Acquisition Parameters of Graphs

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Results with or by

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Kevin G. Milans, Noah Prince, Chris Stocker,
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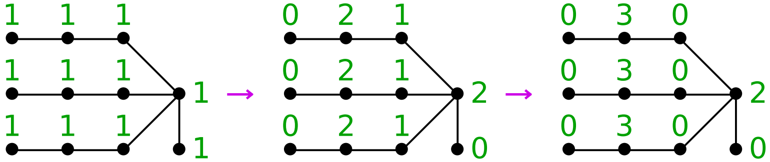
Def. total aquisition number $a_t(G)$ = min size of the final indep. set when each vertex starts with weight 1.

An Example

Ex. $a_t(T) = 4$, 11 vertices.



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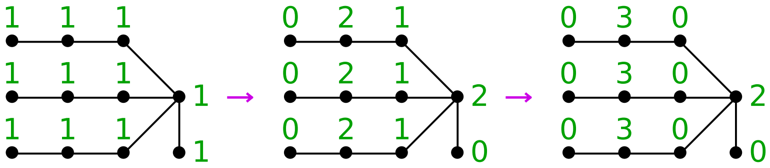

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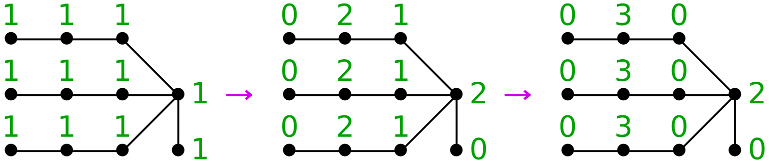

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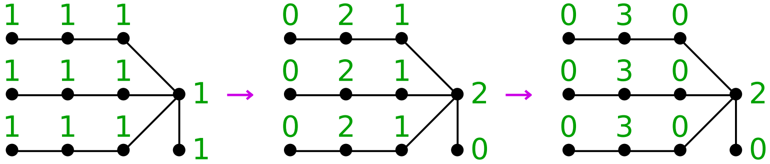

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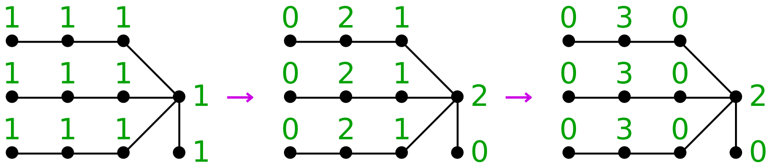

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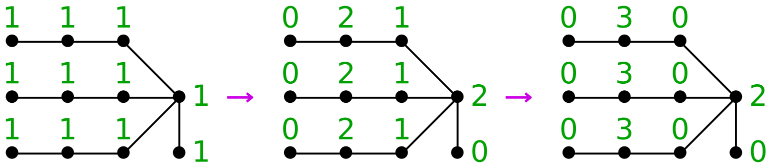

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game acquisition: move all weight, but two players Min and Max alternate moves — $a_g(G)$

Total Acquisition Number — Extremal Problem

All our graphs have n vertices.

Thm. (Lampert–Slater [1995]) If G is connected and nontrivial, then $a_t(G) \le \frac{n+1}{3}$, and this is sharp.

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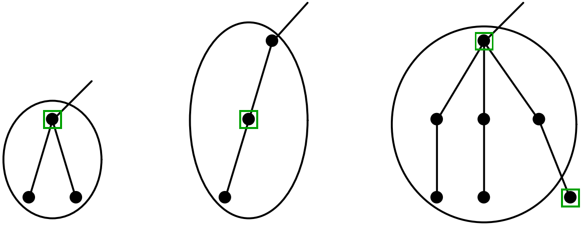
Easy cases:



