

Circular chromatic index of Cartesian products of graphs

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Abstract

The *circular chromatic index* of a graph G , written $\chi'_c(G)$, is the minimum r permitting a function $f: E(G) \rightarrow [0, r)$ such that $1 \leq |f(e) - f(e')| \leq r - 1$ whenever e and e' are incident. Let $G = H \square C_{2m+1}$, where \square denotes Cartesian product and H is an $(s - 2)$ -regular graph of odd order, with $s \equiv 0 \pmod{4}$ (thus G is s -regular). We prove that $\chi'_c(G) \geq s + \lfloor \lambda(1 - 1/s) \rfloor^{-1}$, where λ is the minimum, over all bases of the cycle space of H , of the maximum length of a cycle in the basis. When $H = C_{2k+1}$ and m is large, the lower bound is sharp. In particular, if $m \geq 3k + 1$, then $\chi'_c(C_{2k+1} \square C_{2m+1}) = 4 + \lceil 3k/2 \rceil^{-1}$, independent of m .

1 Introduction

The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colors needed to color the edges so that incident edges receive distinct colors. In the case of a simple graph G (no loops or multiple edges), the famous theorem of Vizing [10] and Gupta [4] yields $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum vertex degree in G .

With only two values available, it is common to say that a graph G is *Class 1* if $\chi'(G) = \Delta(G)$ and *Class 2* otherwise. In this paper we consider a refinement of the chromatic index called the “circular chromatic index”. It equals $\chi'(G)$ when G is Class 1, and otherwise it lies between $\Delta(G)$ and $\chi'(G)$. To define it, we first describe a vertex coloring parameter.

Given a graph G and a real number r , an r -*coloring* of G is a function $f: V(G) \rightarrow [0, r)$ such that $1 \leq |f(x) - f(y)| \leq r - 1$ whenever x and y are adjacent. In essence, the set

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of colors form a circle of circumference r , and the colors assigned to adjacent vertices must differ by at least 1 (in each direction) along the circle.

The *circular chromatic number* of G , written $\chi_c(G)$, is the infimum of all r such that G admits an r -coloring (the infimum can be replaced with minimum). There are many equivalent formulations of $\chi_c(G)$ (see [12, 13] for surveys and many basic results). The definition here is not the most common but is useful for our results. Due to the elementary result that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ [9], the parameter χ_c is a refinement of χ , and this has motivated its extensive study over the past decade.

For a graph G , the *line graph* $L(G)$ is the graph with vertex set $E(G)$ whose adjacency relation is the incidence relation for edges in G . The *circular chromatic index* $\chi'_c(G)$ is defined by $\chi'_c(G) = \chi_c(L(G))$. That is, we seek the smallest r permitting an r -coloring of the edges of G . Since $\chi'(G) = \chi(L(G))$, we have $\chi'(G) - 1 < \chi'_c(G) \leq \chi'(G)$, and χ'_c is a refinement of χ' . From the definition, $\chi'_c(G)$ is at least the maximum number of pairwise incident edges. Thus $\chi'_c(G) = \chi'(G)$ when G is Class 1. Otherwise, $\Delta(G) < \chi'_c(G) \leq \Delta(G) + 1$.

Several papers have been published about χ'_c . It was proved in [2] that all 2-edge-connected graphs with maximum degree at most 3 have circular chromatic index at most $11/3$, except for two small graphs with circular chromatic index 4. In [5], it was proved that 2-edge-connected 3-regular graphs of large girth have circular chromatic index close to 3. This result was generalized in [6]: for any positive integer d , graphs with maximum degree d have circular chromatic index arbitrarily close to d if their girth is sufficiently large.

In this paper, we study the behavior of circular chromatic index under a product operation. Given graphs G and H , the *Cartesian product* $G \square H$ is the graph with vertex set $V(G) \times V(H)$ defined by making the pair (u, v) adjacent to the pair (u', v') if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$. It has long been known that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$ [1, 8, 11]. The argument holds as well for χ_c , so the behavior of χ_c is trivial under the Cartesian product.

The behavior of χ'_c is more interesting. If $G \square H$ is Class 1, then $\chi'_c(G \square H) = \Delta(G \square H)$, so we consider only products that are Class 2. The product is Class 1 when G or H is Class 1 [7] or when G and H both have perfect matchings [7]. To avoid Class 1, let G and H be regular graphs with odd order. The product $G \square H$ is then also regular with odd order, and a regular graph is Class 1 if and only if it has an edge-coloring in which every color class is a perfect matching, which does not exist in $G \square H$.

