Cycles in Color-Critical Graphs

Douglas B. West

Department of Mathematics
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

Joint work with
Benjamin R. Moore

slides and paper at
https://faculty.math.illinois.edu/~west/ (click "Preprints")

Cycles and Colorings

Def. G is k-colorable if $V(G)$ can be labeled using k colors so that adjacent vertices have distinct colors.
Chromatic number $\chi(G)$ = least k s.t. G is k-colorable.

Cycles and Colorings

Def. G is k-colorable if $V(G)$ can be labeled using k colors so that adjacent vertices have distinct colors.
Chromatic number $\chi(G)$ = least k s.t. G is k-colorable.

Thm. (König) Every graph having no odd cycle is bipartite (i.e. 2-colorable).

Cycles and Colorings

Def. G is k-colorable if $V(G)$ can be labeled using k colors so that adjacent vertices have distinct colors.
Chromatic number $\chi(G)$ = least k s.t. G is k-colorable.

Thm. (König) Every graph having no odd cycle is bipartite (i.e. 2-colorable).

Thm. (Stong [2006]) A graph having no vertex belonging to $\binom{k}{2}$ odd cycles is k-colorable.

Cycles and Colorings

Def. G is k-colorable if $V(G)$ can be labeled using k colors so that adjacent vertices have distinct colors.
Chromatic number $\chi(G)$ = least k s.t. G is k-colorable.

Thm. (König) Every graph having no odd cycle is bipartite (i.e. 2-colorable).

Thm. (Stong [2006]) A graph having no vertex belonging to $\binom{k}{2}$ odd cycles is k-colorable.

Thm. (Erdös–Hajnal [1966]) For even k, a graph having no odd cycle of length more than k is k-colorable.

Cycles and Colorings

Def. G is k-colorable if $V(G)$ can be labeled using k colors so that adjacent vertices have distinct colors.
Chromatic number $\chi(G)$ = least k s.t. G is k-colorable.

Thm. (König) Every graph having no odd cycle is bipartite (i.e. 2-colorable).

Thm. (Stong [2006]) A graph having no vertex belonging to $\binom{k}{2}$ odd cycles is k-colorable.

Thm. (Erdös–Hajnal [1966]) For even k, a graph having no odd cycle of length more than k is k-colorable.

Thm. (Tuza [1992]) A graph having no cycle of length $1 \bmod k$ is k-colorable.

Cycles and Colorings

Def. G is k-colorable if $V(G)$ can be labeled using k colors so that adjacent vertices have distinct colors.
Chromatic number $\chi(G)$ = least k s.t. G is k-colorable.

Thm. (König) Every graph having no odd cycle is bipartite (i.e. 2-colorable).

Thm. (Stong [2006]) A graph having no vertex belonging to $\binom{k}{2}$ odd cycles is k-colorable.

Thm. (Erdös–Hajnal [1966]) For even k, a graph having no odd cycle of length more than k is k-colorable.

Thm. (Tuza [1992]) A graph having no cycle of length $1 \bmod k$ is k-colorable.

$\therefore$ non-k-colorable graphs require cycles with length in certain classes. But, how many?

Many Cycles

Def. generalized Kempe chain = a walk following specified colors. (Neumann-Lara [1982], Toft [1995])

Many Cycles

Def. generalized Kempe chain = a walk following specified colors. (Neumann-Lara [1982], Toft [1995])
doubly-critical graph = deleting the endpoints of any edge reduces chromatic number by 2.

Many Cycles

Def. generalized Kempe chain = a walk following specified colors. (Neumann-Lara [1982], Toft [1995])
doubly-critical graph = deleting the endpoints of any edge reduces chromatic number by 2.

Thm. (Kawarabayashi–Pedersen–Toft [2010]) In a doubly-critical graph with chromatic number $k+1$, each edge lies in at least $\prod_{i=1}^{\ell-2}(k-i)$ cycles of length ℓ.

Many Cycles

Def. generalized Kempe chain = a walk following specified colors. (Neumann-Lara [1982], Toft [1995])
doubly-critical graph = deleting the endpoints of any edge reduces chromatic number by 2.

Thm. (Kawarabayashi–Pedersen–Toft [2010]) In a doubly-critical graph with chromatic number $k+1$, each edge lies in at least $\prod_{i=1}^{\ell-2}(k-i)$ cycles of length ℓ.

Thm. (Moore–West [2020+]) For $2 \le r \le k$ and $e \in E(G)$, if $G - e$ is k-colorable and G is not, then the edge e lies in at least $\prod_{i=1}^{r-1}(k-i)$ cycles of length $1 \bmod r$.

Many Cycles

Def. generalized Kempe chain = a walk following specified colors. (Neumann-Lara [1982], Toft [1995])
doubly-critical graph = deleting the endpoints of any edge reduces chromatic number by 2.

Thm. (Kawarabayashi–Pedersen–Toft [2010]) In a doubly-critical graph with chromatic number $k+1$, each edge lies in at least $\prod_{i=1}^{\ell-2}(k-i)$ cycles of length ℓ.

Thm. (Moore–West [2020+]) For $2 \le r \le k$ and $e \in E(G)$, if $G-e$ is k-colorable and G is not, then the edge e lies in at least $\prod_{i=1}^{r-1}(k-i)$ cycles of length $1 \bmod r$.

Cor. Every graph with fewer than $(k-1)!$ cycles of length $1 \bmod k$ is k-colorable.

Many Cycles

Def. generalized Kempe chain = a walk following specified colors. (Neumann-Lara [1982], Toft [1995])
doubly-critical graph = deleting the endpoints of any edge reduces chromatic number by 2.

Thm. (Kawarabayashi–Pedersen–Toft [2010]) In a doubly-critical graph with chromatic number $k+1$, each edge lies in at least $\prod_{i=1}^{\ell-2}(k-i)$ cycles of length ℓ.

Thm. (Moore–West [2020+]) For $2 \le r \le k$ and $e \in E(G)$, if $G-e$ is k-colorable and G is not, then the edge e lies in at least $\prod_{i=1}^{r-1}(k-i)$ cycles of length $1 \bmod r$.

