The adversary degree-associated reconstruction number of double-brooms

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Abstract

A vertex-deleted subgraph of a graph G is a *card*. A *dacard* specifies the degree of the deleted vertex along with the card. The *adversary degree-associated reconstruction number* $\mathrm{adrn}(G)$ is the least k such that every set of k dacards determines G. We determine $\mathrm{adrn}(D_{m,n,p})$, where the *double-broom* $D_{m,n,p}$ with $p \geq 2$ is the tree with $m+n+p$ vertices obtained from a path with p vertices by appending m leaves at one end and n leaves at the other end. We determine $\mathrm{adrn}(D_{m,n,p})$ for all m,n,p. For $2 \leq m \leq n$, usually $\mathrm{adrn}(D_{m,n,p}) = m+2$, except $\mathrm{adrn}(D_{m,m+1,p}) = m+1$ and $\mathrm{adrn}(D_{m,m+2,p}) = m+3$. There are exceptions when $(m,n) = (2,3)$ or $p = 4$. For $m = 1$ the usual value is 4, with exceptions when $p \in \{2,3\}$ or $n = 2$.

Keywords: degree-associated reconstruction, double-broom, adversary reconstruction.

1 Introduction

The Reconstruction Conjecture of Kelly [6, 7] and Ulam [17] has been open for more than 50 years. For a graph G, the subgraph $G - v$ obtained by deleting a vertex $v \in V(G)$ is a *card* of G. Cards are unlabeled; only the isomorphism class is known. The multiset of cards is the *deck* of G. The Reconstruction Conjecture asserts that every graph with at least three vertices is uniquely determined by its deck. Such a graph is *reconstructible*.

For a reconstructible graph G, Harary and Plantholt [5] introduced the *reconstruction number*, denoted by $\mathrm{rn}(G)$; it is the least k such that some multiset of k cards from the deck of G determines G (meaning that every graph not isomorphic to G shares at most $\mathrm{rn}(G)-1$ of these cards with G). Myrvold [13] proposed the *adversary reconstruction number*, denoted by $\mathrm{arn}(G)$; it is the least k such that every multiset of k cards G determines G. Equivalently, it is

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one more than the maximum number of cards that G shares with another graph, and it equals the difference between the number of vertices and the *fault reconstructibility* (Manvel [8]).

A *degree-associated card* or *dacard* is a pair (C, d) consisting of a card and the degree of the missing vertex. The *dadeck* is the multiset of dacards. From the full deck without degrees, it is easy to compute the degrees of the missing vertices; hence the Reconstruction Conjecture is equivalent to reconstructibility from the dadeck. Without having all the cards, the degree of the vertex missing from a card is hard to compute, and the dacard provides more information. Hence graphs may be reconstructible using fewer dacards than cards.

Ramachandran [14] defined the *degree-associated reconstruction number*, denoted $\mathrm{drn}(G)$, to be the minimum number of dacards that determine G. Since any dacard provides the number of edges in its reconstructions, $\mathrm{drn}(G)$ is the same as the "class reconstruction number" of G when the class is the set of graphs with that number of edges, where Harary and Plantholt [5] introduced the *class reconstruction number* of a graph as the minimum number of cards needed to reconstruct it given that it belongs to a particular class.

Barrus and West [1] proved $\mathrm{drn}(G) \geq 3$ for vertex-transitive graphs and $\mathrm{drn}(G) = 2$ for all caterpillars except stars (value 1) and the one 6-vertex tree with vertex degrees $(3, 3, 1, 1, 1, 1)$ (value 3). Spinoza [15] extended this by proving $\mathrm{drn}(G) = 2$ for graphs G obtained from a caterpillar by arbitrary edge subdivisions followed by possibly adding leaf neighbors at the leaves. Barrus and West [1] conjectured that the maximum of $\mathrm{drn}(G)$ over n-vertex graphs is $n/4+2$, achieved by the disjoint union of two copies of the complete bipartite graph $K_{n/4,n/4}$.

Monikandan et al. [11] introduced the degree-associated analogue of $\mathrm{arn}(G)$ (attributing the notion to Ramachandran). When G is reconstructible from its dadeck, the *adversary degree-associated reconstruction number*, denoted $\mathrm{adrn}(G)$, is the least k such that every set of k dacards determines G. From the definition, $\mathrm{drn}(G) \leq \mathrm{adrn}(G)$. Equality holds when G is vertex-transitive, since when the dacards are pairwise isomorphic there is only one way to have a given number of dacards.

The value of adrn is known for complete graphs, complete bipartite graphs, cycles, and wheels [11], and for paths [16]. In a subsequent paper, Monikandan and Sundar Raj [10] determined adrn for double-stars, for subdivisions of stars, and for the disjoint union of t complete n-vertex graphs and s cycles of length m. The proof in [1] that $\mathrm{drn}(G) \leq \min\{r+2, n-r+1\}$ when G is r-regular with n vertices also implies $\mathrm{adrn}(G) \leq \min\{r+2, n-r+1\}$. (For elementary results on the edge version of adversary degree-associated reconstruction, see [9].)

For most graphs, the values of all these parameters are quite small. Müller [12] proved that for almost all graphs, the subgraphs with more than half the vertices (by a constant fraction) are non-isomorphic (this was also shown earlier by Korshunov), and he noted that this implies that almost all graphs are reconstructible. Independently, Bollobás [2] reproved these statements and also showed that G is reconstructible from any three vertex-deleted subgraphs when the induced subgraphs omitting two vertices are nonisomorphic and have no nontrivial automorphisms. This implies not only that $\mathrm{rn}(G) \le 3$ but also $\mathrm{arn}(G) \le 3$ for almost all graphs, as noted also by Myrvold [13]. Using this, Barrus and West [1] obtained $\mathrm{drn}(G) = 2$ for almost all graphs [1].

Since $\mathrm{adrn}(G) \le \mathrm{arn}(G)$, it is thus of some interest to find graphs G where $\mathrm{adrn}(G)$ is large. Bowler, Brown, and Fenner [3] constructed infinite families of pairs of graphs in which the pairs with n vertices have $2\lfloor (n-4)/3 \rfloor$ common dacards, so $\mathrm{adrn}(G)$ can be as large as $2\lfloor (n-4)/3 \rfloor + 1$. They conjecture that this is the largest value for n-vertex graphs. Among vertex-transitive graphs, $G = 2K_{n/4,n/4}$ in [1] achieves $\mathrm{adrn}(G) = \mathrm{drn}(G) = \frac{1}{4}|V(G)| + 2$.

In this paper we determine adrn for all double-brooms. The *double-broom* $D_{m,n,p}$ with $p \ge 2$ is the tree with $m+n+p$ vertices obtained from a path with p vertices by appending m leaf neighbors at one end and n leaf neighbors at the other end. Every double-broom is a caterpillar, so $\mathrm{drn}(D_{m,n,p}) = 2$ (except for stars and $D_{2,2,2}$). We summarize our results below. In particular, since $\mathrm{adrn}(D_{m,m,2}) = m+2$, the value can be as large as $\frac{1}{2}|V(G)| + 1$.