In particular, we consider the product of an odd cycle with a regular graph H of odd order, where the degree of the vertices in H is congruent to 2 modulo 4. We prove that $\chi'_c(H \square C_{2m+1}) \geq s + \lfloor \lambda(1 - 1/s) \rfloor^{-1}$, where λ is the maximum length of the cycles in some basis of the cycle space of H (choosing the basis to make λ smallest gives the best lower bound available from this argument). We also prove that the bound is sharp when H is an odd cycle and m is large. Indeed, $\chi'_c(H \square C_{2m+1})$ always decreases to a limit as m increases. In particular, if $m \geq 3k + 1$, then $\chi'_c(C_{2k+1} \square C_{2m+1}) = 4 + \lceil 3k/2 \rceil^{-1}$, independent of m .

2 Properties of r -Colorings

We view the color set $[0, r)$ for a r -coloring of a graph as the set of real numbers modulo r . Thus we interpret it as a circle C^r of circumference r , by identifying 0 and r . For $a, b \in C^r$, we write $[a, b]_r$ for the set in C^r moving from a to b through increasing values. That is, $[a, b]_r = [a, b]$ when $a \leq b$, while $[a, b]_r = [a, r) \cup [0, b]$ when $a > b$. For convenience, we extend this notation to all real numbers a and b by letting $[a, b]_r = [a \bmod r, b \bmod r]_r$, where $a \bmod r$ and $b \bmod r$ are the remainders of a and b upon division by r . The intervals $[a, b]_r$, $(a, b]_r$ and $(a, b)_r$ are defined similarly. We use $\ell([a, b])$ to denote the length of the interval $[a, b]$, and we define a measure of distance on the circle as $|a - b|_r = \min\{\ell([a, b]_r), \ell([b, a]_r)\}$. An s -clique is a set of s pairwise adjacent vertices.

Lemma 2.1 *Let G be a graph and f be an r -coloring of G , where $r = s + \epsilon$ with $s \in \mathbb{N}$ and $\epsilon < 1/2$. If Q is an s -clique in G and $v \in Q$, then each set $[f(v) + i, f(v) + i + \epsilon]_r$ for $0 \leq i \leq s - 1$ contains the color of exactly one vertex in Q . If X and Y are intersecting s -cliques, then for each $x \in X$ there is a unique $y \in Y$ such that $|f(y) - f(x)|_r \leq \epsilon$.*

Proof. Since the colors on vertices of Q must pairwise differ by at least 1, the i th such color after $f(v)$ must be at least i units later along the circle. It cannot be more than $i + \epsilon$ units later, since $s - i$ subsequent colors are encountered in returning to $f(v)$.

Now consider $v \in X \cap Y$. With $x_0 = y_0 = v$, let x_i be the i th vertex of X whose color is encountered moving upward from $f(v)$ around the circle (similarly define y_i). By the preceding paragraph, both $f(x_i)$ and $f(y_i)$ lie in $[f(v) + i, f(v) + i + \epsilon]_r$, for $1 \leq i \leq s - 1$. Hence they differ by at most ϵ . Furthermore, since $\epsilon < 1/2$, the distance between two such intervals is more than ϵ , so y_i is the only vertex of Y whose color is within ϵ of $f(x_i)$. ■

To facilitate proofs, we interpret vertex colorings as edge-weightings of orientations. Let \vec{G} be an orientation of a graph G . For a weight function $w: E(G) \rightarrow \mathbb{R}$ and a walk W in G , let $w(W)$ denote the sum of the weights along W , where the weight of an edge counts negatively when followed against its direction in \vec{G} .

A *tension* on \vec{G} is a weight function w such that $w(C) = 0$ for every cycle C in G . Given a real number r with $r \geq 2$, an r -*tension* is a tension w such that $1 \leq |w(uv)| \leq r - 1$ for every $uv \in E(G)$. An r -coloring f of G generates an r -tension w on an orientation \vec{G} by letting $w(uv) = f(v) - f(u)$ for each $uv \in E(\vec{G})$.

A *modular r -tension* on an orientation \vec{G} is a weight function $w: E(G) \rightarrow \mathbb{R}$ such that (1) $w(C)$ is a multiple of r whenever C is a cycle in G , and (2) the weight on each edge differs by at least 1 from any multiple of r . Every r -tension is a modular r -tension, so an r -coloring of G generates a modular r -tension on \vec{G} as above.

Conversely, a modular r -tension w on \vec{G} generates an r -coloring f of G as follows. We may assume that G is connected (else do this in each component). Fix a vertex x . For each vertex v , choose an x, v -walk W in G , and choose $f(v) \equiv w(W) \pmod{r}$ with $0 \leq f(v) < r$.