Cor. Every graph with fewer than $(k-1)!$ cycles of length $1 \bmod k$ is k-colorable.

Later we prove a similar result for circular coloring.

Other Cycle Lengths

Thm. (Chen–Ma–Zang [2015]) For $1 \le \ell \le k$ and $k \ge 2$, if G has no cycle of length $\ell \bmod k$, then G is k-colorable if $\ell \ne 2$ and is $(k+1)$-colorable if $\ell = 2$.

Other Cycle Lengths

Thm. (Chen–Ma–Zang [2015]) For $1 \leq \ell \leq k$ and $k \geq 2$, if G has no cycle of length $\ell \bmod k$, then G is k-colorable if $\ell \neq 2$ and is $(k+1)$-colorable if $\ell = 2$.

Many lengths: Sudakov-Verstraëte, Bondy-Vince, Thomassen.

Other Cycle Lengths

Thm. (Chen–Ma–Zang [2015]) For $1 \leq \ell \leq k$ and $k \geq 2$, if G has no cycle of length $\ell \bmod k$, then G is k-colorable if $\ell \neq 2$ and is $(k+1)$-colorable if $\ell = 2$.

Many lengths: Sudakov-Verstraëte, Bondy-Vince, Thomassen.

Thm. (Gao–Huo–Lui–Ma [2019+]) If G is not k-colorable, then G contains cycles with $k-1$ consecutive lengths.

Other Cycle Lengths

Thm. (Chen–Ma–Zang [2015]) For $1 \le \ell \le k$ and $k \ge 2$, if G has no cycle of length $\ell \bmod k$, then G is k-colorable if $\ell \ne 2$ and is $(k+1)$-colorable if $\ell = 2$.

Many lengths: Sudakov-Verstraëte, Bondy-Vince, Thomassen.

Thm. (Gao–Huo–Lui–Ma [2019+]) If G is not k-colorable, then G contains cycles with $k-1$ consecutive lengths.

Also, 3-connected nonbipartite graphs with mindeg $\ge k$ have cycles with $k-1$ consecutive lengths.

Other Cycle Lengths

Thm. (Chen–Ma–Zang [2015]) For $1 \le \ell \le k$ and $k \ge 2$, if G has no cycle of length $\ell \bmod k$, then G is k-colorable if $\ell \ne 2$ and is $(k+1)$-colorable if $\ell = 2$.

Many lengths: Sudakov-Verstraëte, Bondy-Vince, Thomassen.

Thm. (Gao–Huo–Lui–Ma [2019+]) If G is not k-colorable, then G contains cycles with $k-1$ consecutive lengths.

Also, 3-connected nonbipartite graphs with mindeg $\ge k$ have cycles with $k-1$ consecutive lengths.

Sharp: Petersen graph has no 4-, 7-, 10-cycle.

Other Cycle Lengths

Thm. (Chen–Ma–Zang [2015]) For $1 \le \ell \le k$ and $k \ge 2$, if G has no cycle of length $\ell \bmod k$, then G is k-colorable if $\ell \ne 2$ and is $(k+1)$-colorable if $\ell = 2$.

Many lengths: Sudakov-Verstraëte, Bondy-Vince, Thomassen.

Thm. (Gao–Huo–Lui–Ma [2019+]) If G is not k-colorable, then G contains cycles with $k-1$ consecutive lengths.

Also, 3-connected nonbipartite graphs with mindeg $\ge k$ have cycles with $k-1$ consecutive lengths.

Sharp: Petersen graph has no 4-, 7-, 10-cycle.

Thm. (Gao–Huo–Ma [2020+]) When $k \ge 6$, every non-k-colorable graph not having K_{k+1} as a block has cycles with k consecutive lengths.

Other Cycle Lengths

Thm. (Chen–Ma–Zang [2015]) For $1 \le \ell \le k$ and $k \ge 2$, if G has no cycle of length $\ell \bmod k$, then G is k-colorable if $\ell \ne 2$ and is $(k+1)$-colorable if $\ell = 2$.

Many lengths: Sudakov-Verstraëte, Bondy-Vince, Thomassen.

Thm. (Gao–Huo–Lui–Ma [2019+]) If G is not k-colorable, then G contains cycles with $k-1$ consecutive lengths.

Also, 3-connected nonbipartite graphs with mindeg $\ge k$ have cycles with $k-1$ consecutive lengths.

Sharp: Petersen graph has no 4-, 7-, 10-cycle.

Thm. (Gao–Huo–Ma [2020+]) When $k \ge 6$, every non-k-colorable graph not having K_{k+1} as a block has cycles with k consecutive lengths.

Saito [1992] and Dean–Kaneko–Ota–Toft [1991] together did it for $k = 3$; G–H–M includes $k \in \{4, 5\}$ on arXiv.

Our result

Our result

Thm. (Moore–West [2020+]) For $2 \le r \le k$ and $e \in E(G)$, if $G - e$ is k-colorable and G is not, then the edge e lies in at least $\prod_{i=1}^{r-1}(k-i)$ cycles of length $1 \bmod r$.

Our result

Thm. (Moore–West [2020+]) For $2 \le r \le k$ and $e \in E(G)$, if $G - e$ is k-colorable and G is not, then the edge e lies in at least $\prod_{i=1}^{r-1}(k-i)$ cycles of length $1 \bmod r$.

Pf. Fix proper coloring $\phi\colon V(G-e) \to [k]$. Let $e = xy$.

Our result

Thm. (Moore–West [2020+]) For $2 \leq r \leq k$ and $e \in E(G)$, if $G - e$ is k-colorable and G is not, then the edge e lies in at least $\prod_{i=1}^{r-1}(k-i)$ cycles of length $1 \bmod r$.

Pf. Fix proper coloring $\phi\colon V(G-e) \to [k]$. Let $e = xy$.

We obtain a cycle through e for each cyclic list σ of r elements of $[k]$ containing $\phi(x)$:

Our result

Thm. (Moore–West [2020+]) For $2 \le r \le k$ and $e \in E(G)$, if $G - e$ is k-colorable and G is not, then the edge e lies in at least $\prod_{i=1}^{r-1}(k-i)$ cycles of length $1 \bmod r$.