Theorem 1.1. *If* $p \ge 6$*, then* $\mathrm{adrn}(D_{2,3,p}) = 4$*. Otherwise, for* $2 \le m \le n$ *and* $p \ne 4$*, always* $\mathrm{adrn}(D_{m,n,p}) = m+2$*, except* $\mathrm{adrn}(D_{m,m+1,p}) = m+1$ *and* $\mathrm{adrn}(D_{m,m+2,p}) = m+3$*. When* $p=4$ *the values are larger by* 1*. For* $m=1$ *the value is* 4*, except if* $p \in \{2,3\}$ *then* $\mathrm{adrn}(D_{1,n,p}) = 3$ *for* $n \ge 4$ *and* $\mathrm{adrn}(D_{1,1,2}) = 3$*, and if* $p \ge 5$ *with* $p \ne 7$ *then* $\mathrm{adrn}(D_{1,2,p}) = 5$*.*

The lower bounds in Theorem 1.1 come mostly from simple constructions.

Proposition 1.2. *If* $1 \le m \le n$*, then* $\mathrm{adrn}(D_{m,n,p}) \ge m+2$*, except* $\mathrm{adrn}(D_{m,n,p}) \ge m+1$ *for* $m = n-1 \ge 2$*. The stronger bound* $\mathrm{adrn}(D_{m,n,p}) \ge m+3$ *holds for* $m = n-2 \ge 1$*. In each case, the lower bound increases by* 1 *when* $p = 4$*.*

Proof. For the main case, $D_{m,n,p}$ and $D_{m+1,n-1,p}$ have $m+1$ common dacards obtained by deleting leaves. If $m = n-2$, then in fact they have $m+2$ such common dacards. If $m = n-1$, then $D_{m,n,p} \cong D_{m+1,n-1,p}$, but $D_{m,n,p}$ and $D_{m-1,n+1,p}$ have m common dacards obtained by deleting leaves (note that $D_{0,n,p} \cong D_{1,n,p-1}$).

For the special case $p = 4$, besides the common dacards obtained by deleting leaves, the graphs $D_{m,n,4}$ and $D_{m+1,n-1,4}$ also share one dacard obtained by deleting a vertex of degree 2, as do $D_{m,n,p}$ and $D_{m-1,n+1,p}$. Hence when $p = 4$ the lower bound is increased by 1. □

Constructions for stronger lower bounds when $m = 1$ will be provided in Section 4. The arguments for $m = 1$ and $m \ge 2$ are quite different, so we separate them. We study the more general case $m \ge 2$ in Section 3 and the more special case $m = 1$ in Section 4. We begin in Section 2 with some ideas that are common to both cases. The bulk of the paper is the proof that the constructions give the correct values for all (m, n, p).

2 Preliminaries

In arguments for upper bounds, we will always let $\mathcal{S}$ be a given list of dacards of $D_{m,n,p}$ and G be a reconstruction from $\mathcal{S}$. If this forces $G \cong D_{m,n,p}$ whenever $|\mathcal{S}| = k$, then $\text{adrn}(D_{m,n,p}) \le k$, but a single exception yields $\text{adrn}(D_{m,n,p}) > k$.

For $D_{m,n,p}$, we distinguish three types of dacards. A *hub vertex* is a neighbor of a leaf, and a *middle vertex* is a vertex with degree 2. A *hub dacard*, *middle dacard*, or *leaf dacard* is a dacard obtained by deleting a hub, middle, or leaf vertex, respectively.

We refer to the two hub vertices in $D_{m,n,p}$ as the *left hub* (with degree $m+1$) and the *right hub* (with degree $n+1$). A hub vertex with degree 2 is also a middle vertex; this occurs when m or n is 1. In the degenerate case $m = 0$, we write $D_{0,n,p} = D_{1,n,p-1}$ for $p > 1$. Also, $D_{m,n,1}$ is the star $K_{1,m+n}$, where the left hub and right hub are the same vertex.

When $m, n \ge 2$, the leaf dacards are copies of $(D_{m-1,n,p}, 1)$ and $(D_{m,n-1,p}, 1)$. The cards obtained by deleting other vertices are disjoint unions of two brooms, where the *broom* $B_{m,a}$ is the tree with $m+a$ vertices obtained from a path with a vertices by adding m leaf neighbors to one endpoint of the path. The degenerate broom $B_{m,0}$ consists of m isolated vertices.

By symmetry, we focus on $1 \le m \le n$. Since $D_{m,n,p}$ is a tree, the presence of any leaf dacard in $\mathcal{S}$ implies that every reconstruction from $\mathcal{S}$ is a tree. We show first that distinct leaf dacards determine $D_{m,n,p}$ except for $D_{1,2,p}$.

Lemma 2.1. *If $1 \le m < n$ and $n \ge 3$, then two distinct leaf dacards determine $D_{m,n,p}$.*

Proof. Since the leaf dacards are distinct, one has a vertex of degree $n+1$. Hence $\Delta(G) \ge n+1$. Since $\Delta(D_{m,n-1,p}) = \max\{m+1, n\} = n$, reaching maximum degree $n+1$ by adding a leaf to $D_{m,n-1,p}$ requires making the leaf adjacent to a vertex of degree n in $D_{m,n-1,p}$. Since $n \ge 3$, each such choice yields $G \cong D_{m,n,p}$. $\square$

A prior result will be useful when $\mathcal{S}$ contains distinct middle dacards. We provide a short proof for completeness.

Lemma 2.2 ([1])**.** *Let G be a graph with dacards $(F, 2)$ and $(F', 2)$. If F and F' are forests with two components, and the components of F do not have the same sizes as those of F', then G is a tree.*

Proof. If G is not a tree, then the vertex added to F to obtain G has two neighbors in one component of F, forming a cycle C. Since F' is a forest, the vertex deleted from G to form F' must also lie on C. Now the components in F' have the same sizes as those in F. $\square$

We use "+" to denote disjoint union. The middle dacards of $D_{m,n,p}$ are expressions of the form $(B_{m,a} + B_{n,b}, 2)$ for some nonnegative a and b with $a + b = p - 1$, where a or b is 0 if and only if the corresponding hub vertex has degree 2 and is a middle vertex. A middle dacard occurs twice in the dadeck of $D_{m,n,p}$ only if $m = n$, otherwise once. The number of middle dacards is $p - 2$ plus the number of hubs having degree 2.

Lemmas 3.2 and 4.3 give conditions under which a set of middle dacards determines $D_{m,n,p}$. We show first that two middle dacards do not always suffice. For convenience, a *j-vertex* is a vertex with degree j, and a *j-neighbor* of a vertex is a neighbor with degree j.

Proposition 2.3. *The two middle dacards obtained by deleting the 2-neighbors of the hubs do not suffice to reconstruct $D_{m,n,p}$ in the following cases: any $D_{m,n,p}$ when $p \geq 6$, also $D_{m,m,4}$, $D_{m,m,5}$, and $D_{m,m+1,5}$.*

Proof. We provide another graph G having these two dacards, which are $(K_{1,m} + B_{n,p-2}, 2)$ and $(B_{m,p-2} + K_{1,n}, 2)$. When $p \geq 6$, construct G from $D_{m,n,5}$ by growing a path of length $p-5$ from the central vertex. For $D_{m,m,5}$, construct G from $C_5 + K_{1,m}$ by adding $m-1$ leaf neighbors to one vertex of the 5-cycle. For $D_{m,m,4}$, construct G from $C_3 + K_{1,m}$ by adding m leaf neighbors to one vertex of the 3-cycle. For $D_{m,m+1,5}$, construct G from $D_{m,m+1,4}$ by subdividing one edge incident to the vertex of maximum degree. □

The alternative reconstructions of $D_{m,m,p}$ for $p \in \{4, 5\}$ are not trees. This does not contradict Lemma 2.2, because in this case the two given middle dacards are identical. In most cases, we used the middle dacards corresponding to 2-neighbors of the hub vertices.

