Circular Chromatic Index of Cartesian Products of Graphs

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Traffic Lights and Coloring

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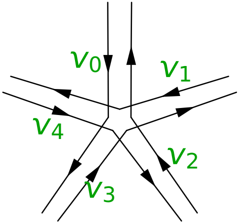

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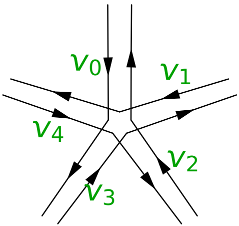

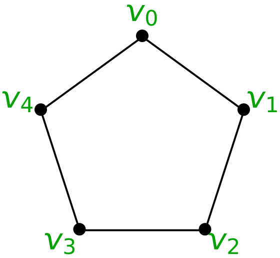

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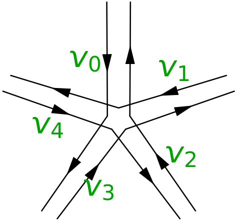

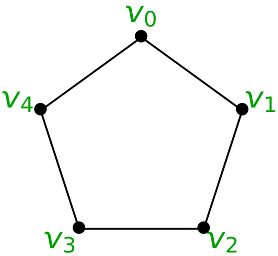

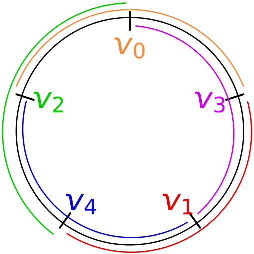

More efficient cycle: 2.5 units.

A Circular Coloring Model

Many equivalent definitions, this one convenient:

Def. r-coloring of G, for real r: a fcn $f\colon V(G) \to [0, r)$ such that $1 \le |f(x) - f(y)| \le r - 1$ when $xy \in E(G)$.

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- (Vince [1988]) infimum is achieved, χ_c is rational.
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• Similar for $\chi_c(G)$ if max avg. degree is bounded.
E.g., triangle-free & $\mathrm{mad}(G) < 12/5 \Rightarrow \chi_c(G) \leq 5/2$, which is sharp (Borodin–Hartke–Ivanova–Kostochka–West [2007+]).

Cartesian Product

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- $G \,\square\, H$ is Class 2 when G and H are regular graphs of odd order (no perfect matching!).

Main Results

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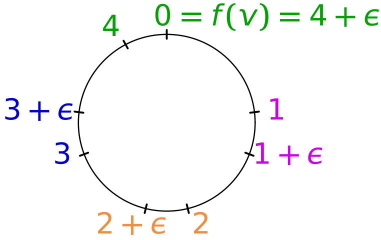
- Always $\chi'_c(H \square C_{2m+1})$ descends to a limit as m grows.

Ideas for Lower Bound, I

Lem. If G has an r-coloring f with $r = s + \epsilon$, and Q is an s-clique containing v, then Q has one vertex with color in each arc $[f(v) + i, f(v) + i + \epsilon]$, for $0 \leq i \leq s - 1$.



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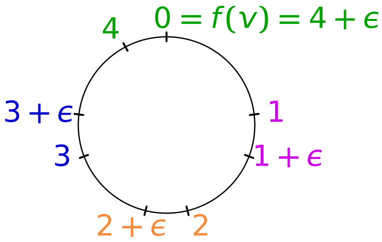

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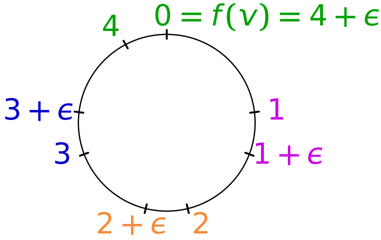

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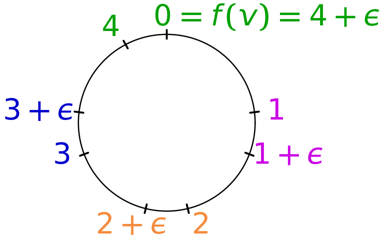

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We study $\chi'_c(H \square C_{2m+1})$ by studying r-colorings of graphs where every edge lies in a copy of K_s.

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We need graphs G_i with $\lambda(G_i) = \lambda(H)$.

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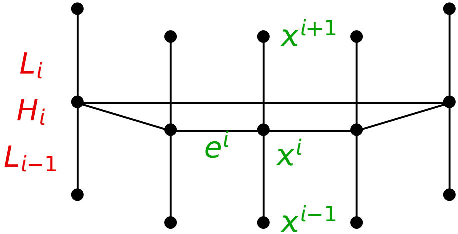

i th link, vertical edges

i th layer, horizontal edges

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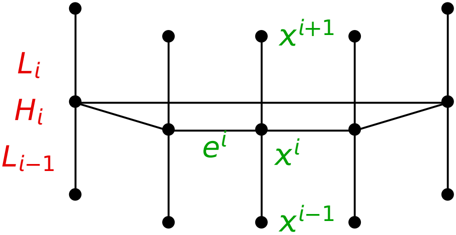