Since w is a modular r -tension, $f(v)$ does not depend on the choice of W , and the colors on adjacent vertices differ by at least 1. We call the resulting f an r -coloring *generated from* w . We say “an” here because the coloring depends on the choice of x , but only by a cyclic permutation. We have shown that $\chi_c(G)$ equals the least r such that some orientation \vec{G} has a modular r -tension.

Our lower bound on $\chi'_c(H \square C_{2m+1})$ uses an analogue of girth, employing a parameter obtained from the cycle space of the graph. We obtain a strong lower bound when all the cycles in some basis of the cycle space are short.

Within the binary vector space of dimension $|E(G)|$ with canonical basis vectors indexed by the edges, the *cycle space* of an undirected graph G is the subspace spanned by the incidence vectors of the cycles. The analogue for an orientation \vec{G} is the real vector space spanned by the signed incidence vectors of the cycles. For each cycle C in G , followed in a given direction, the *signed incidence vector* relative to \vec{G} has 1 or -1 in each position for an edge of C , using -1 if and only if the edge is followed against its direction in \vec{G} .

For any orientation \vec{G} , the same sets of cycles form bases of its cycle space as form bases of the cycle space of the underlying graph G . In either context, the number of nonzero positions in the incidence vector for a cycle is the same. Hence we define the relevant parameter in terms of G . For a basis \mathcal{B} of the cycle space of G , let $\lambda(\mathcal{B})$ denote the maximum length of an element of \mathcal{B} . Let $\lambda(G)$ denote the minimum of $\lambda(\mathcal{B})$ over all bases of the cycle space. Note that $\lambda(G)$ may be larger than the girth of G , but never smaller. The smaller the value of $\lambda(G)$, the larger the lower bound we will obtain on $\chi_c(G)$.

Before embarking on the technical lemmas, we pause to motivate their hypotheses. Let $F = H \square C_{2m+1}$. When H is $(s-2)$ -regular, F is s -regular. Furthermore, the edges incident to any vertex of F become an s -clique in $L(F)$. Conversely, any two adjacent vertices of $L(F)$ correspond to two incident edges in F and hence lie in an s -clique in $L(F)$. Therefore, we can study r -edge-colorings of F by studying r -colorings of $L(F)$, which we do by studying r -colorings of graphs in which every edge lies in a complete subgraph of order s .

Lemma 2.2 *Let G be a graph such that each edge lies in a complete subgraph of order s . Let G have an r -coloring f such that*

$$r < s + \frac{1}{\lfloor \lambda(G)(1 - 1/s) \rfloor}.$$

If \vec{G} is an orientation of G , then setting $w(xy) = \lfloor \ell([f(x), f(y))_r] \rfloor$ for all $xy \in E(\vec{G})$ defines a modular s -tension on \vec{G} .

Proof. Let $\epsilon = r - s$, so $\epsilon < \lfloor \lambda(G)(1 - 1/s) \rfloor^{-1}$. For an edge xy , let Q be an s -clique containing x and y , and let $f(Q) = \{f(v) : v \in Q\}$. Let $t = |f(Q) \cap [f(x), f(y)]_r|$, so $s - t = |f(Q) \cap [f(y), f(x)]_r|$. By Lemma 2.1,

$$t \leq \ell([f(x), f(y)]_r) \leq t + \epsilon \quad \text{and} \quad s - t \leq \ell([f(x), f(y)]_r) \leq s - t + \epsilon. \quad (1)$$

By definition, $w(xy) = \lfloor \ell([f(x), f(y)]_r) \rfloor$, so

$$f(y) \in [f(x) + w(xy), f(x) + w(xy) + \epsilon]_r. \quad (2)$$

By (1), $w(yx) = s - w(xy)$. Let \mathcal{B} be a basis of the cycle space such that $\lambda(\mathcal{B}) = \lambda(G) = k$. To prove that w is a modular s -tension (when restricted to an orientation \vec{G} of G), it suffices to show that $w(C) \equiv 0 \pmod{s}$ for each C whose signed incidence vector lies in \mathcal{B} . (Since $w(yx) = s - w(xy)$, the choice of \vec{G} does not matter.)

Let x_0, \dots, x_{l-1} be the vertices of C in order, and let $x_l = x_0$; note that $l \leq k$. Let $e_i = x_i x_{i+1}$. In testing whether $w(C) \equiv 0 \pmod{s}$, the orientation of the edges along C does not matter; all orientations yield the same congruence class for $w(C)$. Since the same sets of cycles yield bases under each orientation, in studying C we may assume an orientation with each e_i directed from x_i to x_{i+1} . Now $w(C) \equiv \sum_{i=0}^{l-1} w(e_i) \pmod{s}$.

