

A SHORT PROOF OF THE DEGREE BOUND FOR INTERVAL NUMBER

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Received 2 February, 1987

A short proof is given of the fact that every graph has an interval representation of depth 2 in which each vertex v is represented by at most $\lfloor\frac{1}{2}(d(v)+1\rfloor$ intervals, except for an arbitrarily specified vertex w that appears left-most in the representation and is represented by at most $\lceil\frac{1}{2}(d(w)+1)\rceil$ intervals.

A *t-interval representation* of a graph G is a map f assigning to each vertex $v \in V(G)$ the union of at most t intervals on the real line, such that $uv \in E(G)$ if and only if $f(u) \cap f(v) \neq \emptyset$. The *interval number* $i(G)$ of G is the minimum t such that G has t-interval representation. The *depth* of a representation is $\max_{x\in R} |f^{-1}(x)|$; note that any representation of a triangle-free graph has depth 2. An interval in $f(u)$ is *displayed* if it contains some subinterval belonging to no other $f(v)$; a representation of G is *displayed* if every vertex of G is assigned a displayed interval.

Griggs and West [1] proved that $i(G) \leqslant \lceil\frac{1}{2}(\Delta+1)\rceil$, where $\Delta = \max_v d(v)$ and $d(v)$ denotes the degree of vertex v. The result is best possible, with equality attained by any triangle-free regular graph. They proved the upper bound by showing that G has a representation of depth 2 in which a specified vertex w appears leftmost in the representation (i.e. has the smallest image point), and every vertex is assigned at most $\lceil\frac{1}{2}(d(v)+1\rceil$ intervals. Their proof is inductive on the number of edges, with cases depending on whether cycles pass through w. Here we give a short proof of a slightly sharper result.

Theorem. *Given a graph G and a vertex $w \in V(G)$, G has an interval representation of depth 2 in which w is assigned the left-most interval, every interval is displayed, and every vertex is assigned $\lfloor\frac{1}{2}(d(v)+1)\rfloor$ intervals, except that w is assigned $\lceil\frac{1}{2}(d(w)+1)\rceil$ intervals.*

Proof. Form a new graph G' by adding a vertex x and joining it to all vertices of odd degree in G, if there are any. The resulting graph is Eulerian. Let

* Research supported in part by ONR Grant N00014-85K0570 and by NSF grant DMS 8504322.

$C = v_0, v_1, \ldots, v_m$ be an Eulerian circuit with $v_0 = v_m = w$, and if $d(w)$ is odd choose C so the last edge is xw. Assign to v_i the interval $[i - \frac{2}{3}, i + \frac{2}{3}]$. Each interval of this representation of G' (except the first and last) is displayed and accounts for two edges through the vertex. Each $v \in V(G')$ is assigned $\lfloor \frac{1}{2}(d(v) + 1) \rfloor$ intervals, except that w is assigned $\lfloor \frac{1}{2}(d(w) + 1) \rfloor + 1$ intervals. Deleting the intervals for x yields a representation of G. If $d(w)$ is odd, the last interval for w intersects no other interval for G; hence we need only $\lceil \frac{1}{2}(d(w) + 1) \rceil$ intervals for w. □

Reference

[1] J.R. Griggs and D.B. West, Extremal values of the interval number of a graph, SIAM J. Algebraic and Discrete Methods, 1 (1980) 1–7.