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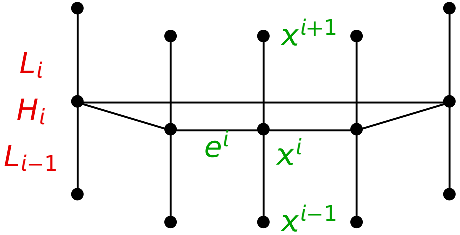

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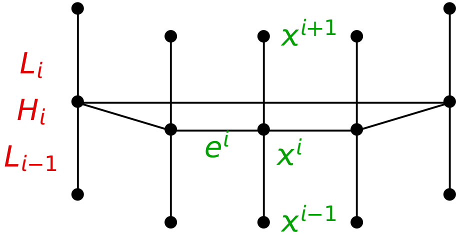

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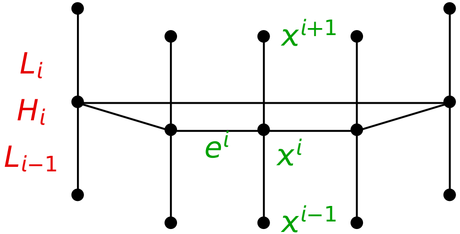

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- The color ptn of $V(G)$ under g does not depend on v^*.

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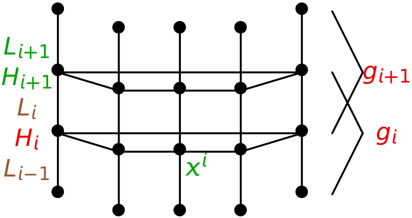

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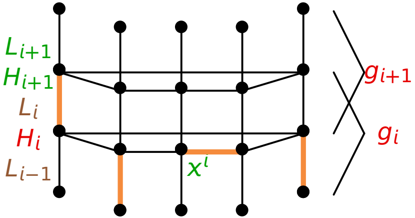

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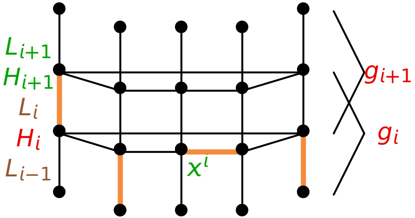

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Let c_i be # colors w odd usage in L_i under g_i; it is odd.

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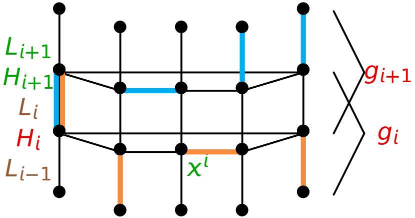

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Let c_i be # colors w odd usage in L_i under g_i; it is odd.

Same # has odd usage under g_{i+1} (same ptn of L_i).

Odd usage in L_i $\Leftrightarrow$ even usage in L_{i+1}, so $c_{i+1} = s - c_i$.

Now $4 \mid s$ $\Rightarrow$ $c_i \neq s/2$, so c_i alternates values but can't!

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Pf. Step 1: Modify f to multiples of $1/q$.

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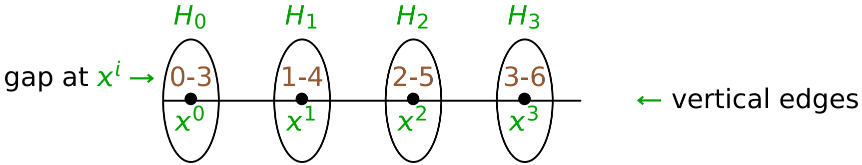
Difference between $f'(x)$ and $f'(y)$ is bounded by the same multiples of $1/q$ as between $f(x)$ and $f(y)$.

Color gaps of length ≥ 3 are preserved.

Construction of Coloring

Step 2: The case $2m + 1 = p$.
Use f on each layer H_i, shifted cyclically by i.
Gaps $(\alpha-1, \alpha+2)$ at x^i and $(\alpha, \alpha+3)$ at x^{i+1} share $[\alpha, \alpha+1]$.



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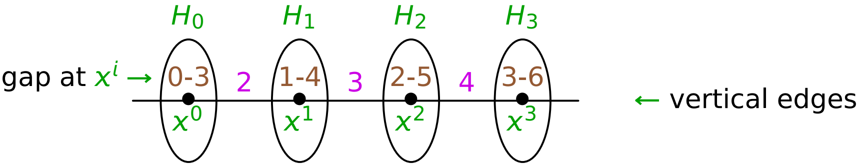
The color on the pth link is one before the first.



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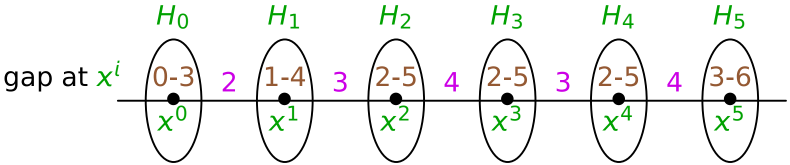

Step 3: The case $2m+1>p$.
Augment the case $2m-1$: insert two new layers next to a fixed H^i with the same r-edge-coloring as H^i. Go down once between the new layers and then continue up.

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Hence the upper bound theorem applies. ■

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Elementary computation shows that these values satisfy the hypotheses of the lemma. ∎

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- The result on toroidal grids shows that "max" cannot be replaced by "min", but in the analogous statement about ordinary edge-chromatic number it can.