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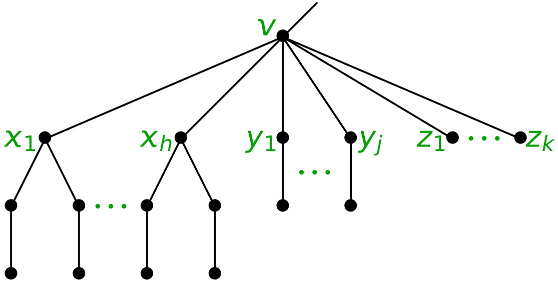

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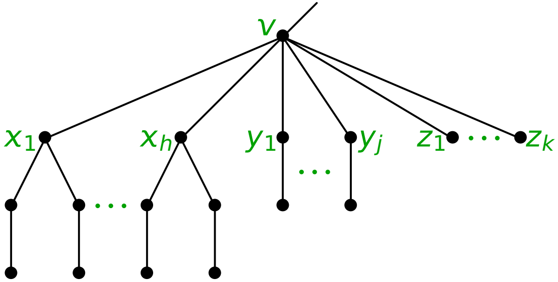
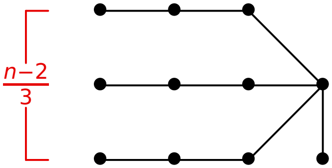


Sharpness:



Lower bound — An Obstruction

- View each initial unit of weight as a token or chip.

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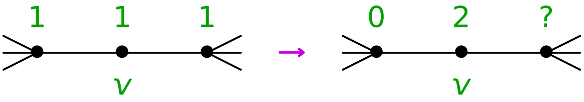

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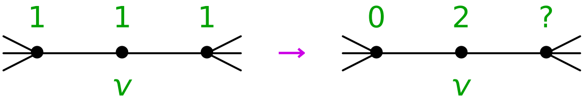

Cor. If the x, y-path in a tree has a vertex of degree 2 adjacent to neither x nor y, then the chips starting at x and y cannot combine.

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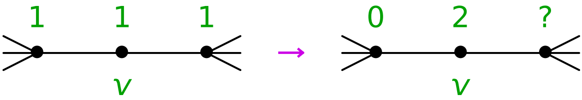

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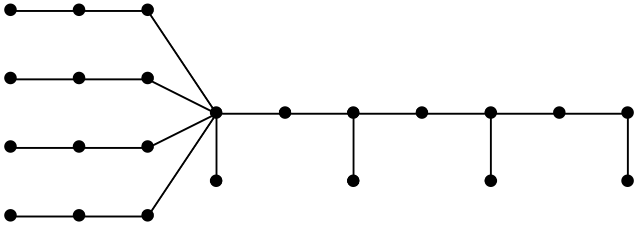
• The lower bounds on a_t that use these observation apply also to a_u.

Trees with $a_t(G)$ Large

Ex. The tree $T_{l,m}$ with $(l, m) = (4, 3)$.



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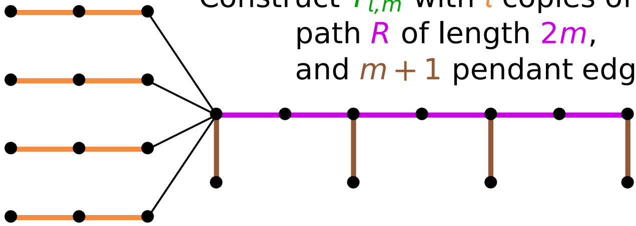

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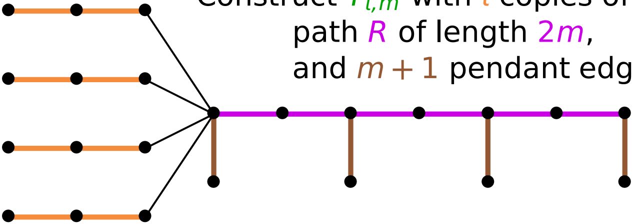