3 The General Case $m \geq 2$

Always let $\mathcal{S}$ be part of the dadeck of $D_{m,n,p}$, and let G be a reconstruction from $\mathcal{S}$. Let a *left leaf* or *right leaf* in $D_{m,n,p}$ be a leaf neighbor of the left or right hub, respectively.

Lemma 3.1. *When $m \geq 2$, any hub dacard and leaf dacard as $\mathcal{S}$ determine $D_{m,n,p}$, except when $m = 2$ and $p \neq 3$ and $\mathcal{S} = \{(D_{1,n,p}, 1), (B_{2,p-1} + nK_1, n + 1)\}$.*

Proof. Let H be the given hub card, consisting of a broom H' and isolated vertices. The graph G arises from H by adding a vertex x. Since $\mathcal{S}$ contains a leaf dacard, G is a tree, so x is adjacent in G to all isolated vertices of H and to one vertex in H'.

For the exceptional case, we provide an alternative reconstruction. Since $m = 2$, the broom H' is $B_{2,p-1}$. Let w and w' be the left leaves. Form G by letting the neighbor of x in H' be w. Note that $G - x = H$ and $G - w' = D_{1,n,p}$. If $p = 3$, then $G \cong D_{2,n,p}$. If $p \neq 3$, then $G \not\cong D_{2,n,p}$, so then $D_{2,n,p}$ is not reconstructible from $\mathcal{S}$.

Now consider the general case. Since $m \geq 2$, the leaf card has diameter $p+1$, and hence $\operatorname{diam} G \geq p+1$. If $p \geq 3$, then $\operatorname{diam} H' = p - 1$ and hence $\operatorname{diam} G \leq p + 1$. If $p = 2$, then $\operatorname{diam} H' = p$. Hence $\operatorname{diam} G = p + 1$ when $p \geq 3$, and $\operatorname{diam} G \in \{3, 4\}$ when $p = 2$.

Let y be the neighbor of x in H'. If $p \geq 3$, then $\operatorname{diam} G = p + 1$ requires y to be a leaf in H'. Let z be the neighbor of y in H'. If $d_{H'}(z) = 2$ or $p = 3$, then $G \cong D_{m,n,p}$. Otherwise,

$d_{H'}(z) \ge 3$ and $p \ge 4$. Now z has two neighbors in G with degree at least 2, and obtaining a double-broom by deleting a leaf requires $d_{H'}(z) = 3$. This is precisely the exceptional case, so that is the only instance with $p \ge 4$ where $D_{m,n,p}$ is not determined by $\mathcal{S}$.

It remains to consider $p = 2$, where H' is a star consisting of a hub vertex and its leaf neighbors. If $\operatorname{diam} G = 3$, then y is the center of H' and $G \cong D_{m,n,2}$. If $\operatorname{diam} G = 4$, then y is a leaf of H'. In this case $G \in \{D_{m-1,n,3}, D_{m,n-1,3}\}$. The leaf card in $\mathcal{S}$ belongs to $\{D_{m-1,n,2}, D_{m,n-1,2}\}$. It must arise also from the double-broom G, so it belongs to $\{D_{m-2,n,3}, D_{m-1,n-1,3}, D_{m,n-2,3}\}$. With $m \ge 2$, the only way the leaf card can lie in both sets is $D_{m-1,n,2} \cong D_{m-2,n,3}$ with $m = 2$. Thus when $p = 2$ we are again in the exceptional case $m = 2$, with $G \ncong D_{m,n,p}$ only when $\mathcal{S}$ is as claimed. □

When $m \ge 2$ and $n = m$, having two distinct middle dacards requires $p \ge 5$.

Lemma 3.2. *When $m \ge 2$, two distinct middle dacards determine $D_{m,n,p}$ when $p \le 5$ (except for $D_{m,m+1,5}$), and otherwise any three middle dacards suffice. Fewer do not suffice.*

Proof. Let $\mathcal{S}$ consist only of middle dacards from $D_{m,n,p}$. The constructions in Proposition 2.3 show that weaker requirements on $\mathcal{S}$ do not suffice.

Every middle dacard has two components, and they are brooms. With distinct middle dacards in $\mathcal{S}$, Lemma 2.2 implies that any reconstruction G is a tree. Hence the vertex x added to a middle dacard has one neighbor in each component of the dacard. Let the *desired neighbors* of x in the two components be vertices that yield $D_{m,n,p}$ when x is adjacent to them. The desired neighbor in each component is a leaf farthest from a highest-degree vertex, except that it may instead be the central vertex when the component is a star.

Proposition 2.3 shows that distinct middle dacards need not determine $D_{m,m+1,5}$. If $\mathcal{S}$ consists of all three middle dacards, then the reconstruction from $(K_{1,m+1} + K_{1,m+2}, 2)$ gives the added vertex x one neighbor in each component. To share the dacards in $\mathcal{S}$, a reconstruction must have three vertices of degree 2. Hence x must be adjacent to a leaf in each component, yielding $D_{m,m+1,5}$. We may henceforth assume that $(m, n, p) \neq (m, m{+}1, 5)$.

Suppose now that there exists $(H, 2) \in \mathcal{S}$ such that both desired neighbors of x are leaves, so $p \ge 5$. The vertex degrees exceeding 2 in H are $n + 1$ and $m + 1$, which is true in each member of $\mathcal{S}$ where the desired neighbors are leaves. Since $(m, n, p) \neq (m, m + 1, 5)$, if G arises by making x adjacent to one of the two high-degree vertices in H, then obtaining a member of $\mathcal{S}$ from G requires deleting a neighbor of that vertex. However, x is the only neighbor whose deletion leaves two brooms, so such a reconstruction G cannot share two dacards with $D_{m,n,p}$.

Hence when $\mathcal{S}$ contains such a dacard $(H, 2)$ we may assume that x is not adjacent in G to either high-degree vertex in H. Let F and F' be the components of H, with x not adjacent to its desired neighbor in F in the reconstruction G (the hub in F may be either hub). We claim that G has at most one 2-vertex z other than x (and z must be in F) such that $G-z$ consists of two brooms. Furthermore, the component of $G-z$ containing F' will be a broom only if x is adjacent to its desired neighbor in F'. Thus x and two vertices of F' contribute 3 to p.

Let $y \in V(F)$ be the neighbor of x in F in the reconstruction G. If y has degree 2 in F, then z must be the neighbor of y on the path from y to the high-degree vertex of F, and the path in F contributes at least four vertices to p (the hub, z, y, and a leaf), so $p \geq 7$. If y is a leaf other than a desired neighbor of x in F, then z must be y, and F must still contribute at least three vertices to p, so $p \geq 6$. These two cases are shown in Figure 1.