Since each edge lies in a complete subgraph of order s , (2) applies to each edge, so $f(x_{i+1}) \in [f(x_i) + w(e_i), f(x_i) + w(e_i) + \epsilon]_r$ for $0 \leq i \leq l-1$. Combining the allowed variations in the intervals for all edges of C yields

$$f(x_0) \in [f(x_0) + w(C), f(x_0) + w(C) + l\epsilon]_r. \quad (3)$$

By symmetry, we may choose $f(x_0) = 0$, which reduces (3) to $0 \in [w(C), w(C) + l\epsilon]_r$.

Since $1 \leq w(e_i) \leq s-1$, we have $l \leq w(C) \leq (s-1)l$. Since $w(C)$ is an integer, by choosing q to be $\lceil w(C)/s \rceil$ or $\lfloor w(C)/s \rfloor$ we can write $w(C) = qs + j$ for integers q and j such that $\lceil l/s \rceil \leq q \leq \lfloor l(1-1/s) \rfloor$ and $|j| \leq s-1$. Now

$$[w(C), w(C) + l\epsilon]_r = [j + qr - q\epsilon, j + qr + (l-q)\epsilon]_r = [j - q\epsilon, j + (l-q)\epsilon]_r.$$

Since $q \leq \lfloor l(1-1/s) \rfloor$, we have $q\epsilon \leq \lfloor l(1-1/s) \rfloor \epsilon \leq \lfloor k(1-1/s) \rfloor \epsilon < 1$. Similarly, $q \geq \lceil l/s \rceil$ yields $(l-q)\epsilon \leq \lfloor l(1-1/s) \rfloor \epsilon < 1$. Since $0 \in [w(C), w(C) + l\epsilon]_r \subseteq (j-1, j+1)_r$, we thus have $j = 0$. That is, $w(C) \equiv 0 \pmod{s}$. Thus w is a modular s -tension on \vec{G} . \blacksquare

The conclusion of Lemma 2.2 states that G is s -colorable. This is impossible if G is s -regular with odd order, so the lemma implies that $\chi_c(G) \geq s + \lfloor \lambda(G)(1-1/s) \rfloor^{-1}$. With $G = L(H \square C_{2m+1})$, we obtain a lower bound for $\chi'_c(H \square C_{2m+1})$, but it is not the lower bound we seek. The cycle space for G contains copies of the cycle space for H , but it is larger, and it may be that $\lambda(G) > \lambda(H)$, so the bound may be weaker than desired. To improve the bound, we will study subgraphs of G where we can control the value of λ . Before introducing these subgraphs, we prove a technical lemma about the color classes of the colorings generated from the modular s -tension produced by Lemma 2.2.

Lemma 2.3 *Let G be a graph such that each edge lies in a complete subgraph of order s . Suppose that G has an r -coloring f such that*

$$r < s + \frac{1}{\lfloor \lambda(G)(1-1/s) \rfloor}.$$

For a fixed vertex $v^* \in V(G)$ and any $x \in V(G)$, let $g(x) = \lfloor \ell([f(v^*), f(x)]_r) \rfloor$. This function g is a proper (integer) s -coloring of G that satisfies the following property: $g(x) = g(x')$ if and only if G has a vertex list (x, \dots, x') in which any consecutive entries v and v' satisfy $d_G(v, v') = 2$ and $|f(v) - f(v')|_r < 1/2$.

Proof. Call a list (x, \dots, x') with the specified properties an x, x' -skiplist.

Let \vec{G} be an orientation of G . By Lemma 2.2, setting $w(xy) = \lfloor \ell([f(x), f(y)]_r) \rfloor$ for all $xy \in E(\vec{G})$ defines a modular s -tension w on \vec{G} , and g is an s -coloring of G generated from w . Since the values of w are integers in $\{0, \dots, s-1\}$, in fact g is a proper (integer) s -coloring of G .

Vertices y and y' with $|f(y) - f(y')|_r < 1/2$ must be nonadjacent. If they have a common neighbor z , then

$$f(z) \in [f(y) + w(yz), f(y) + w(yz) + \epsilon]_r \cap [f(y') + w(y'z), f(y') + w(y'z) + \epsilon]_r.$$

If $|w(y'z) - w(yz)| \geq 1$, then the intervals on the right are disjoint, since $\epsilon < 1/2$ and $|f(y) - f(y')|_r < 1/2$. Therefore $w(yz) = w(y'z)$, which yields $g(y) = g(y')$. Therefore, all vertices in an x, x' -skiplist have the same color under g ; in particular, $g(x) = g(x')$.