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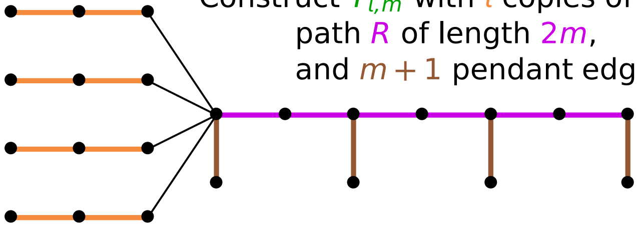

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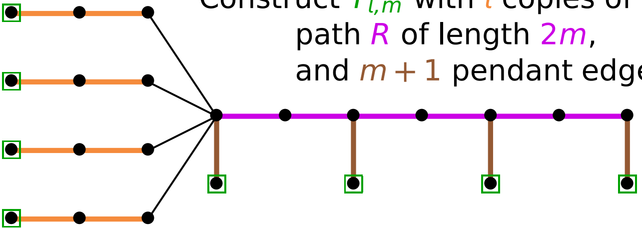

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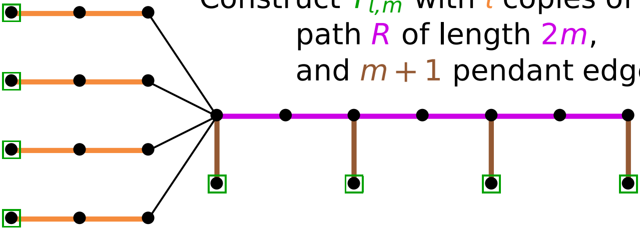

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Thm. For $d \geq 3$ and $k \geq 6$, there is a tree T with
$\Delta(T) = d$, $\mathrm{diam}\, T \geq k$, and $a_u(T) = a_t(T) = \frac{|V(T)|+1}{3}$.

The Extremal Graphs

Thm. (LeSaulnier–West [2013]) An n-vertex graph G satisfies $a_t(G) = \frac{n+1}{3}$ if and only if G is a tree obtained from P_2 by iteratively growing a 3-edge path from a neighbor of a leaf.

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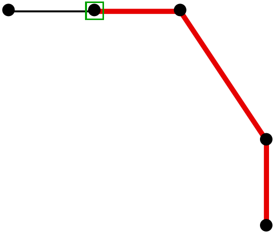

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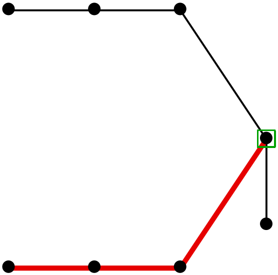

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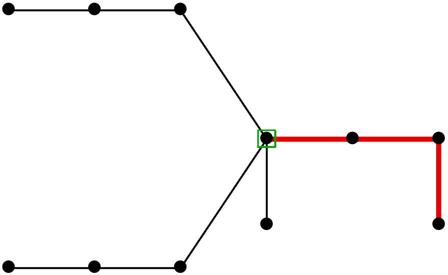

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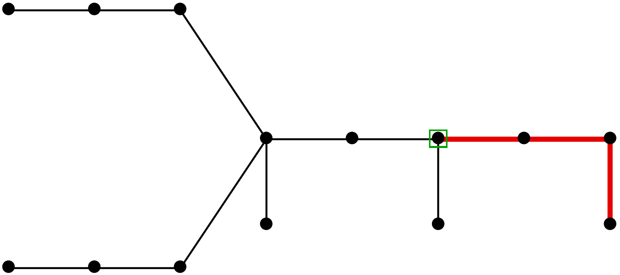

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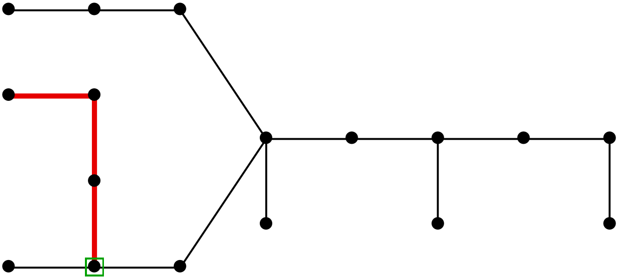