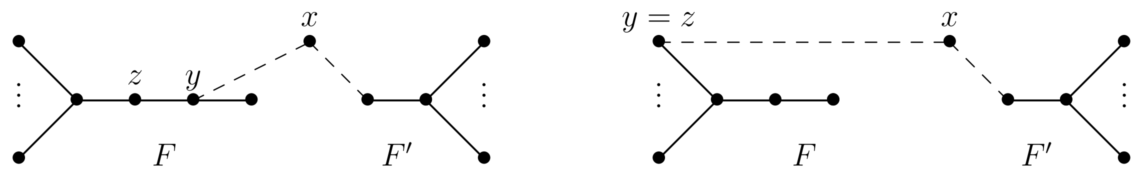

Figure 1: Cases for the neighbor of x in F (Lemma 3.2)

Whenever $|\mathcal{S}| \geq 3$, and for $D_{m,m,5}$ when $|\mathcal{S}| = 2$, there exists $(H, 2)$ of the specified form in $\mathcal{S}$. We have shown that a reconstruction other than $D_{m,n,p}$ then shares at most two middle dacards with $D_{m,n,p}$. Hence $|\mathcal{S}| \geq 3$ suffices when $p \geq 6$, and $|\mathcal{S}| \geq 2$ suffices when $p \in \{4, 5\}$ unless the two middle dacards in $\mathcal{S}$ both arise by deleting 2-neighbors of the hub vertices. When $m = n$ these cards are identical, and by Proposition 2.3 they do not suffice.

The remaining case is thus $p \in \{4, 5\}$ with $m < n$, and $\mathcal{S} = \{(H_1, 2), (H_2, 2)\}$, where $H_1 = B_{m,1} + B_{n,p-2}$ and $H_2 = B_{m,p-2} + B_{n,1}$. Lemma 2.2 still guarantees that G is a tree. Note that $B_{r,1}$ and $B_{r,2}$ are stars, with centers of degrees r and $r+1$, respectively. Let G be reconstructed from H_2 by adding one vertex x of degree 2. Since H_1 has an $(n+1)$-vertex, $\Delta(G) \geq n+1$. Hence x must be adjacent in G to a vertex of degree n in H_2.

For $p = 4$, if the other neighbor of x in H_2 is not a leaf, then G has only one 2-vertex. Hence the other neighbor is a leaf, and $G \cong D_{m,n,4}$.

For $p = 5$, by Proposition 2.3 these two dacards do not suffice when $n \in \{m, m+1\}$, so we may assume $n > m+1$. Let y be the other neighbor of x in H_2. If y is the one 2-vertex in $B_{m,3}$, then G has only one 2-vertex. If y is the $(m+1)$-vertex in H_2, then deleting the only 2-vertex other than x leaves an isolated vertex, which does not occur in a middle dacard of $D_{m,n,5}$. Finally, if y is a leaf neighbor of the $(m+1)$-vertex in H_2, then $G - y \cong B_{m-1,3} + B_{n+1,1}$. Since $n > m+1$, this yields $G - y \notin \{H_1, H_2\}$. Hence x must be made adjacent to its desired neighbor in $B_{m,3}$, and $G \cong D_{m,n,5}$. □

Lemma 3.3. *When $m \geq 2$, every reconstruction from a hub dacard and a middle dacard of $D_{m,n,p}$ is a tree.*

Proof. Let G be such a reconstruction. Since the middle dacard has no isolated vertex, reconstructing G from the hub dacard requires making the added vertex x adjacent to all the isolated vertices of the hub dacard. The remaining added edge must then give x a neighbor in the other component, making G a tree. □

Lemma 3.4. *For $m \geq 2$, the graph $D_{m,n,p}$ is reconstructible from its two hub dacards and any one middle dacard.*

Proof. Since $\mathcal{S}$ has a middle dacard, $p \geq 3$. We reconstruct G from the left hub dacard $(mK_1 + D_{1,n,p-2}, m+1)$, letting y be the $(n+1)$-vertex. The added vertex x must be adjacent to the m isolated vertices and one vertex v in the nontrivial component. If $v = y$, then G has an $(n+2)$-vertex, which is forbidden by the right hub dacard in $\mathcal{S}$. If v is the most distant leaf from y, which yields the same graph as any leaf when $p = 3$, then $G \cong D_{m,n,p}$.

Hence $p > 3$. If v is a leaf neighbor of y, then deleting y does not create n isolated vertices. If v is a 2-vertex that is not a neighbor of y, then the nontrivial component of $G - y$ is not a broom. If x is adjacent to the 2-neighbor of y, then deleting any 2-vertex yields a graph containing $D_{2,n,3}$, which is not a subgraph of any middle dacard of $D_{2,n,p}$. □

Lemma 3.5. *If $n \geq 2$ and $p \geq 2$ with $p \neq 4$, then* $\mathrm{adrn}(D_{2,n,p}) = 4$*, except* $\mathrm{adrn}(D_{2,3,p}) = 3$ *for $2 \leq p \leq 5$ and* $\mathrm{adrn}(D_{2,4,p}) = 5$ *for all p. When $p = 4$, each answer is larger by* 1.

Proof. Lower bounds. For $p \neq 4$, consider various n. The graph $D_{2,2,p}$ shares three leaf dacards with $D_{1,3,p}$. The graph $D_{2,3,p}$ shares two leaf dacards with $D_{1,4,p}$, which gives the desired lower bound when $p \leq 5$. The graph $D_{2,4,p}$ shares four leaf dacards with $D_{3,3,p}$. For $n \geq 5$, the graph $D_{2,n,p}$ shares three leaf dacards with $D_{3,n-1,p}$. When $p = 4$, each pair given also has a common middle dacard. When $p \geq 6$, the graph $D_{2,n,p}$ shares one leaf dacard and two other dacards with the graph obtained from $D_{1,n,p}$ by appending an edge at the vertex having distance 3 from the left hub; this improves the lower bound to 4 when $n = 3$.

Upper bounds. Let $\mathcal{S}$ consist of the specified number of dacards from $D_{2,n,p}$, and let G be a reconstruction from $\mathcal{S}$. We first show that G is a tree. When $\mathcal{S}$ contains a leaf dacard, the claim is immediate. Otherwise, $|\mathcal{S}| \geq 3$ implies that $\mathcal{S}$ contains a hub dacard and a middle dacard or three middle dacards, of which two must be distinct. Now Lemma 3.3 or Lemma 2.2 implies that G is a tree. We organize the cases by pairs of dacards in $\mathcal{S}$. Consider a set $\mathcal{S}$ of the specified size that does not determine $D_{2,n,p}$.

Claim 0: *$\mathcal{S}$ does not contain distinct leaf dacards.* Lemma 2.1 suffices.