Pf. Fix proper coloring $\phi\colon V(G-e) \to [k]$. Let $e = xy$.

We obtain a cycle through e for each cyclic list σ of r elements of $[k]$ containing $\phi(x)$: $\# = \prod_{i=1}^{r-1}(k-i)$.

Our result

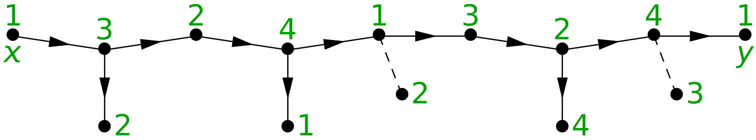
Thm. (Moore–West [2020+]) For $2 \le r \le k$ and $e \in E(G)$, if $G - e$ is k-colorable and G is not, then the edge e lies in at least $\prod_{i=1}^{r-1}(k-i)$ cycles of length $1 \bmod r$.

Pf. Fix proper coloring $\phi\colon V(G-e) \to [k]$. Let $e = xy$.

We obtain a cycle through e for each cyclic list σ of r elements of $[k]$ containing $\phi(x)$: $\# = \prod_{i=1}^{r-1}(k-i)$.

Tool: σ-subdigraph D_σ has vertex set $V(G)$, putting $uv \in E(D_\sigma) \iff uv \in E(G)$ and $\sigma(\phi(u)) = \phi(v)$.

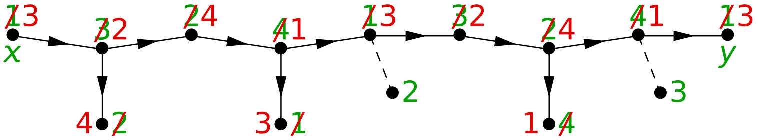
$F =$ digraph of paths from x in D_σ. Ex: $\sigma = (1, 3, 2, 4)$.



Recoloring

Let $\phi' : V(G-e) \to [k]$ by $\begin{cases} \phi'(u) = \sigma(\phi(u)) & \text{if } u \in V(F) \\ \phi'(u) = \phi(u) & \text{if } u \notin V(F). \end{cases}$

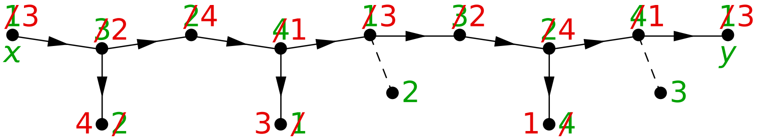
Ex. $\sigma = (1, 3, 2, 4)$.



Recoloring

Let $\phi' : V(G-e) \to [k]$ by $\begin{cases} \phi'(u) = \sigma(\phi(u)) & \text{if } u \in V(F) \\ \phi'(u) = \phi(u) & \text{if } u \notin V(F). \end{cases}$

Ex. $\sigma = (1, 3, 2, 4)$.

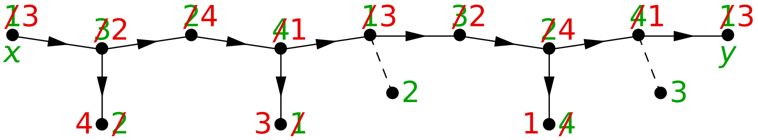


For edge uv, if $\phi'(u) = \phi(v)$, then $v \in V(F)$ and the color on v also changes. Thus ϕ' is proper k-coloring of $G-e$.

Recoloring

Let $\phi' : V(G-e) \to [k]$ by $\begin{cases} \phi'(u) = \sigma(\phi(u)) & \text{if } u \in V(F) \\ \phi'(u) = \phi(u) & \text{if } u \notin V(F). \end{cases}$

Ex. $\sigma = (1, 3, 2, 4)$.



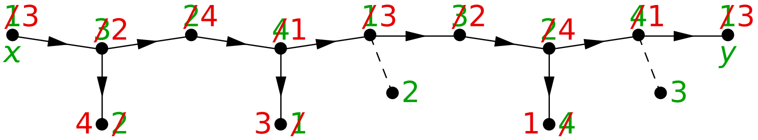
For edge uv, if $\phi'(u) = \phi(v)$, then $v \in V(F)$ and the color on v also changes. Thus ϕ' is proper k-coloring of $G-e$.

G not k-colorable $\Rightarrow$ $\phi(x) = \phi(y)$. Also $\phi'(x) \neq \phi(x)$.

Recoloring

Let $\phi' : V(G-e) \to [k]$ by $\begin{cases} \phi'(u) = \sigma(\phi(u)) & \text{if } u \in V(F) \\ \phi'(u) = \phi(u) & \text{if } u \notin V(F). \end{cases}$

Ex. $\sigma = (1,3,2,4)$.



For edge uv, if $\phi'(u) = \phi(v)$, then $v \in V(F)$ and the color on v also changes. Thus ϕ' is proper k-coloring of $G-e$.

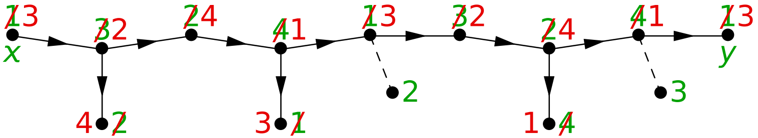
G not k-colorable $\Rightarrow$ $\phi(x) = \phi(y)$. Also $\phi'(x) \neq \phi(x)$.

If $y \notin V(F)$, then ϕ' would be a proper k-coloring of G.

Recoloring

Let $\phi' : V(G-e) \to [k]$ by $\begin{cases} \phi'(u) = \sigma(\phi(u)) & \text{if } u \in V(F) \\ \phi'(u) = \phi(u) & \text{if } u \notin V(F). \end{cases}$

Ex. $\sigma = (1,3,2,4)$.



For edge uv, if $\phi'(u) = \phi(v)$, then $v \in V(F)$ and the color on v also changes. Thus ϕ' is proper k-coloring of $G-e$.