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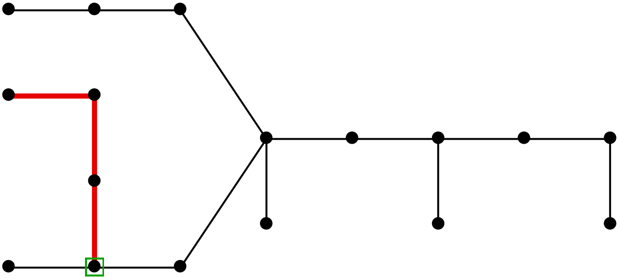

• The graphs G such that $a_u(G) = \frac{n+1}{3}$ are precisely those such that $a_t(G) = \frac{n+1}{3}$.

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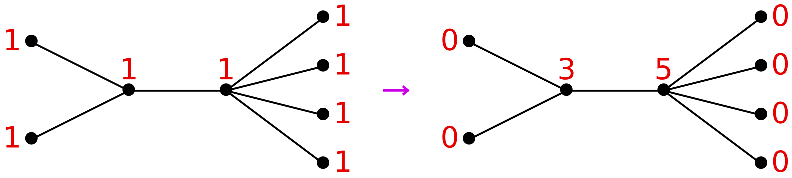

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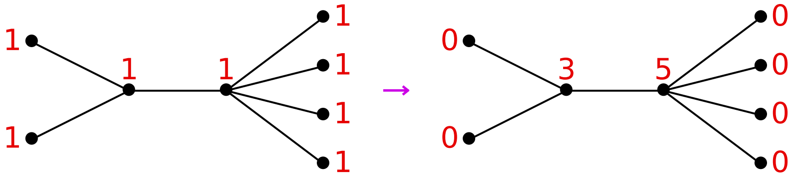

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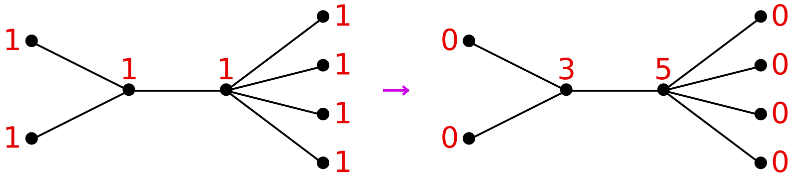

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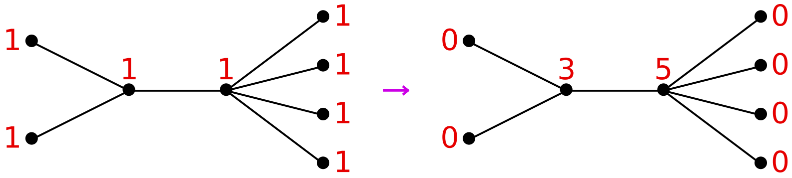

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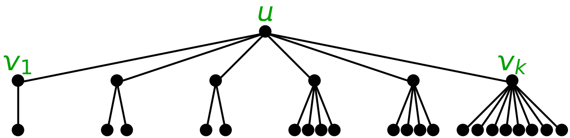
- When $\mathrm{diam}(T) = 5$, delete the central edge and use the result for trees of diameter 4.

Trees with Diameter 4, Upper Bound



Thm. $a_t(T) \leq \sqrt{n \lg n}$.