Claim 1: *$\mathcal{S}$ does not contain both a hub dacard and a leaf dacard.* If $\mathcal{S}$ contains both, then Lemma 3.1 suffices, unless $p \neq 3$ and $\mathcal{S}$ contains the left leaf and right hub dacards. In the exceptional case, if $p = 2$, then the only reconstruction from the right hub dacard other than $D_{2,n,2}$ is $D_{1,n,3}$, which shares only two dacards with $D_{2,n,2}$. When $p \geq 4$, the only reconstruction from the right hub dacard $(B_{2,p-1} + nK_1, n+1)$ that shares a leaf dacard with $D_{2,n,p}$ is obtained by making the added vertex x adjacent to a leaf neighbor of the 3-vertex in $B_{2,p-1}$. This shares no other dacard with $D_{2,n,p}$ except for one middle dacard when $p = 6$. Hence $|\mathcal{S}| \geq 3$ for $2 \leq p \leq 5$ and $|\mathcal{S}| \geq 4$ for $p \geq 6$ suffice.

Claim 2: *$\mathcal{S}$ does not contain the two left leaf dacards.* Since $m = 2$, the leaf card $D_{1,n,p}$ is a broom. Alternative reconstructions are obtained by adding a pendant edge at a vertex other than the left hub. If the edge is appended at a 2-vertex, then no other leaf deletion yields a broom. If it is appended at the left leaf to form $D_{1,n,p+1}$, then only one leaf deletion returns the diameter to $p + 1$.

When we add the pendant edge at a right leaf, no other leaf deletion yields a broom unless $p = 2$, but in this case the third dacard is a hub or a right leaf dacard. When $n = 2$ the right leaf dacard of $D_{2,n,2}$ is the same as the left, but the alternative reconstruction has this dacard only twice in its deck.

Finally, appending the edge at the right hub yields $D_{1,n+1,p}$. The $n+1$ right leaf cards are the same as $D_{1,n,p}$, but here $\mathcal{S}$ contains also $|\mathcal{S}| - 2$ middle dacards (by Claim 1). If $|\mathcal{S}| \geq 4$, then $D_{1,n+1,p}$ requires a middle dacard whose card has a vertex of degree $n + 2$, but $D_{2,n,p}$ has no such dacard. Finally, if $|\mathcal{S}| = 3$, then the third dacard must be $(P_{p-1} + K_{1,n+1}, 2)$. This is a dacard of $D_{2,n,p}$ only when $p = 4$. However, for $D_{2,n,4}$ we are given $|\mathcal{S}| > 3$.

Claim 3: *$\mathcal{S}$ does not contain two right leaf dacards.* With two right leaf dacards, by Claim 2 we may assume $n \geq 3$. The right leaf card is $D_{2,n-1,p}$. Alternative reconstructions are obtained by adding a pendant edge at a vertex other than the right hub. If the edge is appended at a 2-vertex or at a leaf, then no other leaf deletion yields a double-broom.

In the remaining case, the pendant edge is added at the left leaf, yielding $D_{3,n-1,p}$; being different from $D_{2,n,p}$ requires $n > 3$. When $p \neq 4$ and $n \geq 5$, this graph shares only three dacards with $D_{2,n,p}$, all being copies of $(D_{2,n-1,p}, 1)$, but we are given $|\mathcal{S}| = 4$. Although $D_{3,3,p}$ shares four dacards with $D_{2,4,p}$, in that case we have $|\mathcal{S}| = 5$. When $p = 4$, the two graphs also share one middle dacard, but we have $|\mathcal{S}| = 5$ when $n \neq 4$ and $|\mathcal{S}| = 6$ for $D_{2,4,4}$.

Claim 4: *$\mathcal{S}$ does not contain three middle dacards.* If $p \geq 5$, then three middle dacards determine $D_{2,n,p}$, by Lemma 3.2. If $p \leq 4$, then three middle dacards do not exist.

Claim 5: $|\mathcal{S}| = 3$. By Claims 0, 2, and 3, $\mathcal{S}$ contains at most one leaf dacard. By Claim 4, $\mathcal{S}$ contains at most two middle dacards. By Claim 1, $\mathcal{S}$ does not contain both a leaf and a hub dacard. Hence $|\mathcal{S}| \leq 3$, with equality possible only by having two middle dacards and one leaf or hub dacard, by Lemma 3.4.

The remaining case. The only cases where $|\mathcal{S}| = 3$ are $D_{2,3,p}$ for $p \in \{2, 3, 5\}$. Among these, only $D_{2,3,5}$ has two middle dacards. We consider this case.

Suppose first that $\mathcal{S}$ contains the middle dacard $(K_{1,3} + K_{1,4}, 2)$. If the added vertex x is adjacent to leaves in both components, then $G \cong D_{2,3,5}$. If it is adjacent to both centers, then G has only one 2-vertex. Hence x is adjacent to a leaf in one star and the center of the other star. If it is adjacent to the center of $K_{1,4}$, then deleting the other 2-vertex leaves a 5-vertex, which does not exist in a dacard of $D_{2,3,5}$. Finally, if x is adjacent to the center of $K_{1,3}$, then the only two dacards obtained by deleting 2-vertices of G are isomorphic, but $\mathcal{S}$

does not contain two isomorphic such dacards.

Hence the middle dacards in $\mathcal{S}$ are obtained by deleting the 2-neighbors of the hubs. We reconstruct G by adding x to $P_3 + D_{1,3,2}$. Suppose first that x is adjacent to the center of the path component. If $G \ncong D_{2,3,5}$, then the only choice for the neighbor of x in the other component that yields a graph sharing two middle dacards with $D_{2,3,5}$ is a leaf neighbor of the 4-vertex. However, the resulting graph G shares no other dacard with $D_{2,3,5}$.

Hence x must be adjacent to a leaf of the path component. The other middle dacard has two 3-vertices. To obtain two vertices with degree at least 3 in G, the newly added vertex x must be adjacent to the only 2-vertex in $D_{1,3,2}$. Deleting any 2-vertex other than x from this graph leaves adjacent vertices with degree at least 3, which do not exist in $D_{2,3,5}$. $\square$

Lemma 3.6. *If* $3 \le m \le n$ *and* $p \ne 4$*, then* $\mathrm{adrn}(D_{m,n,p}) = m+2$*, except* $\mathrm{adrn}(D_{m,m+1,p}) = m+1$ *and* $\mathrm{adrn}(D_{m,m+2,p}) = m+3$*. When* $p = 4$*, each value is larger by* 1*.*

Proof. Lower bounds. Since $D_{m,m+1,p}$ and $D_{m-1,m+2,p}$ share m leaf dacards, $\mathrm{adrn}(D_{m,m+1,p}) \ge m+1$. When $n \ne m+1$, the graphs $D_{m,n,p}$ and $D_{m+1,n-1,p}$ share at least $m+1$ leaf dacards, so $\mathrm{adrn}(D_{m,n,p}) \ge m+2$. In the special case $n = m+2$, they share $m+2$ leaf dacards, so $\mathrm{adrn}(D_{m,m+2,p}) \ge m+3$. When $p = 4$, in each case the pairs also share one middle dacard, increasing the lower bound by 1.

Upper bounds. In each case, let $\mathcal{S}$ consist of the specified number of dacards from $D_{m,n,p}$. Since $m \ge 3$, we have $|\mathcal{S}| \ge 4$. When $\mathcal{S}$ contains two distinct leaf dacards or a leaf dacard and a hub dacard, we invoke Lemma 2.1 or Lemma 3.1, respectively. If $\mathcal{S}$ contains two hub dacards and a middle dacard, then Lemma 3.4 suffices. If $\mathcal{S}$ contains three middle dacards, then Lemma 3.2 suffices.