G not k-colorable $\Rightarrow$ $\phi(x) = \phi(y)$. Also $\phi'(x) \neq \phi(x)$.

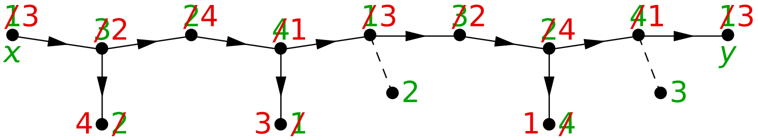
If $y \notin V(F)$, then ϕ' would be a proper k-coloring of G.

$\therefore$ $y \in V(F)$, length of an x,y-path in F is a multiple of r. Add e to complete a cycle of length $1 \bmod r$.

Recoloring

Let $\phi' : V(G-e) \to [k]$ by $\begin{cases} \phi'(u) = \sigma(\phi(u)) & \text{if } u \in V(F) \\ \phi'(u) = \phi(u) & \text{if } u \notin V(F). \end{cases}$

Ex. $\sigma = (1,3,2,4)$.



For edge uv, if $\phi'(u) = \phi(v)$, then $v \in V(F)$ and the color on v also changes. Thus ϕ' is proper k-coloring of $G-e$.

G not k-colorable $\Rightarrow$ $\phi(x) = \phi(y)$. Also $\phi'(x) \neq \phi(x)$.

If $y \notin V(F)$, then ϕ' would be a proper k-coloring of G.

$\therefore$ $y \in V(F)$, length of an x,y-path in F is a multiple of r. Add e to complete a cycle of length $1 \bmod r$.

Since ϕ is fixed, the paths are distinct for distinct σ. ∎

Comments

Thm. Brewster–McGuinness–Moore–Noel [2016]
If $G - e$ is k-colorable and G is not, then $G - e$ has at least $(k-1)!/2$ cycles with lengths divisible by k.

Comments

Thm. Brewster–McGuinness–Moore–Noel [2016]
If $G - e$ is k-colorable and G is not, then $G - e$ has at least $(k-1)!/2$ cycles with lengths divisible by k.

Thm. Moore–West [2020+] For $3 \le r \le k$ and $e \in E(G)$, if $G - e$ is k-colorable and G is not, then $G - e$ has at least $\frac{1}{2}\prod_{i=1}^{r-1}(k-i)$ cycles with lengths divisible by r.

Comments

Thm. Brewster–McGuinness–Moore–Noel [2016]
If $G - e$ is k-colorable and G is not, then $G - e$ has at least $(k-1)!/2$ cycles with lengths divisible by k.

Thm. Moore–West [2020+] For $3 \le r \le k$ and $e \in E(G)$, if $G - e$ is k-colorable and G is not, then $G - e$ has at least $\frac{1}{2}\prod_{i=1}^{r-1}(k-i)$ cycles with lengths divisible by r.

Pf. Again use D_σ, but recolor one vertex at a time working back from the sinks in D_σ. For each σ, we obtain a proper k-coloring of G or a cycle in D_σ whose length is a multiple of r. ■

Comments

Thm. Brewster–McGuinness–Moore–Noel [2016]
If $G - e$ is k-colorable and G is not, then $G - e$ has at least $(k-1)!/2$ cycles with lengths divisible by k.

Thm. Moore–West [2020+] For $3 \le r \le k$ and $e \in E(G)$, if $G - e$ is k-colorable and G is not, then $G - e$ has at least $\frac{1}{2}\prod_{i=1}^{r-1}(k-i)$ cycles with lengths divisible by r.

Pf. Again use D_σ, but recolor one vertex at a time working back from the sinks in D_σ. For each σ, we obtain a proper k-coloring of G or a cycle in D_σ whose length is a multiple of r. ∎

Thm. If G is not k-colorable, then G has at least $k!/2$ cycles with length $1 \bmod k$, with equality only when they all have length $k+1$.

Comments

Thm. Brewster–McGuinness–Moore–Noel [2016]
If $G - e$ is k-colorable and G is not, then $G - e$ has at least $(k-1)!/2$ cycles with lengths divisible by k.

Thm. Moore–West [2020+] For $3 \le r \le k$ and $e \in E(G)$, if $G - e$ is k-colorable and G is not, then $G - e$ has at least $\frac{1}{2}\prod_{i=1}^{r-1}(k-i)$ cycles with lengths divisible by r.

Pf. Again use D_σ, but recolor one vertex at a time working back from the sinks in D_σ. For each σ, we obtain a proper k-coloring of G or a cycle in D_σ whose length is a multiple of r. ■

Thm. If G is not k-colorable, then G has at least $k!/2$ cycles with length $1 \bmod k$, with equality only when they all have length $k+1$.

Pf. Probabilistic, with random vertex ordering, using Tuza's strengthening of Minty's Theorem. ■

H-coloring and Circular Coloring

Def. homomorphism to H is a map $\phi\colon V(G) \to V(H)$ such that $uv \in E(G) \Rightarrow \phi(u)\phi(v) \in E(H)$.

H-coloring and Circular Coloring

Def. homomorphism to H is a map $\phi\colon V(G) \to V(H)$ such that $uv \in E(G) \Rightarrow \phi(u)\phi(v) \in E(H)$.

Ex. Proper k-coloring is just homomorphism to K_k.

H-coloring and Circular Coloring

Def. homomorphism to H is a map $\phi\colon V(G) \to V(H)$ such that $uv \in E(G) \Rightarrow \phi(u)\phi(v) \in E(H)$.

Ex. Proper k-coloring is just homomorphism to K_k.

Def. circular clique $K_{k:d}$ has vertex set $\mathbf{Z}_k$ and edge set $\{ij\colon d \le j - i \le k - d\}$.

H-coloring and Circular Coloring

Def. homomorphism to H is a map $\phi\colon V(G) \to V(H)$ such that $uv \in E(G) \Rightarrow \phi(u)\phi(v) \in E(H)$.

Ex. Proper k-coloring is just homomorphism to K_k.