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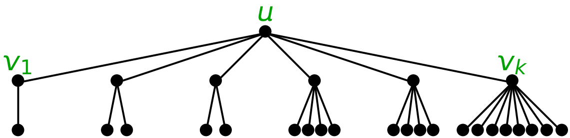

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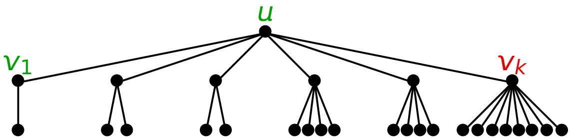

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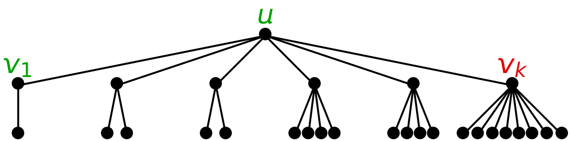

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Case 3: let w_i = weight on u before processing v_i;
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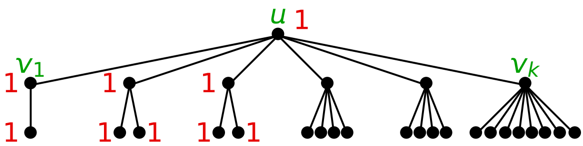

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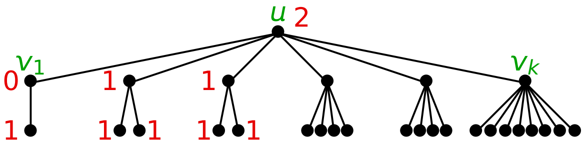

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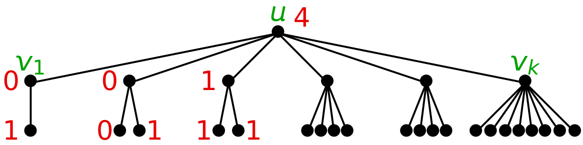

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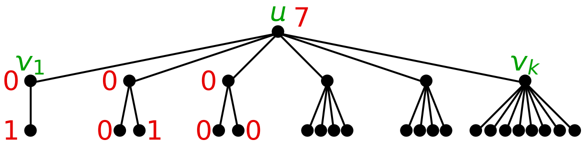

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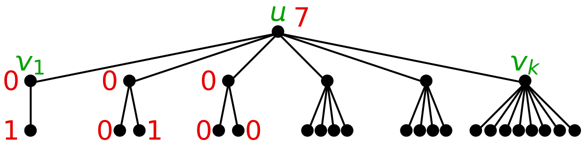

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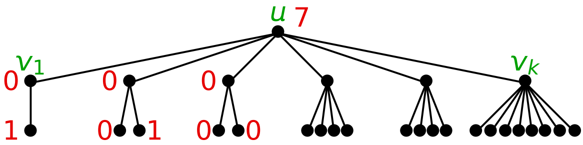

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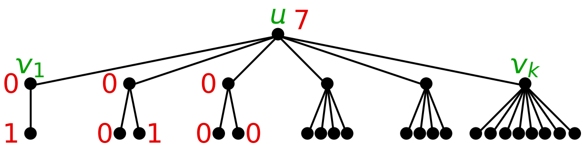

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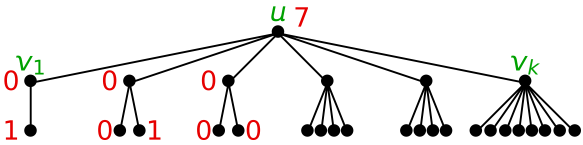

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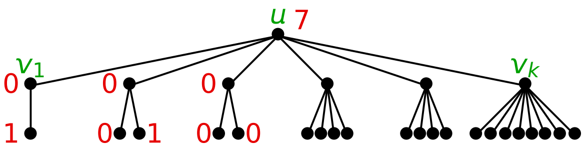

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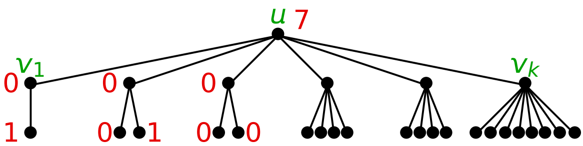

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Hence $a_t(T) < \frac{2n}{k} \lg \frac{2n}{k} < \sqrt{n \lg n}$. ∎