Now a hub dacard forbids leaf dacards, so $|\mathcal{S}| \ge 4$ requires three middle dacards or a second hub dacard plus a middle dacard. Hence we may assume that $\mathcal{S}$ contains no hub dacard. If $\mathcal{S}$ contains at most one leaf dacard, then $\mathcal{S}$ contains three middle dacards. Hence we may assume that $\mathcal{S}$ contains two identical leaf dacards.

If $\mathcal{S}$ has two copies of the right leaf dacard $D_{m,n-1,p}$, then the only possible alternative reconstruction is $D_{m+1,n-1,p}$, which is isomorphic to $D_{m,n,p}$ when $n = m+1$. When $n \notin \{m+1, m+2\}$ and $p \ne 4$, these two graphs share only $m+1$ copies of this dacard and no other dacards. When $n = m+2$, they share $m+2$ copies of this dacard and no others. When $p = 4$, in each case they also share one middle dacard, but then $|\mathcal{S}|$ is larger by 1.

If $\mathcal{S}$ has two copies of the left leaf dacard $D_{m-1,n,p}$, then we may assume $n > m$ by symmetry. The only possible alternative reconstruction is $D_{m-1,n+1,p}$. When $p \ne 4$, this graph and $D_{m,n,p}$ share m copies of this dacard and no other dacards, and hence $\mathrm{adrn}(D_{m,m+1,p}) \le m{+}1$. When $p = 4$, the two graphs also share one middle dacard. $\square$

Lemmas 3.5 and 3.6 yield adrn for all double-brooms that are not brooms.

Theorem 3.7. *If $p \geq 6$, then* $\text{adrn}(D_{2,3,p}) = 4$*. Otherwise, if $2 \leq m \leq n$ and $p \neq 4$, then* $\text{adrn}(D_{m,n,p}) = m + 2$*, except* $\text{adrn}(D_{m,m+1,p}) = m + 1$ *and* $\text{adrn}(D_{m,m+2,p}) = m + 3$*. When $p = 4$, each value increases by* 1*.*

4 The Special Case $m = 1$

The problem is a bit more difficult when $m = 1$ (brooms), partly because deleting the left leaf vertex decreases the diameter (when $p \geq 2$). We first consider the special case $n = 1$, where the broom is a path. This special case was also computed in [16].

Lemma 4.1. *If $p \geq 3$, then* $\text{adrn}(D_{1,1,p}) = 4$*. Also* $\text{adrn}(D_{1,1,2}) = 3$*.*

Proof. Note that $D_{1,1,2} = P_4$, which shares two dacards with $K_{1,3}$. Three dacards include a leaf and a middle dacard, which determine P_4.

Hence we may assume $p \geq 3$. For the lower bound, $D_{1,1,p}$ and $D_{1,2,p-1}$ have three common dacards: $(P_{p-2} + P_3, 2)$ and two copies of $(P_{p+1}, 1)$.

For the upper bound, let $\mathcal{S}$ consist of four dacards from $D_{1,1,p}$. Reconstructing G from a dacard in $\mathcal{S}$ by making the added vertex x adjacent to anything other than its desired neighbor(s) creates a vertex y of degree 3. Since no card of $D_{1,1,p}$ has a vertex of degree at least 3 and no dacard in $\mathcal{S}$ is obtained by deleting a 3-vertex, every dacard of $D_{1,1,p}$ obtained from G by deleting a vertex must be obtained by deleting a neighbor of y. Since there are only three such vertices, $|\mathcal{S}| = 4$ requires $G \cong D_{1,1,p}$. □

We next give a few constructions for lower bounds.

Proposition 4.2. *The double-broom $D_{1,n,p}$ has three middle dacards that do not determine it when (n, p) is $(1, 6)$, $(2, 6)$, or any (n, p) with $p \geq 7$.*

Proof. For $D_{1,1,6}$, which is just an 8-vertex path, the three dacards obtained by deleting the hub vertices and one 2-neighbor of a hub vertex are shared with the graph obtained from a 5-vertex path by growing a path of length 3 from the central vertex.

For $D_{1,2,6}$, the three dacards obtained by deleting the left hub vertex and the two 2-neighbors of the hub vertices are shared with the graph obtained from an 8-vertex path by adding a leaf neighbor of one of the central vertices.

For $D_{1,n,p}$ with $p \geq 7$, the three dacards obtained by deleting the 2-vertices with distances 0, 1, and 5 from the left hub are shared with the graph obtained from $D_{1,n,p-2}$ by growing a path of length 2 from the vertex at distance 2 from its left hub. □

In light of Lemma 4.1, we may henceforth assume $n > 1$. With $m = 1$, this implies that the middle dacards are distinct. Next we prove an analogue of Lemma 3.2, giving conditions under which a collection of middle dacards determines $D_{1,n,p}$.

Lemma 4.3. *When $n > 1$, $D_{1,n,p}$ is determined by any two middle dacards when $p \in \{3,4\}$, any three middle dacards when $p \in \{5,6\}$ (except for $D_{1,2,6}$), and any four middle dacards otherwise. There is exactly one set of three middle dacards that does not determine $D_{1,2,7}$.*

Proof. The need for four middle dacards is shown by the constructions in Proposition 4.2.

Every middle dacard has two components (a path and a broom). The dacards are distinct, so every reconstruction is a tree (Lemma 2.2), and the added vertex x has one neighbor in each component. Let $(L, 2)$ be the dacard of $D_{1,n,p}$ obtained by deleting the left hub, and let $(R, 2)$ be the dacard obtained by deleting the 2-neighbor of the right hub.

With $n > 1$, having two middle dacards requires $p \geq 3$. When $p = 3$, there are only two middle dacards, $(L, 2)$ and $(R, 2)$. Each reconstruction from $(L, 2)$ has a vertex of degree at least $n + 1$. Hence the missing vertex when reconstructing from $(R, 2)$ must be made adjacent to the unique n-vertex in R (and to a vertex of K_2), yielding $D_{1,n,3}$.

When $p = 4$, there are exactly three middle dacards. Suppose first that $\mathcal{S} = \{(L, 2),(R, 2)\}$. Reconstruct G by adding x to R. The vertex of degree $n+1$ in L requires making x adjacent to the center of $K_{1,n}$ in R (when $n = 2$, each component of R is $K_{1,n}$, so the choice of which center is used does not matter). If x is adjacent to the center of the other component, then G has only one 2-vertex; otherwise, $G \cong D_{1,n,4}$.

In the last case for $p = 4$, and for $p \geq 5$ with $|\mathcal{S}| \geq 3$, we have $(H, 2) \in \mathcal{S}$ with $H \notin \{L, R\}$.

Case 1: *G is formed by making x adjacent to a leaf in each component of H.* The resulting graph G also arises by attaching $n - 1$ leaf neighbors at an internal vertex v of a path P with $p + 2$ vertices. If $G \not\cong D_{1,n,p}$, then v is not a neighbor of a leaf of P. Also P contains x and its neighbors. Altogether, P contains at least four vertices on one side of v and at least two on the other, so $p + 2 \geq 7$ (see Figure 2).