Def. circular clique $K_{k:d}$ has vertex set $\mathbf{Z}_k$ and edge set $\{ij\colon d \le j - i \le k - d\}$.

Vince [1988]: (k, d)-coloring is homomorphism to $K_{k,d}$.

H-coloring and Circular Coloring

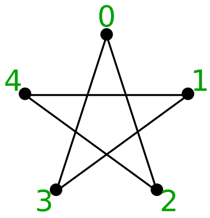
Def. homomorphism to H is a map $\phi\colon V(G) \to V(H)$ such that $uv \in E(G) \Rightarrow \phi(u)\phi(v) \in E(H)$.

Ex. Proper k-coloring is just homomorphism to K_k.

Def. circular clique $K_{k:d}$ has vertex set $\mathbf{Z}_k$ and edge set $\{ij\colon d \le j - i \le k - d\}$.

Vince [1988]: (k,d)-coloring is homomorphism to $K_{k,d}$.

Ex. C_5 is $(5,2)$-colorable.



H-coloring and Circular Coloring

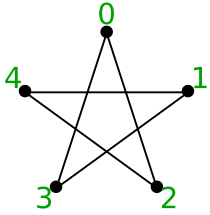
Def. homomorphism to H is a map $\phi\colon V(G) \to V(H)$ such that $uv \in E(G) \Rightarrow \phi(u)\phi(v) \in E(H)$.

Ex. Proper k-coloring is just homomorphism to K_k.

Def. circular clique $K_{k:d}$ has vertex set $\mathbf{Z}_k$ and edge set $\{ij\colon d \le j - i \le k - d\}$.

Vince [1988]: (k, d)-coloring is homomorphism to $K_{k,d}$.

Ex. C_5 is $(5, 2)$-colorable.



circular chromatic number
$\chi_c(G)$ = least k/d such that G is (k, d)-colorable.

H-coloring and Circular Coloring

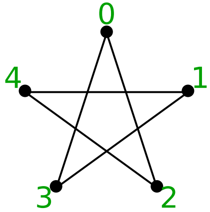
Def. homomorphism to H is a map $\phi\colon V(G) \to V(H)$ such that $uv \in E(G) \Rightarrow \phi(u)\phi(v) \in E(H)$.

Ex. Proper k-coloring is just homomorphism to K_k.

Def. circular clique $K_{k:d}$ has vertex set $\mathbf{Z}_k$ and edge set $\{ij\colon d \le j - i \le k - d\}$.

Vince [1988]: (k, d)-coloring is homomorphism to $K_{k,d}$.

Ex. C_5 is $(5, 2)$-colorable.



circular chromatic number
$\chi_c(G)$ = least k/d such that G is (k, d)-colorable.

Vince [1988]: $\lceil \chi_c(G) \rceil = \chi(G)$.

H-coloring and Circular Coloring

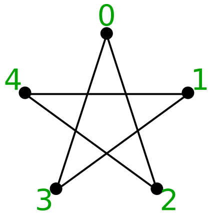
Def. homomorphism to H is a map $\phi\colon V(G) \to V(H)$ such that $uv \in E(G) \Rightarrow \phi(u)\phi(v) \in E(H)$.

Ex. Proper k-coloring is just homomorphism to K_k.

Def. circular clique $K_{k:d}$ has vertex set $\mathbf{Z}_k$ and edge set $\{ij\colon d \le j - i \le k - d\}$.

Vince [1988]: (k,d)-coloring is homomorphism to $K_{k,d}$.

Ex. C_5 is $(5,2)$-colorable.



circular chromatic number
$\chi_c(G)$ = least k/d such that G is (k,d)-colorable.

Vince [1988]: $\lceil \chi_c(G) \rceil = \chi(G)$.

Surveys: Zhu ['01, '06]

Extension of Tuza's Result

Zhu [2002]: A sufficient condition for (k, d)-colorability.

Cor. Given $\gcd(k, d) = 1$, choose s so $sd \equiv 1 \bmod k$. If G has no cycle with length congruent to $is \bmod k$ for any $i \in \{1, ..., 2d - 1\}$, then G is (k, d)-colorable.

Extension of Tuza's Result

Zhu [2002]: A sufficient condition for (k, d)-colorability.

Cor. Given $\gcd(k, d) = 1$, choose s so $sd \equiv 1 \bmod k$.
If G has no cycle with length congruent to $is \bmod k$ for any $i \in \{1, ..., 2d - 1\}$, then G is (k, d)-colorable.

Setting $d = 1$ yields Tuza's result.

Extension of Tuza's Result

Zhu [2002]: A sufficient condition for (k, d)-colorability.

Cor. Given $\gcd(k, d) = 1$, choose s so $sd \equiv 1 \bmod k$. If G has no cycle with length congruent to $is \bmod k$ for any $i \in \{1, \ldots, 2d - 1\}$, then G is (k, d)-colorable.

Setting $d = 1$ yields Tuza's result.

Our result: It suffices to exclude $i \in \{1, \ldots, d\}$.

Extension of Tuza's Result

Zhu [2002]: A sufficient condition for (k, d)-colorability.

Cor. Given $\gcd(k, d) = 1$, choose s so $sd \equiv 1 \bmod k$.
If G has no cycle with length congruent to $is \bmod k$ for any $i \in \{1, \ldots, 2d - 1\}$, then G is (k, d)-colorable.

Setting $d = 1$ yields Tuza's result.

Our result: It suffices to exclude $i \in \{1, \ldots, d\}$.

Sharpness: When $k = 2d + 1$, we have $-2d \equiv 1 \bmod k$, so $s = -2$. Note that $K_{(2d+1):d} = C_{2d+1}$.

Extension of Tuza's Result

Zhu [2002]: A sufficient condition for (k, d)-colorability.

Cor. Given $\gcd(k, d) = 1$, choose s so $sd \equiv 1 \bmod k$. If G has no cycle with length congruent to $is \bmod k$ for any $i \in \{1, \ldots, 2d-1\}$, then G is (k, d)-colorable.

Setting $d = 1$ yields Tuza's result.