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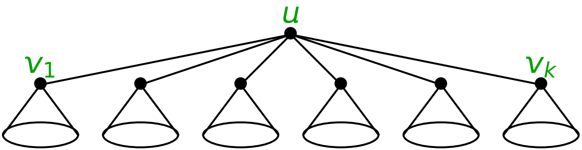

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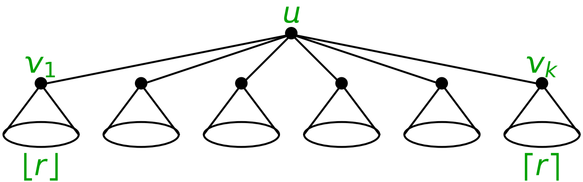

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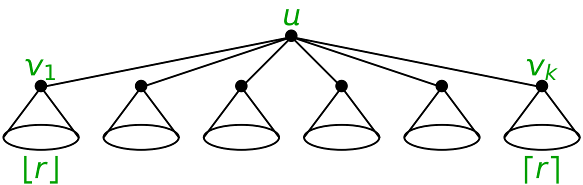

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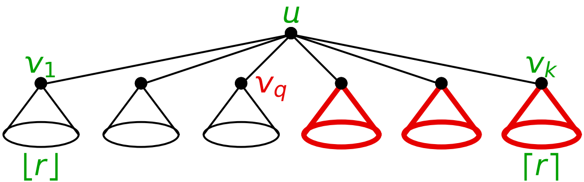

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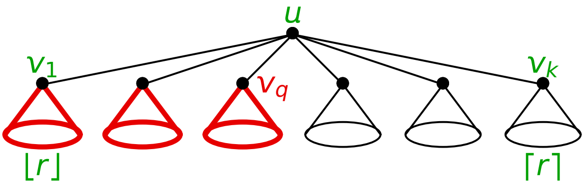

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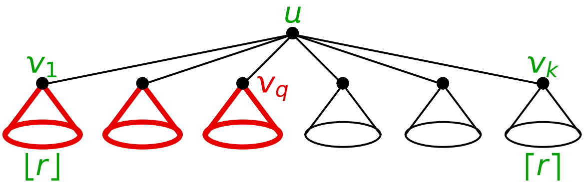

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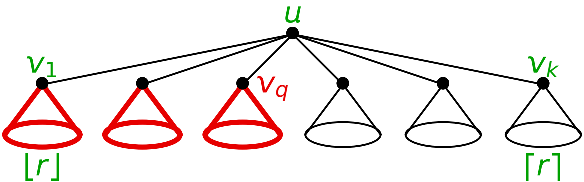

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Thm. If $G \ne C_5$ and $d(u) + d(v) \ge n - 1$ whenever $uv \notin E(G)$, then $a_t(G) = 1$.

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Perhaps $c = 2$. This suffices for Moore graphs, polarity graphs, and graphs without 4-cycles.

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For $k = 2$, we know $a_t(C_n) = \lceil n/4 \rceil$, but another construction has a larger value.

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Ques. What is the maximum of $a_t(G)$ when G is a connected n-vertex graph with minimum degree k?

Always $a_t(G) \leq \frac{1+\ln(k+1)}{k+1} n$, since $a_t(G) \leq \gamma(G)$.

For $k = 2$, we know $a_t(C_n) = \lceil n/4 \rceil$, but another construction has a larger value.

For a binary tree with triangles appended at the leaves, $\delta(G) = 2$ but $a_t(G) > (\frac{1}{4} + \frac{1}{1024})n$.

