Cutwidth of Triangular Grids *

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Abstract

When the vertices of an n-vertex graph G are numbered by the integers 1 through n, the *length* of an edge is the difference between the numbers on its endpoints. Two edges *overlap* if the larger of their lower numbers is less than the smaller of their upper numbers. The *bandwidth* of G is the minimum, over all numberings, of the maximum length of an edge. The *cutwidth* of G is the minimum, over all numberings, of the maximum number of pairwise overlapping edges. The bandwidth of triangular grids was determined by Hochberg, McDiarmid, and Saks in 1995. We show that the cutwidth of the triangular grid with side-length l is $2l$.

Keywords: graph labeling, cutwidth, bandwidth, triangular grid.

1 Introduction

The bandwidth and cutwidth problems for graphs are optimization problems applied in VLSI design, network communications, and other areas involving graph layout (see [3, 4, 6, 14]). We compute the cutwidth for certain special graphs; first we formulate the problem.

Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G. A *numbering* (or *labeling*) of an n-vertex graph G is a bijection $f\colon V(G) \to \{1, \dots, n\}$. Given a numbering f of G, let

$$c(G, f) = \max_{1 \le i < n} \left|\{uv \in E(G)\colon f(u) \le i < f(v)\}\right|.$$

When f is viewed as embedding G in a path, $c(G, f)$ is the maximum number of pairwise overlapping edges, measuring congestion. The *cutwidth* of G, denoted $c(G)$, is $\min\{c(G, f)\colon f$ is a numbering of $G\}$. A numbering f that minimizes $c(G, f)$ is *optimal*.

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Similarly, for a given numbering f, let

$$B(G, f) = \max\{|f(u) - f(v)|\colon\ uv \in E(G)\}.$$

When f is viewed as embedding G in a path, $B(G, f)$ is the maximum length (dilation) of an edge. The *bandwidth* of G, denoted $B(G)$, is $\min\{B(G, f)\colon\ f \text{ is a numbering of } G\}$.

Much work has been done on computing bandwidth and cutwidth of special graphs, especially graphs relevant in the areas of application. Let P_n and C_n denote the path and cycle with n vertices. The *cartesian product* of graphs G and H, denoted $G\square H$, is the graph with vertex set $V(G) \times V(H)$ in which two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. The early results include

- $B(P_m\square P_n) = \min\{m, n\}$ for $m, n \geq 2$.
- $B(P_m\square C_n) = \min\{2m, n\}$ for $m \geq 2, n \geq 3$.
- $B(C_m\square C_n) = 2\ \min\{m, n\} - \delta_{m,n}$ for $m, n \geq 3$,

where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise. The first two of these results were obtained by Chvátalová [5]; the third was obtained by Li, Tao, and Shen [10].

The cutwidth of these graphs has also been computed [13, 16]:

- $c(P_m\square P_n) = \min\{m, n\} + 1$ for $m \geq 2, n \geq 3$.
- $c(P_m\square C_n) = \min\{2m, n + 1\} + 1$ for $m, n \geq 3$.
- $c(C_m\square C_n) = 2\ \min\{m, n\} + 2$ for $m, n \geq 3$.

In these examples, always $B(G) < c(G)$. Equality holds when $G = C_n$. Only some sparse graphs (such as some trees [12]) are known to satisfy $c(G) \leq B(G)$; it would be interesting to determine which graphs satisfy this inequality.

The cutwidth was also computed for "meshes"[17]. Polynomial-time algorithms are known for computing cutwidth on trees [18] and for recognizing when $c(G) \leq k$ [7, 15]. An exact formula for cutwidth on trees called "iterated caterpillars" appears in [12]. The cutwidth of n-dimensional hypercubes was studied in [2, 11].

The *triangular grid* T_l is the graph whose vertices are the nonnegative integer triples with sum l such that vertices (x, y, z) and (x', y', z') are adjacent if and only if $|x - x'| + |y - y'| + |z - z'| = 2$. That is, two vertices are adjacent when they agree in one coordinate and differ by 1 in the other two coordinates. M.L. Weaver and the third author conjectured $B(T_l) = l + 1$. Using topological methods (Sperner's Lemma), Hochberg, McDiarmid, and Saks [9] proved this as a special case of a more general result computing the bandwidth of a family of triangulations of planar discs.