Our result: It suffices to exclude $i \in \{1, \ldots, d\}$.

Sharpness: When $k = 2d + 1$, we have $-2d \equiv 1 \bmod k$, so $s = -2$. Note that $K_{(2d+1):d} = C_{2d+1}$.

The odd cycles w. lengths from $2d - 1$ to 1 have lengths congruent to is for $1 \leq i \leq d$.

Extension of Tuza's Result

Zhu [2002]: A sufficient condition for (k, d)-colorability.

Cor. Given $\gcd(k, d) = 1$, choose s so $sd \equiv 1 \bmod k$.
If G has no cycle with length congruent to $is \bmod k$ for any $i \in \{1, \ldots, 2d-1\}$, then G is (k, d)-colorable.

Setting $d = 1$ yields Tuza's result.

Our result: It suffices to exclude $i \in \{1, \ldots, d\}$.

Sharpness: When $k = 2d+1$, we have $-2d \equiv 1 \bmod k$, so $s = -2$. Note that $K_{(2d+1):d} = C_{2d+1}$.

The odd cycles w. lengths from $2d-1$ to 1 have lengths congruent to is for $1 \le i \le d$. Not C_{2d+1}-colorable.

Extension of Tuza's Result

Zhu [2002]: A sufficient condition for (k, d)-colorability.

Cor. Given $\gcd(k, d) = 1$, choose s so $sd \equiv 1 \bmod k$. If G has no cycle with length congruent to $is \bmod k$ for any $i \in \{1, \ldots, 2d-1\}$, then G is (k, d)-colorable.

Setting $d = 1$ yields Tuza's result.

Our result: It suffices to exclude $i \in \{1, \ldots, d\}$.

Sharpness: When $k = 2d+1$, we have $-2d \equiv 1 \bmod k$, so $s = -2$. Note that $K_{(2d+1):d} = C_{2d+1}$.

The odd cycles w. lengths from $2d-1$ to 1 have lengths congruent to is for $1 \le i \le d$. Not C_{2d+1}-colorable.

Hence we cannot omit any of these lengths in a sufficient condition for $(2d+1, d)$-colorability.

Using D_σ Again

Thm. Given $\gcd(k, d) = 1$ and $k > 2d$, choose s so $sd \equiv 1 \bmod k$. If $G - e$ is (k, d)-colorable and G is not, then e lies in a cycle with length congruent to $is \bmod k$ for some $i \in \{1, \ldots, d\}$.

Using D_σ Again

Thm. Given $\gcd(k,d)=1$ and $k>2d$, choose s so $sd\equiv 1 \bmod k$. If $G-e$ is (k,d)-colorable and G is not, then e lies in a cycle with length congruent to $is \bmod k$ for some $i\in\{1,\ldots,d\}$.

Pf. Fix a (k,d)-coloring ϕ of $G-e$. Let $e=xy$.

Using D_σ Again

Thm. Given $\gcd(k,d)=1$ and $k>2d$, choose s so $sd\equiv 1 \bmod k$. If $G-e$ is (k,d)-colorable and G is not, then e lies in a cycle with length congruent to $is \bmod k$ for some $i\in\{1,\ldots,d\}$.

Pf. Fix a (k,d)-coloring ϕ of $G-e$. Let $e=xy$.

By symmetry, $\phi(y)=0$, so $\phi(x)\in\{0,\pm 1,\ldots,\pm(d-1)\}$.

Using D_σ Again

Thm. Given $\gcd(k,d)=1$ and $k>2d$, choose s so $sd\equiv 1 \bmod k$. If $G-e$ is (k,d)-colorable and G is not, then e lies in a cycle with length congruent to $is \bmod k$ for some $i\in\{1,\dots,d\}$.

Pf. Fix a (k,d)-coloring ϕ of $G-e$. Let $e=xy$.

By symmetry, $\phi(y)=0$, so $\phi(x)\in\{0,\pm1,\dots,\pm(d-1)\}$.

Special color permutation $\sigma=(0,d,2d,\dots,-d)$.

Using D_σ Again

Thm. Given $\gcd(k,d)=1$ and $k>2d$, choose s so $sd\equiv 1 \bmod k$. If $G-e$ is (k,d)-colorable and G is not, then e lies in a cycle with length congruent to $is \bmod k$ for some $i\in\{1,\ldots,d\}$.

Pf. Fix a (k,d)-coloring ϕ of $G-e$. Let $e=xy$.

By symmetry, $\phi(y)=0$, so $\phi(x)\in\{0,\pm1,\ldots,\pm(d-1)\}$.

Special color permutation $\sigma=(0,d,2d,\ldots,-d)$.

Define D_σ and F as before (paths from x via σ-chains).

Using D_σ Again

Thm. Given $\gcd(k,d)=1$ and $k>2d$, choose s so $sd\equiv 1 \bmod k$. If $G-e$ is (k,d)-colorable and G is not, then e lies in a cycle with length congruent to $is \bmod k$ for some $i\in\{1,\ldots,d\}$.

Pf. Fix a (k,d)-coloring ϕ of $G-e$. Let $e=xy$.

By symmetry, $\phi(y)=0$, so $\phi(x)\in\{0,\pm1,\ldots,\pm(d-1)\}$.

Special color permutation $\sigma=(0,d,2d,\ldots,-d)$.

Define D_σ and F as before (paths from x via σ-chains).

Let $\phi'\colon V(G-e)\to \mathbf{Z}_k$ by $\begin{cases}\phi'(u)=\phi(u)+1 & \text{if } u\in V(F)\\ \phi'(u)=\phi(u) & \text{if } u\notin V(F).\end{cases}$

Using D_σ Again

Thm. Given $\gcd(k,d)=1$ and $k>2d$, choose s so $sd\equiv 1 \bmod k$. If $G-e$ is (k,d)-colorable and G is not, then e lies in a cycle with length congruent to $is \bmod k$ for some $i\in\{1,\dots,d\}$.

Pf. Fix a (k,d)-coloring ϕ of $G-e$. Let $e=xy$.

