Cycles in Color-Critical Graphs

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Joint work with
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slides and paper at
https://faculty.math.illinois.edu/~west/ (click "Preprints")

Cycles and Colorings

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$\therefore$ non-k-colorable graphs require cycles with length in certain classes. We will guarantee many cycles.

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Pf. Induction on n. For $n \le k$, G is k-colorable.

For $n > k$, consider some $v \in V(G)$.

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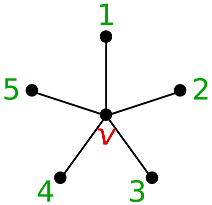

By the ind. hypoth., $G - v$ has a proper k-coloring ϕ, which extends to v unless all colors appear on $N(v)$.

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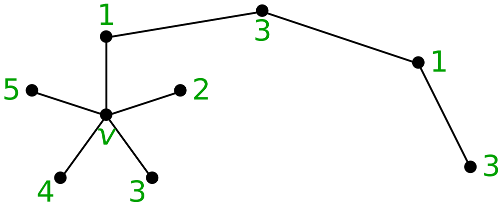

To remove color i from $N(v)$, switch colors i and j on the components of $G_{i,j}$ containing nbrs of v colored i, where $G_{i,j}$ is the subgraph induced by verts. colored i or j by ϕ.

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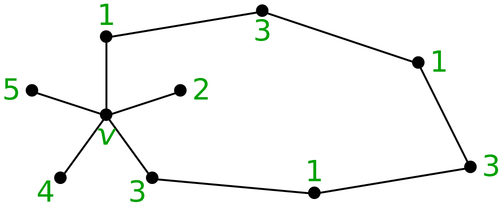

This works unless some neighbors of v with colors i and j are joined by a path alternating colors i and j under f, called a Kempe chain. With v, this forms an odd cycle.

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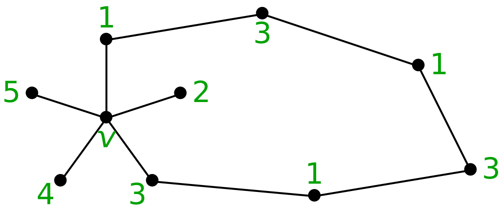

This works unless some neighbors of v with colors i and j are joined by a path alternating colors i and j under f, called a Kempe chain. With v, this forms an odd cycle.

We obtain a proper k-coloring of G unless there is such a path for every pair $\{i, j\}$ of colors, yielding $\binom{k}{2}$ odd cycles through v. ■

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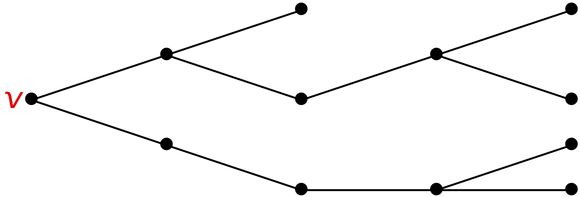
Pf. If T is a spanning tree of G maximizing the sum of the distances from the root v, then every edge of $G - E(T)$ joins a vertex to an ancestor.



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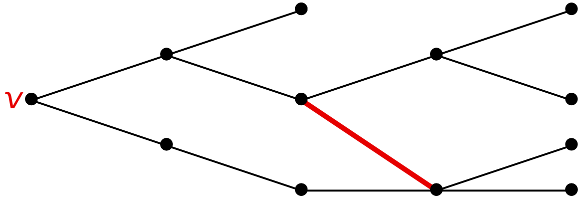

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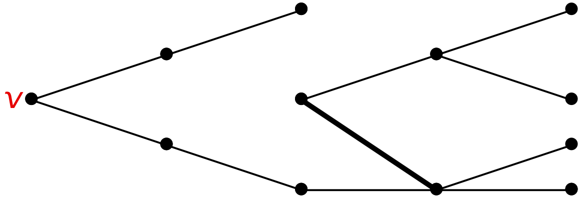