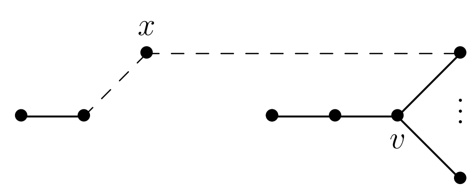

Figure 2: A potential reconstruction for Case 1 of Lemma 4.3

With $p \geq 5$, we have $|\mathcal{S}| \geq 3$. Every middle dacard of $D_{1,n,p}$ except $(R, 2)$ has a vertex of degree $n + 1$ with at most one non-leaf neighbor. Such a dacard can be obtained from G only by deleting a vertex at distance 2 from v along P. Having two such vertices of degree 2 requires $p \geq 6$, and there are never more than two such vertices. Even if $(R, 2) \in \mathcal{S}$, when $|\mathcal{S}| = 4$ we are finished. Since we are allowing $|\mathcal{S}| = 4$ for $D_{1,2,6}$, the only case remaining for $|\mathcal{S}| = 3$ is $p = 6$ and $n \geq 3$ with $(R, 2) \in \mathcal{S}$. When obtaining G from R instead of H, the added vertex x' must be adjacent to the n-vertex. If the neighbor of x' in the other component of R is not a leaf, then the longest path in G has $p+1$ vertices, but G already contains P with $p+2$ vertices.

When $p = 7$ and $n = 2$, this case does not give rise to two distinct middle dacards besides the deletion of the 2-neighbor of the right hub. Having two 2-vertices at distance 2 from v along P requires v to be the center of P, but then the two resulting dacards are identical.

Case 2: *Some neighbor of x in G (call it y) has degree at least 2 in H.* We first prove that y cannot be the $(n+1)$-vertex in H. This would give y degree $n+2$ in G, but every dacard of $D_{1,n,p}$ has no such vertex. Hence every middle dacard of $D_{1,n,p}$ must arise from G by deleting a 2-neighbor of y, and there are fewer than $|\mathcal{S}|$ such vertices.

We conclude that y must be a 2-vertex in H. Now obtaining a path and a broom by deleting a 2-vertex of G requires deleting a 2-neighbor of y. Having one such vertex other than x requires $p \geq 6$, and having two requires $p \geq 7$. Hence $|\mathcal{S}|$ is larger than the number of dacards that G shares with $D_{1,n,p}$.

For $D_{1,2,7}$, obtaining three distinct cards shared with G in this way happens only when x is the 2-neighbor of the left hub in $D_{1,2,7}$ and G is the graph in Proposition 4.2. □

The next lemma considers the double-stars with $m = 1$, done also in [10]; we include it here for completeness.

Lemma 4.4. *Always* $\mathrm{adrn}(D_{1,n,2}) = 3$*, except* $\mathrm{adrn}(D_{1,2,2}) = \mathrm{adrn}(D_{1,3,2}) = 4$.

Proof. Proposition 1.2 yields the lower bounds except for $n = 2$, where we use that $D_{1,2,2}$ and $D_{1,1,3}$ have three common dacards. Now consider the upper bounds. Let $\mathcal{S}$ have the specified size. Since $|\mathcal{S}| > p$, we have a leaf dacard and G is a tree.

When $n = 1$, we have a middle dacard, and it has only one reconstruction.

When $n = 2$, we have four of the five dacards, so $(D_{1,1,2}, 1)$ is a dacard. The only alternative reconstruction is P_5, where any four dacards include at least two whose deleted vertex has degree 2; however, $D_{1,2,2}$ has only one 2-vertex.

When $n \geq 3$, distinct leaf dacards are forbidden, and the right hub dacard has only one tree reconstruction. With $|\mathcal{S}| \geq 3$, the right hub and left leaf dacards are thus forbidden. For $n = 3$, we obtain $(K_1 + K_{1,3}, 2)$ as a dacard. Its only alternative reconstruction is $D_{1,2,3}$, which has only one copy of $(D_{1,2,2}, 1)$ as a dacard.

For $n \geq 4$, at least two copies of $(D_{1,n-1,2}, 1)$ are dacards. The alternative reconstructions are $D_{1,n-1,3}$, $D_{2,n-1,2}$, and the graph obtained from P_5 by giving the central vertex $n-2$ leaf neighbors. Each of these graphs shares at most two dacards with $D_{1,n,2}$. □

For the remaining cases with $m = 1$, we may assume $n \geq 2$ and $p \geq 3$. The arguments for $p = 3$ and $p \geq 4$ are different, so we separate the proofs.

Lemma 4.5. *If $n \geq 4$, then* $\mathrm{adrn}(D_{1,n,3}) = 3$*; if $n \in \{2, 3\}$, then* $\mathrm{adrn}(D_{1,n,3}) = 4$.

Proof. Proposition 1.2 yields the lower bounds when $n \geq 3$. For $n = 2$, the graphs $D_{1,2,3}$ and $D_{1,1,4}$ have three common dacards.

For the upper bounds, let $\mathcal{S}$ consist of the given number of dacards of $D_{1,n,3}$, and let G be another reconstruction from $\mathcal{S}$. Since $|\mathcal{S}| \geq 4$ and only the right hub dacard has more than two components, a leaf dacard or application of Lemma 2.2 implies that G is a tree.

Suppose that $\mathcal{S}$ contains the right hub dacard $(P_3 + nK_1, n + 1)$. Since G is a tree, G arises by making the center of the star $K_{1,n}$ adjacent to the center vertex v of P_3, yielding $D_{2,n,2}$. The dacards of $D_{1,n,3}$ shared with $D_{2,n,2}$ are only the left leaf and right hub dacards of $D_{1,n,3}$, but $|\mathcal{S}| > 2$.

Hence we may assume that $\mathcal{S}$ omits the right hub dacard. If both middle dacards are present, then Lemma 4.3 suffices. Without the right hub dacard, at most one dacard in $\mathcal{S}$ is not a leaf dacard. Therefore, $(D_{1,n-1,3}, 1) \in \mathcal{S}$.

For $n = 2$, the alternative reconstructions G from $D_{1,1,3}$ are $G \in \{P_6, P_5'\}$, where P_5' arises from P_5 by appending a leaf at the central vertex. Since $|\mathcal{S}| = 4$, we have all three leaf dacards. However, P_6 does not have enough leaf dacards, and P_5' fails because the left leaf dacard $(D_{1,2,2}, 1)$ of $D_{1,2,3}$ appears only once in $\mathcal{S}$.

For $n \geq 3$, Lemma 2.1 applies when the left leaf dacard is present, so we may assume $|\mathcal{S}| - 1$ copies of $(D_{1,n-1,3}, 1)$ in $\mathcal{S}$. For $n = 3$, that leaves only $G = D_{2,2,3}$. The fourth dacard deletes a vertex of degree 2, but in $D_{1,3,3}$ this leaves a vertex of degree at least 3, and in the alternative reconstruction $D_{2,2,3}$ it does not.