Conversely, suppose that $g(x) = g(x')$. Let v_0, \dots, v_t be the vertices along an x, x' -path in G , with $x = v_0$ and $x' = v_t$. For $0 \leq i \leq t-1$, let X_i be an s -clique of G containing v_i and v_{i+1} . Select auxiliary vertices x_0, \dots, x_t as follows. Having selected x_0, \dots, x_{i-1} (starting with $x_0 = v_0 = x$), observe that $v_i \in X_{i-1} \cap X_i$. By Lemma 2.1, there is a unique vertex $x_i \in X_i$ with $|f(x_i) - f(x_{i-1})|_r \leq \epsilon < 1/2$. Applying the preceding paragraph with $y = x_i$ and $y' = x_{i-1}$ yields $g(x_i) = g(x_{i-1})$. Finally, $x_t = x'$, since $x_t, x' \in X_t$ and $g(x_t) = g(x) = g(x')$. Now (x_0, \dots, x_t) is an x, x' -skiplist. \blacksquare

The crucial consequence of Lemma 2.3 is that the partition of G into color classes under g does not depend on the choice of v^* .

3 A Lower Bound on $\chi'_c(H \square C_{2m+1})$

We specialize again to the study of $\chi'_c(H \square C_{2m+1})$. When H is $(s-2)$ -regular with odd order, the product $H \square C_{2m+1}$ is s -regular with odd order and hence is Class 2. Thus $\chi'_c(H \square C_{2m+1}) > s$. We improve this lower bound when s is divisible by 4.

Let $V(C_{2m+1}) = \{v_0, \dots, v_{2m}\}$, indexed in order; treat subscripts modulo $2m+1$. The i th layer H_i of $H \square C_{2m+1}$ is the subgraph induced by $V(H) \times \{v_i\}$. Each layer H_i is isomorphic to H . For $e \in E(H)$ and $x \in V(H)$, let e^i and x^i denote the copies of e and x in H_i . We call $\bigcup_{i=0}^{2m} E(H_i)$ the *horizontal edges* of $H \square C_{2m+1}$.

For $x \in V(H)$, let l_x^i denote the edge $x^i x^{i+1}$ in $H \square C_{2m+1}$. Let $L_i = \{l_x^i : x \in V(H)\}$; we call L_i the i th *link* of $H \square C_{2m+1}$ and call $\bigcup_{i=0}^{2m} L_i$ the *vertical edges* of $H \square C_{2m+1}$.

In a graph G whose vertices all have degree s or 1 , any two incident edges are incident at a vertex of degree s . Therefore, in $L(G)$ every edge lies in a complete subgraph of order s . We will be applying the results of Section 2 to subgraphs of $H \square C_{2m+1}$ having the form $L_{i-1} \cup H_i \cup L_i$, where every vertex has degree s or 1 . We also need the following observation.

Lemma 3.1 *For any graph G , the equality $\lambda(L(G)) = \lambda(G)$ holds.*

Proof. Since cycles in G turn into cycles in $L(G)$ and must be spanned by any basis for $L(G)$, we have $\lambda(L(G)) \geq \lambda(G)$. Also, a basis for the cycle space of G (indexed by edges) can be augmented to a basis for the cycle space of $L(G)$ (indexed by vertices) by adding the incidence vectors of triangles in $L(G)$ consisting of three edges in G having a common endpoint. The added vectors have weight 3, so $\lambda(L(G)) \leq \lambda(G)$. \blacksquare

Theorem 3.2 *If H is an $(s-2)$ -regular graph of odd order, where $4 \mid s$, then*

$$\chi'_c(H \square C_{2m+1}) \geq s + \frac{1}{\lfloor \lambda(H)(1 - 1/s) \rfloor}.$$

Proof. If not, then $H \square C_{2m+1}$ has an $(s + \epsilon)$ -edge-coloring f , where $\epsilon < \lfloor \lambda(H)(1 - 1/s) \rfloor^{-1}$.

Let G_i be the subgraph of $L(H \square C_{2m+1})$ induced by $L_{i-1} \cup E(H_i) \cup L_i$ (as defined above). Each edge of G_i lies in a complete subgraph of order s . Let \mathcal{T} be the set of triangles in G_i . If \mathcal{B} is a basis of the cycle space of $L(H_i)$, then $\mathcal{B} \cup \mathcal{T}$ contains a basis of the cycle space of G_i . Thus $\lambda(G_i) = \lambda(L(H_i)) = \lambda(L(H)) = \lambda(H)$, using $H_i \cong H$ and Lemma 3.1.