By symmetry, $\phi(y)=0$, so $\phi(x)\in\{0,\pm 1,\dots,\pm(d-1)\}$.

Special color permutation $\sigma=(0,d,2d,\dots,-d)$.

Define D_σ and F as before (paths from x via σ-chains).

Let $\phi'\colon V(G-e)\to \mathbf{Z}_k$ by $\begin{cases}\phi'(u)=\phi(u)+1 & \text{if } u\in V(F)\\ \phi'(u)=\phi(u) & \text{if } u\notin V(F).\end{cases}$

- ϕ' is a (k,d)-coloring of $G-e$.

Using D_σ Again

Thm. Given $\gcd(k,d)=1$ and $k>2d$, choose s so $sd\equiv 1 \bmod k$. If $G-e$ is (k,d)-colorable and G is not, then e lies in a cycle with length congruent to $is \bmod k$ for some $i\in\{1,\dots,d\}$.

Pf. Fix a (k,d)-coloring ϕ of $G-e$. Let $e=xy$.

By symmetry, $\phi(y)=0$, so $\phi(x)\in\{0,\pm1,\dots,\pm(d-1)\}$.

Special color permutation $\sigma=(0,d,2d,\dots,-d)$.

Define D_σ and F as before (paths from x via σ-chains).

Let $\phi'\colon V(G-e)\to \mathbf{Z}_k$ by $\begin{cases}\phi'(u)=\phi(u)+1 & \text{if } u\in V(F)\\ \phi'(u)=\phi(u) & \text{if } u\notin V(F).\end{cases}$

- ϕ' is a (k,d)-coloring of $G-e$. Consider $vw\in E(G-e)$.

Using D_σ Again

Thm. Given $\gcd(k,d)=1$ and $k>2d$, choose s so $sd \equiv 1 \bmod k$. If $G-e$ is (k,d)-colorable and G is not, then e lies in a cycle with length congruent to $is \bmod k$ for some $i \in \{1,\dots,d\}$.

Pf. Fix a (k,d)-coloring ϕ of $G-e$. Let $e=xy$.

By symmetry, $\phi(y)=0$, so $\phi(x) \in \{0,\pm1,\dots,\pm(d-1)\}$.

Special color permutation $\sigma=(0,d,2d,\dots,-d)$.

Define D_σ and F as before (paths from x via σ-chains).

Let $\phi' \colon V(G-e) \to \mathbf{Z}_k$ by $\begin{cases} \phi'(u)=\phi(u)+1 & \text{if } u \in V(F) \\ \phi'(u)=\phi(u) & \text{if } u \notin V(F). \end{cases}$

- ϕ' is a (k,d)-coloring of $G-e$. Consider $vw \in E(G-e)$.

$\phi'(w)-\phi'(v)=\phi(w)-\phi(v)$ if $w,v \in V(F)$ or $w,v \notin V(F)$.

Using D_σ Again

Thm. Given $\gcd(k,d)=1$ and $k>2d$, choose s so $sd\equiv 1 \bmod k$. If $G-e$ is (k,d)-colorable and G is not, then e lies in a cycle with length congruent to $is \bmod k$ for some $i\in\{1,\ldots,d\}$.

Pf. Fix a (k,d)-coloring ϕ of $G-e$. Let $e=xy$.

By symmetry, $\phi(y)=0$, so $\phi(x)\in\{0,\pm1,\ldots,\pm(d-1)\}$.

Special color permutation $\sigma=(0,d,2d,\ldots,-d)$.

Define D_σ and F as before (paths from x via σ-chains).

Let $\phi'\colon V(G-e)\to\mathbf{Z}_k$ by $\begin{cases}\phi'(u)=\phi(u)+1 & \text{if } u\in V(F)\\ \phi'(u)=\phi(u) & \text{if } u\notin V(F).\end{cases}$

- ϕ' is a (k,d)-coloring of $G-e$. Consider $vw\in E(G-e)$.

$\phi'(w)-\phi'(v)=\phi(w)-\phi(v)$ if $w,v\in V(F)$ or $w,v\notin V(F)$.

If $v\in V(F)$, $w\notin V(F)$, then $\phi(w)-\phi(v)\in\{d+1,\ldots,k-d\}$.

Using D_σ Again

Thm. Given $\gcd(k,d)=1$ and $k>2d$, choose s so $sd\equiv 1 \bmod k$. If $G-e$ is (k,d)-colorable and G is not, then e lies in a cycle with length congruent to $is \bmod k$ for some $i\in\{1,\dots,d\}$.

Pf. Fix a (k,d)-coloring ϕ of $G-e$. Let $e=xy$.

By symmetry, $\phi(y)=0$, so $\phi(x)\in\{0,\pm1,\dots,\pm(d-1)\}$.

Special color permutation $\sigma=(0,d,2d,\dots,-d)$.

Define D_σ and F as before (paths from x via σ-chains).

Let $\phi'\colon V(G-e)\to \mathbf{Z}_k$ by $\begin{cases}\phi'(u)=\phi(u)+1 & \text{if } u\in V(F)\\ \phi'(u)=\phi(u) & \text{if } u\notin V(F).\end{cases}$

- ϕ' is a (k,d)-coloring of $G-e$. Consider $vw\in E(G-e)$.

$\phi'(w)-\phi'(v)=\phi(w)-\phi(v)$ if $w,v\in V(F)$ or $w,v\notin V(F)$.

If $v\in V(F)$, $w\notin V(F)$, then $\phi(w)-\phi(v)\in\{d+1,\dots,k-d\}$.
Now $\phi'(w)-\phi'(v)\in\{d,\dots,k-d-1\}$, proper.

How does recoloring help?

Consider cases $\phi(x) = j$ for $0 \leq j \leq d - 1$.

How does recoloring help?

Consider cases $\phi(x) = j$ for $0 \leq j \leq d - 1$.

(For $\phi(x) = -j$, add j to all colors and exchange x and y.)

How does recoloring help?

Consider cases $\phi(x) = j$ for $0 \leq j \leq d-1$.

(For $\phi(x) = -j$, add j to all colors and exchange x and y.)