For $n \geq 4$, the only places to attach a leaf to $D_{1,n-1,3}$ to yield two copies of $(D_{1,n-1,3}, 1)$ in $\mathcal{S}$ are at the hubs. If at the left hub, then the resulting graph does not have three copies of $(D_{1,n-1,3}, 1)$ in its dadeck, and the dacard obtained by deleting its middle vertex has two components with at least three vertices, which does not occur for $D_{1,n,3}$. Attaching at the right hub yields $D_{1,n,3}$, as desired. □

Lemma 4.6. *If $p \geq 4$ and $n \geq 2$, then* $\mathrm{adrn}(D_{1,n,p}) = 4$*, except* $\mathrm{adrn}(D_{1,2,p}) = 5$ *for $p \geq 5$ with $p \neq 7$.*

Proof. Lower bounds. For $p \geq 4$, the graphs $D_{1,n,p}$ and G have three common dacards, where G is obtained from $D_{1,n,p-1}$ by adding a leaf neighbor to the 2-neighbor of the right hub. The common dacards are obtained from $D_{1,n,p}$ by deleting the right hub, the leaf neighbor of the left hub, and the 2-vertex at distance 3 from the right hub (which requires $p \geq 4$).

When $p \geq 5$ and $p \neq 7$, the graph $D_{1,2,p}$ shares four dacards with the graph G obtained from P_{p+2} by appending a leaf at a vertex that has distance 4 from an end of the path. The four dacards are obtained from $D_{1,2,p}$ by deleting respectively a right leaf and the vertices at distances 1, 5, and $p - 2$ from the right hub along the path of length p. When $p = 5$, one of these is the left leaf, so $p \geq 5$ is needed for the construction. When $p = 7$, two of the vertices specified to be deleted are the same, so this construction does not provide a set of four dacards that do not determine $D_{1,2,7}$.

Upper bounds. Let $\mathcal{S}$ consist of the specified number of dacards of $D_{1,n,p}$, and let G be a reconstruction from $\mathcal{S}$. Since $|\mathcal{S}| \geq 3$ and only the right hub dacard has more than two components, a leaf dacard or an application of Lemma 2.2 implies that G is a tree.

Suppose that $\mathcal{S}$ contains the right hub dacard $(nK_1 + P_p, n+1)$. Since G is a tree, G must be formed by making the center of the star $K_{1,n}$ adjacent to a non-leaf vertex v in P_p. The other dacards in $\mathcal{S}$ have at most one vertex of degree greater than 2, and when there is such a vertex it has only one non-leaf neighbor. Such a subgraph can arise from G only by deleting a neighbor of v on the path with p vertices, since $p \geq 4$. Hence there are at most two such subgraphs, contradicting $|\mathcal{S}| = 4$.

Since $n \geq 2$, the middle dacards are all different. If $\mathcal{S}$ contains no leaf dacard, then we have at least two distinct middle dacards when $p \geq 3$ and at least four when $p \geq 5$. In this case Lemma 4.3 suffices. Hence $\mathcal{S}$ contains a leaf dacard.

Case 1: *$D_{1,n,p}$ with $n \geq 3$ and $p \geq 4$.* If $\mathcal{S}$ contains exactly one copy of the right leaf dacard, then $|\mathcal{S}| = 4$ requires two middle dacards (since the right hub dacard is not present). Hence there is a middle dacard with maximum degree $n+1$, so $\Delta(G) \geq n+1$, and the only reconstruction from the right leaf dacard is $D_{1,n,p}$.

Suppose then that $\mathcal{S}$ contains at least two copies of $(D_{1,n-1,p}, 1)$. The only tree G other than $D_{1,n,p}$ that has at least two copies of $(D_{1,n-1,p}, 1)$ as dacards is $D_{2,n-1,p}$. However, $D_{2,n-1,p}$ shares no middle dacards with $D_{1,n,p}$. Hence the four dacards in $\mathcal{S}$ must include at least three copies of $(D_{1,n-1,p}, 1)$, which occur in $D_{2,n-1,p}$ only when $n = 3$. Now the fourth dacard can only be the left leaf dacard, and Lemma 2.1 suffices.

We may therefore assume that $\mathcal{S}$ does not contain the right leaf dacard. With the right hub dacard also forbidden, $\mathcal{S}$ has at least three middle dacards. Now Lemma 4.3 suffices unless $p \geq 7$ and $\mathcal{S}$ has exactly three middle dacards and the left leaf dacard.

Forming G from the left leaf dacard by attaching the missing vertex x to the high-degree vertex yields degree higher than appears in any dacard. Attaching it to a 2-vertex v yields a 3-vertex and an $(n+1)$-vertex. Hence obtaining a middle dacard in $\mathcal{S}$ from G requires deleting a 2-neighbor of v, of which there are at most 2. The last possibility is that G is obtained from $D_{1,n,p-1}$ by attaching x to a leaf neighbor of the right hub. Only one middle dacard of $D_{1,n,p}$ occurs in the dadeck of this graph.

Case 2: *$D_{1,2,p}$ with $p \geq 4$.* If $\mathcal{S}$ contains both copies of the right leaf dacard, then the only alternative reconstruction sharing these dacards is $D_{1,1,p+1}$. This graph shares only three dacards with $D_{1,2,p}$. Hence $\mathcal{S}$ contains at most one right leaf dacard. With the right hub dacard forbidden and $|\mathcal{S}| = 4$, there are at least two middle dacards, which with Lemma 4.3 completes the proof when $p = 4$.

Now consider $p \geq 5$. If $\mathcal{S}$ contains four middle dacards, then Lemma 4.3 suffices. Hence we may assume that $\mathcal{S}$ has at most one copy of the right leaf dacard and at most three middle dacards.

For $p \neq 7$, we have $|\mathcal{S}| = 5$, which requires that $\mathcal{S}$ contains the left and right dacards along with three middle dacards. The right leaf dacard is a path, and the only alternative reconstruction G that has the left leaf dacard of $D_{1,2,p}$ as a dacard is obtained by adding a leaf neighbor at a vertex v with distance 2 from the end of the path. Now a middle dacard of $D_{1,2,p}$ can only be obtained from G by deleting the 2-vertex at distance 2 from v.

For $p = 7$, the same argument eliminates the case where $\mathcal{S}$ consists of two middle dacards and the left and right leaf dacards. The leaf dacards force the same alternative reconstruction G, which has only one middle dacard of $D_{1,2,7}$ in its dadeck.

In the last case $\mathcal{S}$ consists of three middle dacards of $D_{1,2,7}$ and one leaf dacard. If the three middle dacards do not suffice, then they are as described in Proposition 4.2, and the only alternative reconstruction arises from $D_{1,2,5}$ by growing a path of length 2 from the vertex at distance 2 from the left hub. This tree shares no leaf dacard with $D_{1,2,7}$. □

Lemmas 4.1, 4.4, 4.5, and 4.6 complete the computation of $\mathrm{adrn}(D_{m,n,p})$ when $m = 1$.

Theorem 4.7. *Always* $\mathrm{adrn}(D_{1,n,p}) = 4$*, except that if* $p \in \{2, 3\}$ *then* $\mathrm{adrn}(D_{1,n,p}) = 3$ *for* $n \geq 4$ *and* $\mathrm{adrn}(D_{1,1,2}) = 3$*, and if* $p \geq 5$ *with* $p \neq 7$ *then* $\mathrm{adrn}(D_{1,2,p}) = 5$.

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