For each G_i , Lemma 2.3 states that the function g_i defined by fixing $v^* \in V(G_i)$ and setting $g_i(x) = \lfloor \ell([f(v^*), f(x)]_r) \rfloor$ for all $x \in V(G_i)$ is a proper (integer) s -coloring of G_i . Since this g_i depends only on the global r -coloring f and the choice of v^* , the restrictions to L_i of the partitions of $V(G_i)$ and $V(G_{i+1})$ into color classes under g_i and g_{i+1} are the same when v^* is chosen to be an element of L_i .

Furthermore, Lemma 2.3 implies that the partition of $V(G_i)$ into color classes does not depend on the choice of v^* ; it is determined only by values of f . We conclude that no matter how v_i^* and v_{i+1}^* are chosen in specifying g_i and g_{i+1} , the resulting partitions of L_i into color classes are the same.

Each vertex x^i of the product has two incident vertical edges, namely l_x^i and l_x^{i-1} . We say that a color j is a *vertical color at x^i* if some vertical edge incident to x^i has color j under g_i . For each $x^i \in V(H_i)$, the s incident edges of G_i have distinct colors. Therefore a color j is a vertical color at x^i if and only if no edge of H_i incident to x^i has color j under g_i . Since H has odd order, and the number of vertices of H_i incident to edges of H_i with color j is even, we conclude that j is a vertical color at an odd number of vertices of H_i . In other words, in the partition of $L_{i-1} \cup L_i$ formed by the color classes under g_i , each class has odd size.

Let C_i^+ [respectively, C_i^-] be the set of colors used by g_i on an odd number of edges of L_i [respectively, L_{i-1}]. Since each class under g_i has odd size in $L_i \cup L_{i-1}$, we conclude that $j \in C_i^-$ if and only if $j \notin C_i^+$.

Since $|L_i|$ and $|L_{i-1}|$ are odd, it follows that $|C_i^+|$ and $|C_i^-|$ are also odd. Since $|C_i^+| + |C_i^-| = s$ and s is divisible by 4, it follows that $|C_i^+| \neq |C_i^-|$. Since g_i and g_{i+1} induce the same partitions of L_i , it follows that $|C_{i+1}^-| = |C_i^+|$, and hence also $|C_{i+1}^+| = |C_i^-|$. Now the values of $|C_i^+|$ must alternate between two distinct values as i runs through all $2m + 1$ subscripts, which is impossible since $2m + 1$ is odd. ■

4 An Upper Bound on $\chi'_c(H \square C_{2m+1})$

In this section, we obtain an upper bound on $\chi'_c(H \square C_{2m+1})$ for some H . As a consequence, we show that $\chi'_c(H \square C_{2m+1}) - \Delta(H \square C_{2m+1})$ can be bounded above by a number that is arbitrarily close to $\chi'_c(H) - \Delta(H)$ by making m sufficiently large.

We show first that increasing m cannot increase the circular chromatic index. We simply use the coloring of one layer on three consecutive layers in the larger graph and re-use the colorings on its neighboring links.

Lemma 4.1 *If $m' \geq m$, then $\chi'_c(H \square C_{2m'+1}) \leq \chi'_c(H \square C_{2m+1})$.*

Proof. It suffices to prove that $\chi'_c(H \square C_{h+2}) \leq \chi'_c(H \square C_h)$ for all h . Let f be an r -edge-coloring of $H \square C_h$. Form an r -edge-coloring f' of $H \square C_{h+2}$ as follows. Color the layers H_0, \dots, H_{h-1} and links L_0, \dots, L_{h-1} as under f . Color the layers H_h and H_{h+1} the same as H_{h-1} . Color the links L_h and L_{h+1} the same as L_{h-2} and L_{h-1} , respectively. Now the colors on any two incident edges of $H \square C_{h+2}$ under f' are also colors on two incident edges of $L(H \square C_h)$ under f . Thus f' is also an r -edge-coloring. ■

Since $\chi'_c(H \square C_{2m+1}) \geq \Delta(H \square C_{2m+1}) = \Delta(H) + 2$ for all m , Lemma 4.1 implies that $\chi'_c(H \square C_{2m+1})$ has a limit as $m \rightarrow \infty$. In Section 5 we show that this limit is attained when H is an odd cycle, and we compute its value.