Our main result is the following.

Theorem 1.1. *$c(T_l) = 2l$.*

Similar arguments yield the cutwidths of rectangular grids with added diagonal edges.

2 Preliminaries

Our graphs have no loops or multi-edges. For a set S of vertices in a graph G, let $\overline{S} = V(G) - S$. The *neighborhood* of S, denoted $N(S)$, is $\{v \in \overline{S}\colon\ uv \in E(G) \text{ for some } u \in S\}$; note that $N(S) \subseteq \overline{S}$. The *boundary* of S is $N(\overline{S})$ (some authors refer to $N(S)$ as the boundary of S). The *coboundary* of S, denoted $\partial(S)$ following the notation of [1], is the set of edges in G that have endpoints in both S and $\overline{S}$.

The definitions yield an immediate rephrasing of $c(G, f)$.

Observation 2.1. *If f is a numbering of a graph G, and $S_i = \{v \in V(G)\colon\ f(v) \le i\}$, then $c(G, f) = \max_{1\le i<n} |\partial(S_i)|$.*

We may draw T_l in the plane by putting vertex (x, y, z) at point (x, y) and using (x, y) as the name of the vertex. Now the vertex set is the set of nonnegative integer pairs with sum at most l, and two vertices (x, y) and (x', y') are adjacent if and only if (a) $|x-x'|+|y-y'| = 1$ or (b) $|x - x'| + |y - y'| = 2$ and $x + y = x' + y'$. Figure 1 shows T_4 drawn in this way.

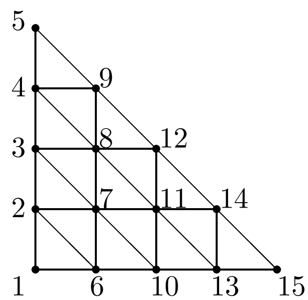

Figure 1: T_4, with an optimal numbering.

With this embedding, the edges of T_l lie in three sets of parallel lines, yielding three partitions of $V(T_l)$. The *horizontal sets* or *rows* have second coordinate fixed; let $P_j = \{(x, j) \in V(T_l)\colon\ 0 \le x \le l - j\}$. The *vertical sets* or *columns* have first coordinate fixed; let $Q_i = \{(i, y) \in V(T_l)\colon\ 0 \le y \le l - i\}$. The *slanted sets* or *diagonals* have the sum of the two coordinates fixed; let $R_k = \{(x, y) \in V(T_l)\colon\ x + y = l - k\}$. Each edge of T_l joins two vertices in one of these sets; the edges accordingly are *horizontal*, *vertical*, or *slanted* edges, respectively.

Lemma 2.2. *$c(T_l) \le 2l$.*

Proof. We number the vertices according to the lexicographical order on the names (x, y), beginning with $(0, 0), \dots, (0, l)$ as indicated in Figure 1. That is, the ordering is by columns. In this ordering, $\partial(S_i)$ contains at most one edge from each horizontal line, at most one edge from each slanted line, and at most one vertical edge. Furthermore, the only case when each nontrivial horizontal line and each nontrivial slanted line contributes one edge is $i = l + 1$, and in this case $\partial(S_i)$ contains no vertical line. Hence always $\partial(S_i) \le 2l$. □

For the lower bound, we first follow the method used by Chvátalová [5] in computing $B(P_m \Box P_n)$, showing that compressing a set in one direction does not increase the size of its coboundary. Given $S \subseteq V(T_l)$, let $a_j = |P_j \cap S|$ for $0 \le j \le l$. The *left-shift* of S is $\{(x, j) \in V(T_l)\colon\ x < a_j\}$.

Lemma 2.3. *If S' is the left-shift of a set S, then $|\partial(S')| \le |\partial(S)|$.*

Proof. Let E_j be the set of edges in T_l with both endpoints in P_j or endpoints in P_j and P_{j+1}. Since $E(T_l) = \bigcup_{0\le j<l} E_j$, it suffices to prove $|\partial(S') \cap E_j| \le |\partial(S) \cap E_j|$ for all j.