Claim: If $\phi(x) = j$ and $\phi(y) = 0$, then xy lies on a cycle with length $is \bmod k$ for some $i \in \{1, \ldots, d-j\}$.

How does recoloring help?

Consider cases $\phi(x) = j$ for $0 \le j \le d - 1$.

(For $\phi(x) = -j$, add j to all colors and exchange x and y.)

Claim: If $\phi(x) = j$ and $\phi(y) = 0$, then xy lies on a cycle with length $is \bmod k$ for some $i \in \{1, \ldots, d - j\}$.

x, y-path of length r in F $\Rightarrow$ $rd \equiv -j \bmod k$, so $r \equiv -js$.

How does recoloring help?

Consider cases $\phi(x) = j$ for $0 \le j \le d-1$.

(For $\phi(x) = -j$, add j to all colors and exchange x and y.)

Claim: If $\phi(x) = j$ and $\phi(y) = 0$, then xy lies on a cycle with length $is \bmod k$ for some $i \in \{1, \dots, d-j\}$.

x, y-path of length r in $F \Rightarrow rd \equiv -j \bmod k$, so $r \equiv -js$.
Resulting cycle length: $r + 1 \equiv -js + ds \equiv (d-j)s \bmod k$.

How does recoloring help?

Consider cases $\phi(x) = j$ for $0 \leq j \leq d-1$.

(For $\phi(x) = -j$, add j to all colors and exchange x and y.)

Claim: If $\phi(x) = j$ and $\phi(y) = 0$, then xy lies on a cycle with length $is \bmod k$ for some $i \in \{1, ..., d-j\}$.

x, y-path of length r in $F \Rightarrow rd \equiv -j \bmod k$, so $r \equiv -js$.
Resulting cycle length: $r + 1 \equiv -js + ds \equiv (d-j)s \bmod k$.

Base Case $j = d - 1$. If $y \notin V(F)$, then ϕ' is a (k, d)-coloring of G, since $\phi'(x) = d$ and $\phi'(y) = 0$.

How does recoloring help?

Consider cases $\phi(x) = j$ for $0 \leq j \leq d-1$.

(For $\phi(x) = -j$, add j to all colors and exchange x and y.)

Claim: If $\phi(x) = j$ and $\phi(y) = 0$, then xy lies on a cycle with length $is \bmod k$ for some $i \in \{1, \ldots, d-j\}$.

x, y-path of length r in $F \Rightarrow rd \equiv -j \bmod k$, so $r \equiv -js$.
Resulting cycle length: $r + 1 \equiv -js + ds \equiv (d-j)s \bmod k$.

Base Case $j = d-1$. If $y \notin V(F)$, then ϕ' is a (k, d)-coloring of G, since $\phi'(x) = d$ and $\phi'(y) = 0$.
$\therefore$ $y \in V(F)$, and cycle through yx has length $1s \bmod k$.

How does recoloring help?

Consider cases $\phi(x) = j$ for $0 \leq j \leq d-1$.

(For $\phi(x) = -j$, add j to all colors and exchange x and y.)

Claim: If $\phi(x) = j$ and $\phi(y) = 0$, then xy lies on a cycle with length $is \bmod k$ for some $i \in \{1, \ldots, d-j\}$.

x, y-path of length r in $F \Rightarrow rd \equiv -j \bmod k$, so $r \equiv -js$.
Resulting cycle length: $r + 1 \equiv -js + ds \equiv (d-j)s \bmod k$.

Base Case $j = d-1$. If $y \notin V(F)$, then ϕ' is a (k, d)-coloring of G, since $\phi'(x) = d$ and $\phi'(y) = 0$.
$\therefore$ $y \in V(F)$, and cycle through yx has length $1s \bmod k$.

Induction step $j < d-1$. If $y \in V(F)$, then cycle through yx has length $(d-j)s \bmod k$.

How does recoloring help?

Consider cases $\phi(x) = j$ for $0 \le j \le d-1$.

(For $\phi(x) = -j$, add j to all colors and exchange x and y.)

Claim: If $\phi(x) = j$ and $\phi(y) = 0$, then xy lies on a cycle with length $is \bmod k$ for some $i \in \{1, \dots, d-j\}$.

x, y-path of length r in $F \Rightarrow rd \equiv -j \bmod k$, so $r \equiv -js$.
Resulting cycle length: $r + 1 \equiv -js + ds \equiv (d-j)s \bmod k$.

Base Case $j = d-1$. If $y \notin V(F)$, then ϕ' is a (k, d)-coloring of G, since $\phi'(x) = d$ and $\phi'(y) = 0$.
$\therefore$ $y \in V(F)$, and cycle through yx has length $1s \bmod k$.

Induction step $j < d-1$. If $y \in V(F)$, then cycle through yx has length $(d-j)s \bmod k$. Hence $y \notin V(F)$.

How does recoloring help?

Consider cases $\phi(x) = j$ for $0 \leq j \leq d-1$.

(For $\phi(x) = -j$, add j to all colors and exchange x and y.)

Claim: If $\phi(x) = j$ and $\phi(y) = 0$, then xy lies on a cycle with length $is \bmod k$ for some $i \in \{1, \ldots, d-j\}$.

x, y-path of length r in $F \Rightarrow rd \equiv -j \bmod k$, so $r \equiv -js$.
Resulting cycle length: $r + 1 \equiv -js + ds \equiv (d-j)s \bmod k$.

Base Case $j = d-1$. If $y \notin V(F)$, then ϕ' is a (k, d)-coloring of G, since $\phi'(x) = d$ and $\phi'(y) = 0$.
$\therefore\ y \in V(F)$, and cycle through yx has length $1s \bmod k$.

Induction step $j < d-1$. If $y \in V(F)$, then cycle through yx has length $(d-j)s \bmod k$. Hence $y \notin V(F)$.

Now ϕ' is a (k, d)-coloring of $G - e$ with $\phi'(x) = j+1$ and $\phi'(y) = 0$. Induction hypothesis yields cycle through xy with length $is \bmod k$ for some $i \in \{1, \ldots, d-j-1\}$. ■