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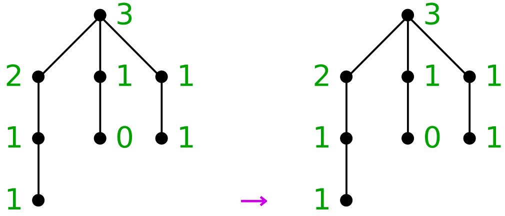

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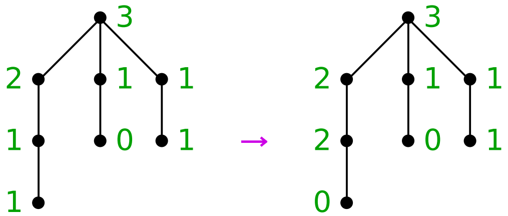

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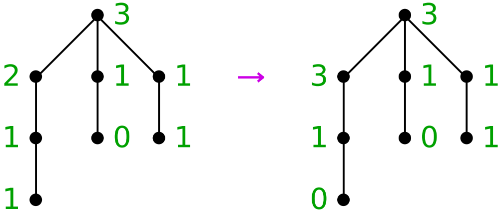

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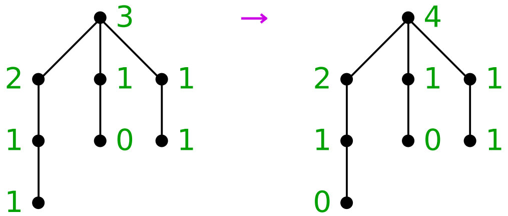

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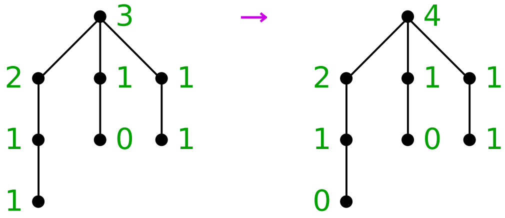

Thm. $a_u(T) = 1 \iff$ an ascending tree can be produced.

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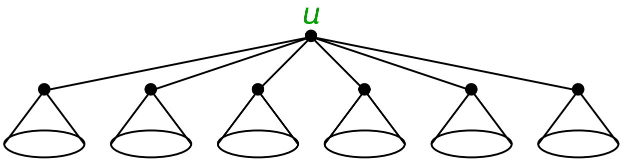

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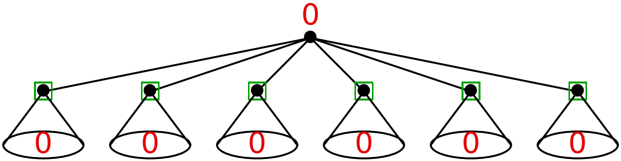

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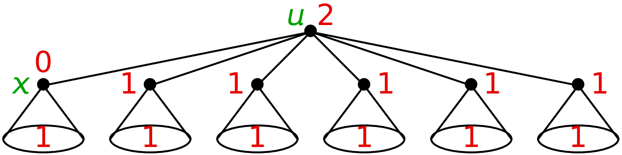

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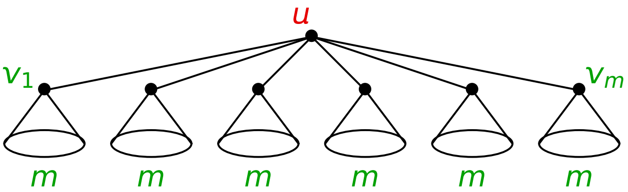

Lower Bound: Make tree with $d(u) = m$ and $d(v) = m$ for $v \in N(u)$, so $n = m^2 + 1$. The first move involving u makes at least m components with positive weight.

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Thm. (Wenger) There is an infinite family of trees with maximum degree 5 and unit acquisition number 1.

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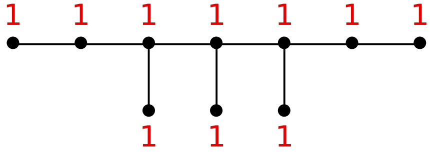

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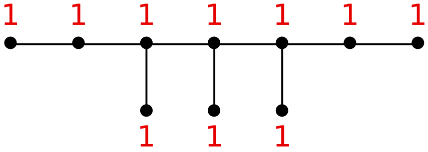
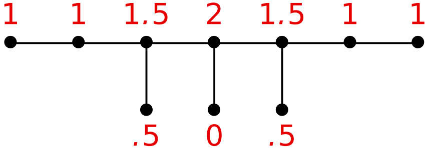


Fractional moves create an ascending tree: $a_f(G) = 1$.