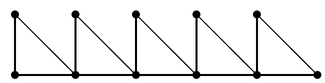

Figure 2: The subgraph of T_l with edge set E_{l-5}.

Let $V_j = P_j \cup P_{j+1}$. Consider the subgraph of T_l with vertex set V_j and edge set E_j; it consists of a chain of $l - j$ edge-disjoint triangles (see Figure 2). Let $S_j = S \cap V_j$ and $S'_j = S' \cap V_j$. Observe that $\partial(S) = \bigcup_{0=j}^{l}(\partial S_j \cap E_j)$, and similarly for S'. Hence it suffices to prove $|\partial(S'_j) \cap E_j| \le |\partial(S_j) \cap E_j|$ for all j, which we do by induction on $l - j$.

For $l - j = 0$ there is nothing to prove; $E_l = \varnothing$. Consider $j < l$. Since the coboundary of a set and its complement are the same, we may assume that S_j contains at least two of the three vertices in the leftmost triangle of E_j. We may also assume $a_j \ge 2$, by inspection.

If the leftmost vertices in P_j and P_{j+1} are both present, then they also lie in S'_j. Delete them and apply the induction hypothesis to the smaller graph. Replacing the two missing vertices adds nothing to $\partial(S'_j) \cap E_j$ (since $a_j \ge 2$), so the desired inequality holds. The symmetric argument also applies if the rightmost vertices in P_j and P_{j+1} are present, since we can obtain the left-shift by first right-shifting and then reversing left and right. The right-shift and left-shift have coboundaries of the same size.

In the remaining case, S_j contains $(1, j)$ and omits one vertex v among $\{(0, j), (0, j+1)\}$; also, S_j does not contain the rightmost vertices of both rows. Let v' be the rightmost vertex of S_j in the row containing v. Form the set $\hat{S}$ from S_j by substituting v for v'.

Note that $\hat{S}$ and S_j have the same left shift, S'_j. Since $\hat{S}$ contains both leftmost vertices, the previous case yields $|\partial(S'_j) \cap E_j| \le |\partial(\hat{S}) \cap E_j|$. Hence it suffices to prove $|\partial(\hat{S}) \cap E_j| \le |\partial(S_j) \cap E_j|$. Introducing v eliminates two edges from the coboundary. Deleting v' cannot introduce three edges into the coboundary with no deletions unless v' is the rightmost vertex in P_{j+1} and the rightmost vertex in P_j is also present, but we have excluded that possibility from this case. □

By symmetry, we may consider shifting along any of the three sets of parallel lines, and the analogous argument to Lemma 2.3 implies that the size of the coboundary does not increase.

For example, if $b_i = |Q_i \cap S|$ for $0 \leq i \leq l$, then the *down-shift* of S is $\{(i, y) \in V(T_l)\colon\ y < b_i\}$. If $c_k = |R_k \cap S|$, then the *slant-shift* of G is $\{(k - z, z) \in V(T_l)\colon\ z < c_k\}$.

Lemma 2.4. *For $S \subseteq V(T_l)$, if S' is the left-shift of S, and S'' is the down-shift of S', then S'' is unchanged under left-shift, and $|\partial(S'')| \leq |\partial(S)|$.*

Proof. Since S' is left-shifted, the list $\{|Q_i \cap S'|\colon\ 0 \leq i \leq l\}$ is nonincreasing, and $|Q_i \cap S''| = |Q_i \cap S'|$ for all i. Since S'' is down-shifted, the list $\{|P_j \cap S''|\colon\ 0 \leq j \leq l\}$ is nonincreasing also. Hence $(i, j) \in S''$ implies $(i - 1, j) \in S''$ for $i \geq 1$, and left-shifting does not change S''.

Successive shifts do not increase the size of the coboundary. □

3 Proof of the main result

A set $S \subseteq V(T_l)$ is *condensed* if it is invariant under left-shift and down-shift.