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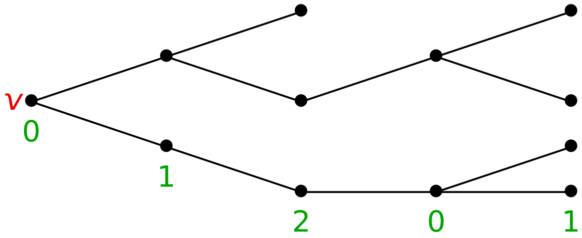

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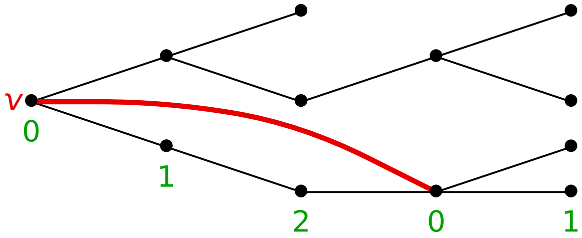

Give color $i \bmod k$ to the vertices with $d_T(v, u) = i$.

If this is not a proper k-coloring, then some edge outside T completes a cycle with length $1 \bmod k$. ∎

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Cor. Every graph with fewer than $(k-1)!$ cycles of length $1 \bmod k$ is k-colorable.

Later we apply a similar argument to circular coloring.

Other Cycle Lengths

Thm. (Chen–Ma–Zang [2015]) For $1 \leq \ell \leq k$ and $k \geq 2$, if G has no cycle of length $\ell \bmod k$, then G is k-colorable if $\ell \neq 2$ and is $(k+1)$-colorable if $\ell = 2$.

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Saito [1992] and Dean–Kaneko–Ota–Toft [1991] together did it for $k = 3$; G–H–M includes $k \in \{4, 5\}$ on arXiv.

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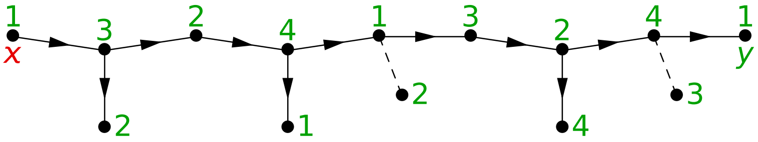
Tool: Digraph D_σ has vertex set $V(G)$, defined by $uv \in E(D_\sigma) \iff uv \in E(G)$ and $\sigma(\phi(u)) = \phi(v)$.

Let $F =$ paths from x in D_σ. Ex: $\sigma = (1, 3, 2, 4)$.



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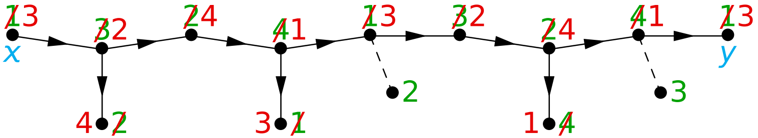

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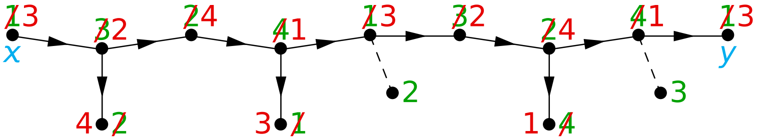

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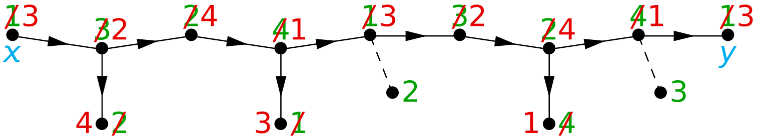

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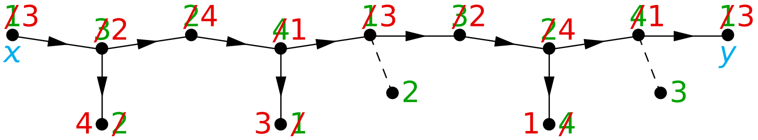

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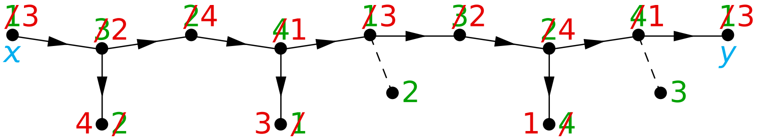

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$\therefore$ $y \in V(F)$, length of an x, y-path in F is a multiple of r.
Add e to complete a cycle of length $1 \bmod r$.