To prove the upper bound, we need a standard result about circular coloring.

Lemma 4.2 (See [13]) *If a graph G has a r -coloring f with $r = p/q$ where $p, q \in \mathbb{N}$, then it has an r -coloring f' such that the colors under f' are multiples of $1/q$, and such that if $xy \in E(G)$, then $|f'(x) - f'(y)|_r$ differs by less than $1/q$ from $|f(x) - f(y)|_r$.*

Proof. Let $f'(x) = \lfloor qf(x) \rfloor / q$ (such multiplication arguments were used as early as [3]). Note that $f'(x)$ is the largest multiple of $1/q$ that does not exceed $f(x)$. Under this transformation, $|f'(x) - f'(y)|_r$ equals $|f(x) - f(y)|$ if the latter is a multiple of $1/q$. Otherwise, the difference shifts to the next larger or next smaller multiple of $1/q$.

In particular, if the colors assigned to two vertices differ by at least a/q before the transformation, for some positive integer a (such as $a = q$), then they also differ by at least a/q after the transformation. Thus f' is an r -coloring. ■

Given an r -edge-coloring of a graph H , a *color gap* for a vertex x of H is a maximal open interval on the circle C^r that contains no color used on an edge incident to x .

Theorem 4.3 *Let H be a graph having a p/q -edge-coloring f such that every vertex x of H has a color gap of length at least 3. If p is odd and $2m + 1 \geq p$, then $\chi'_c(H \square C_{2m+1}) \leq p/q$.*

Proof. By Lemma 4.1, it suffices to prove this when $2m + 1 = p$. By Lemma 4.2 (applied to $L(H)$), we may assume that each $f(e)$ is a multiple of $1/q$, still with each vertex having a color gap of length at least 3 (using $a = 3q$ in that argument). For each $x \in V(H)$, let $(a_x, b_x)_{p/q}$ be a color gap under f with length at least 3.

We produce a p/q -edge-coloring ϕ for $H \square C_{2m+1}$. We use the same coloring f in each layer, except that the colors in each layer increase by one unit from the colors on the corresponding edges in the previous layer. Since $2m = p - 1 = q(p/q) - 1$, the colors on layer H_0 are also one unit (modulo p/q) above the corresponding colors on H_{2m} . This is achieved by letting $\phi(e^i) = f(e) + i \pmod{p/q}$ for each $e \in E(H)$ and $0 \leq i \leq 2m$.

It now suffices to use the color gaps to fit in colors for the vertical edges. Specifically, we set $\phi(l_x^i) = a_x + 2 + i \pmod{p/q}$ for each $x \in V(H)$ and $0 \leq i \leq 2m$. Since no horizontal edge at x^i receives a color in $(a_x + i, a_x + i + 3)$, the colors $a_x + i + 1$ and $a_x + i + 2$ are available for l_x^{i-1} and l_x^i , respectively, when viewed from x^i . Furthermore, ϕ achieves this assignment simultaneously for the vertical edges at all x^j . Hence for all incident edges, the assigned colors differ by at least 1. \blacksquare

For any graph G , let $\partial(G) = \chi'_c(G) - \Delta(G)$. Thus G is Class 1 if and only if $\partial(G) = 0$, and otherwise $0 < \partial(G) \leq 1$.

Corollary 4.4 *For any graph H , $\lim_{m \rightarrow \infty} \partial(H \square C_{2m+1}) \leq \partial(H)$.*

Proof. The limit exists, using $\Delta(H \square C_{2m+1}) \geq \Delta(H) + 2$ and Lemma 4.1. It suffices to show, given $\epsilon > 0$, that $\partial(H \square C_{2m+1}) \leq \partial(H) + \epsilon$ when m is sufficiently large.

Choose $p, q \in \mathbb{N}$ with p odd such that $\chi'_c(H) \leq p/q \leq \chi'_c(H) + \epsilon$. Let f be a p/q -edge-coloring of H . Also f can be viewed as a $(p/q + 2)$ -edge-coloring of H . For $x \in V(H)$, let b_x and a_x be the minimum and maximum colors in $[0, p/q)$ used on edges incident to x , respectively. Since $\ell((a_x, b_x)_{p/q}) \geq 1$, also $\ell((a_x, b_x)_{p/q+2}) \geq 3$. Relative to f as a $(p/q + 2)$ -edge-coloring, each vertex of H thus has a color gap of length at least 3. By Theorem 4.3, $\chi'_c(H \square C_{2m+1}) \leq p/q + 2 \leq \Delta(H \square C_{2m+1}) + \epsilon$ when $2m + 1 \geq p$. \blacksquare