Lemma 3.1. *There is an optimal numbering f^* such that $\{v \in V(T_l)\colon\ f(v) \leq i\}$ is condensed whenever $1 \leq i \leq |V(T_l)|$.*

Proof. Observe first that if $S \subseteq \hat{S} \subseteq V(T_l)$, then the shift of S is contained in the shift of $\hat{S}$ (in any direction). Let f be an optimal numbering, and let $S_i = \{v \in V(T_l)\colon\ f(v) \leq i\}$. Note that $S_1 \subseteq \cdots \subseteq S_n$, where $n = |V(T_l)|$. Letting S'_i be the down-shift of the left-shift of S_i, we therefore also have $S'_1 \subseteq \cdots \subseteq S'_n$; let $S'_0 = \varnothing$. Since $|S'_i| = i$, the set S'_i is obtained by adding a single element v_i to S'_{i-1}, for $1 \leq i \leq n$. Define f^* by letting $f^*(v_i) = i$.

Since $S'_i = \{v \in V(T_l)\colon\ f(v) \leq i\}$, the specified sets are condensed. Also, by Lemma 2.4, $|\partial(S'_i)| \leq |\partial(S_i)|$ for all i. Hence $c(T_l, f^*) \leq c(T_l, f)$, and f^* is an optimal numbering. □

By Lemma 2.2, the following lower bound completes the proof of Theorem 1.1.

Theorem 3.2. $c(T_l) \geq 2l$.

Proof. Let f^* be an optimal numbering as guaranteed by Lemma 3.1. Let $n = |V(T_l)|$, and let $S_i = \{v \in V(T_l)\colon\ f^*(v) \leq i\}$ for $1 \leq i \leq n$; each S_i is condensed. Since $c(T_l, f^*) = \max_{1 \leq i < n} |\partial(S_i)|$, it suffices to show $|\partial(S_i)| \geq 2l$ for some i.

For $i \geq 2$, we obtain S_i from S_{i-1} by adding one vertex. Since $S_1 = \{(0, 0)\}$ and $S_n = V(T_l)$, there exists k such that S_k is the first set in the list that intersects R_1. Furthermore, S_k contains no element of R_0, the longest diagonal, since all the sets are condensed. Let U be the slant-shift of S_k. Since $|\partial(U)| \leq |\partial(S_k)|$, it suffices to show $|\partial(U)| \geq 2l$.

Since S_k is condensed, also U is condensed, by Lemma 2.4. Since $S_k \cap R_1 \neq \varnothing$, we have $(l-1, 0) \in U$. Since U is left-shifted, U intersects $Q_0, \ldots, Q_{l-1}$. Since $U \cap R_0 = \varnothing$, it follows that $\partial(U)$ has a vertical edge in each of these columns.

Let $b_i = |U \cap Q_i|$. Because U is left-shifted, $b_i \geq b_{i+1}$ for $i < l$. Because U is slant-shifted, $b_i \leq b_{i+1} + 1$ for $i < l$. Hence $b_i \in \{b_{i+1}, b_{i+1} + 1\}$. If $b_i = b_{i+1}$, then the slanted edge from

$(i, b_i + 1)$ to $(i + 1, b_i)$ is in $\partial(U)$. If $b_i = b_{i+1} + 1$, then the horizontal edge from (i, b_i) to $(i + 1, b_i)$ is in $\partial(U)$. Thus $\partial(U)$ also has a horizontal or a slanted edge joining Q_i to Q_{i+1} for $i < l$. We conclude that $|\partial(U)| \geq 2l$. □

A similar argument determines the cutwidth for rectangular grids with diagonal edges added, as shown in Figure 3. If m and n are the numbers of rows and columns in the grid, then the cutwidth is $2 \min\{m, n\} - 1$ (note that the parameter l for the largest triangular grid contained in the graph is $\min\{m, n\} + 1$). For $m \leq n$, an optimal numbering is given by the lexicographic order on the vertices as in Lemma 2.2, suggested also in the figure. The lower bound again extracts an optimal numbering whose initial sets are condensed, and then the same lower bound argument applies to the set U.

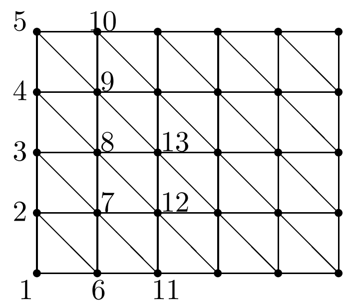

Figure 3: Triangulated rectangular grid $R_{5,6}$.

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