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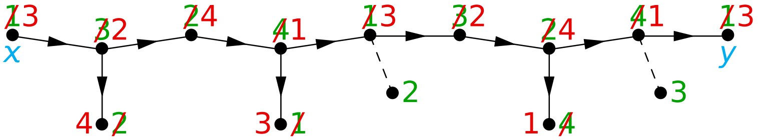

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Since ϕ is fixed, the paths are distinct for distinct σ. ■

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Thm. Brewster–McGuinness–Moore–Noel [2016]
If $G - e$ is k-colorable and G is not, then $G - e$ has at least $(k-1)!/2$ cycles with lengths divisible by k.

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Thm. Moore–West [2021] For $3 \le r \le k$ and $e \in E(G)$, if $G - e$ is k-colorable and G is not, then $G - e$ has at least $\frac{1}{2}\prod_{i=1}^{r-1}(k-i)$ cycles with lengths divisible by r.

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Pf. Again ϕ a fixed proper k-coloring of $G - e$; $e = xy$; σ a cycle of r colors including $\phi(x)$; D_σ and F as before.

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σ and σ^{-1} may yield the same cycle: divide by 2. ■

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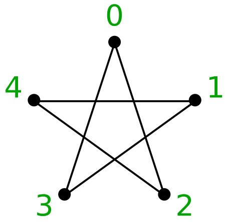

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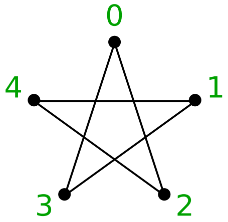

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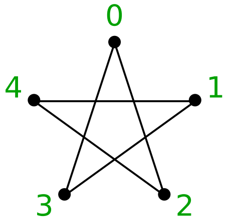

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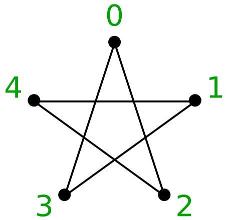

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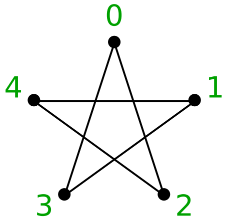

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Zhu [2002]: A sufficient condition for (k, d)-colorability.

Cor. Given $\gcd(k, d) = 1$, choose s so $sd \equiv 1 \bmod k$. If G has no cycle with length congruent to $is \bmod k$ for any $i \in \{1, \ldots, 2d - 1\}$, then G is (k, d)-colorable.

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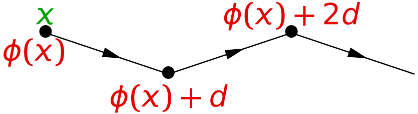

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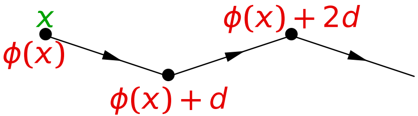

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For these cases, we will use induction on $d - j$.
If $y \notin V(F)$, then $\phi'(x) - \phi'(y) = \phi(x) - \phi(y) + 1$.

Details of the Proof

Claim: Let ϕ be a (k, d)-coloring of $G - xy$, and G have none. If $\phi(x) = j$ and $\phi(y) = 0$, then xy lies on a cycle with length $is \bmod k$ for some $i \in \{1, \dots, d-j\}$.

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Now ϕ' is a (k,d)-coloring of $G-e$ with $\phi'(x)=j+1$ and $\phi'(y)=0$. Induction hypothesis yields cycle through xy with length $is \bmod k$ for some $i\in\{1,\ldots,d-j-1\}$. ∎

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Pf. Probabilistic, with random vertex ordering, using Tuza's strengthening of Minty's Theorem. ■

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Fraction bad is $\le \frac{2(qk+1)q^{qk+1}}{(qk+1)!} = \frac{2q^{qk+1}}{(qk)!} \le \frac{2}{k!}$. ■

Thank you
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