Recall that $H \square H'$ is Class 1 when H or H' is Class 1. That is, $\partial(H) = 0$ or $\partial(H') = 0$ implies $\partial(H \square H') = 0$. It is natural to ask if $\partial(H \square H') \leq \min\{\partial(H), \partial(H')\}$ always holds. It does not, by the following example. Let $H = C_{2k+1}$ and $H' = C_{2m+1}$. Since $\chi'_c(C_{2m+1}) = 2 + 1/m$, we can make $\partial(H')$ arbitrarily small. However, $\lambda(H) = 2k + 1$, so Theorem 3.2 yields $\partial(H \square H') \geq \lfloor (6k + 3)/4 \rfloor^{-1} = \lceil 3k/2 \rceil^{-1}$, independent of m .

On the other hand, $\lceil 3k/2 \rceil^{-1} < k^{-1} = \partial(C_{2k+1})$. Based on this and Theorem 4.3 and other examples, we propose the following conjecture.

Conjecture 4.5 For any graphs H and H' , $\partial(H \square H') \leq \max\{\partial(H), \partial(H')\}$.

5 Tightness of the lower bound

As noted above, Theorem 3.2 implies that $\chi'_c(C_{2k+1} \square C_{2m+1}) \geq 4 + \lceil 3k/2 \rceil^{-1}$ for all m . In this section, we prove that the bound is sharp when $m \geq 3k + 1$. This proves Conjecture 4.5 for products of two odd cycles when one is at least three times as long as the other.

Lemma 5.1 *If there exist integers α, β, q with $0 < q \leq m/2$ such that $|\alpha| + |\beta| = 2k + 1$ and $\alpha q + \beta(q + 1) \equiv 0 \pmod{4q + 1}$, then $\chi'_c(C_{2k+1} \square C_{2m+1}) \leq 4 + 1/q$.*

Proof. By Theorem 4.3 with $p = 4q + 1$, it suffices to produce a $(4 + 1/q)$ -edge-coloring f of C_{2k+1} such that every vertex x of C_{2k+1} has a color gap of length at least 3. Since C_{2k+1} is 2-regular, and we use a color circle of length $4 + 1/q$, the condition on f becomes “If e and e' are incident edges in C_{2k+1} , then $1 \leq |f(e') - f(e)|_{(4+1/q)} \leq 1 + 1/q$.” Multiplying by q , we further transform this to seeking integers z_1, \dots, z_{2k+1} modulo $4q + 1$ such that neighboring integers differ by q or $q + 1$.

In the hypothesis, we may assume by symmetry that $\alpha \geq 0$. We construct the first α and last $|\beta|$ integers as separate arithmetic progressions, with common difference q for the first α and $q + 1$ for the last $|\beta|$. For $1 \leq i \leq \alpha$, let $z_i = iq$ (this portion is empty if $\alpha = 0$). For $1 \leq i \leq |\beta|$, let $z_{\alpha+i} = \alpha q + \epsilon i(q + 1)$, where $\epsilon = 1$ if $\beta > 0$ and $\epsilon = -1$ if $\beta < 0$.

The construction enforces the needed differences until just before the end; we need only compare z_{2k+1} and z_1 . Since $z_{2k+1} = \alpha q + \beta(q + 1) \equiv 0 \pmod{4q + 1}$, indeed z_{2k+1} and z_1 differ by q . ■

Theorem 5.2 *If $m \geq 3k + 1$, then $\chi'_c(C_{2k+1} \square C_{2m+1}) = 4 + \lceil 3k/2 \rceil^{-1}$.*

Proof. We have noted that Theorem 3.2 gives the lower bound. It suffices to find integers α, β, q satisfying the hypotheses of Lemma 5.1 with $q = \lceil 3k/2 \rceil = \lfloor (6k + 3)/4 \rfloor$.

Let $r = \lfloor (k - 1)/2 \rfloor$, so $k = 2r + s$ with $1 \leq s \leq 2$. Now $q = 3r + s + 1$. Let $\alpha = s - 1$ and $\beta = -(4r + s + 2)$. We have $|\alpha| + |\beta| = (4r + 2s + 1) = 2k + 1$ and

$$\alpha q + \beta(q + 1) = (s - 1)q - (4r + s + 2)(q + 1) = -(4q + 1)(r + 1),$$

where the last computation uses $q = 3r + s + 1$. Thus $\alpha q + \beta(q + 1) \equiv 0 \pmod{4q + 1}$, and Lemma 5.1 applies. ■

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