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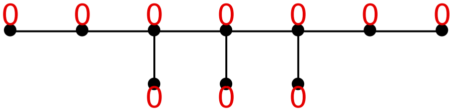

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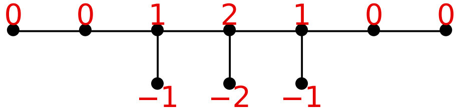

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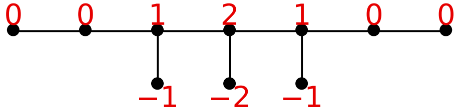

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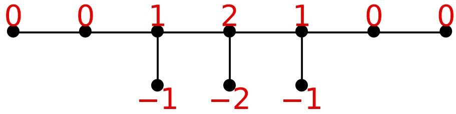

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3) Inductively produce an ascending tree in this model.

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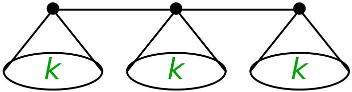
Max first kills one end of the spine and combines the two remaining spine vertices in the second round.



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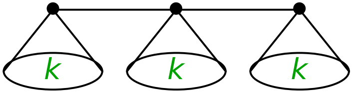

Ques. What is $\max a_g(T)$ among n-vertex trees?

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When is $a_g(G) - \hat{a}_g(G)$ positive? How big can it be?

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Pf. Max makes a king in Y: $t = 1$, $p = m - 1$, $s = n - 1$. ■

A Strategy for Min

Lem. Given: p pawns and q kings in X, s pawns and t kings in Y, $q, t \geq 1$ and $s \geq p$. Let $r = \max\{0, s - p - q + 1\}$. If Max moves next, then Min ensures ending with $\leq \max\{q, t+r\}$ live vertices.

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Thm. $\hat{a}_g(K_{m,n}) \leq n-m+1$.

Pf. Max initially makes a king; Min makes a king on the other side. Now $q=t=1$, $p=m-2$, $s=n-2$. ■

Min-Start: The Balanced Case

Thm. $a_g(K_{n,n}) = 2$.

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Lower Bound: Max makes a king on the same side.
The Max-strategy Lemma applies:
$s + t - p = (n-2) + 2 - (n-2)$.

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Lower Bound: Max makes a king on the same side. The Max-strategy Lemma applies: $s+t-p = (n-2)+2-(n-2)$.

Upper Bound: Max makes another king or absorbs a pawn. Min ensures that each side has a king. Applying the Min-strategy Lemma, $\max\{q, t+(s-p-q+1)\}$ is $\max\{2, 1+(-2+1)\}$ or $\max\{1, 1+(1-1+1)\}$. ■

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Idea of general upper bound:
After first making a king in X, Min can absorb any kings made by Max in Y to achieve $a_g(K_{m,n}) \leq m$.

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Better: Min creates q kings in X to employ the bound $\max\{q, t+(s-p-q+1)\}$ in the Min-strategy Lemma.

Min-Start: Upper Bound when $n - m < 3m$

Thm. $a_g(K_{m,n}) \leq \left\lfloor \frac{n-m}{3} \right\rfloor + 2$.

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Each round takes one pawn from each side, plus an extra pawn from Y for each king made in X.

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When X has $\left\lceil \frac{n-m}{3} \right\rceil + 1$ kings, Min makes a king in Y. With Max to move, the Min-strategy Lemma applies with $q = \left\lceil \frac{n-m}{3} \right\rceil + 1$, $t = 1$, and $s - p \le (n-m) - (q-2)$.

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When X has $\left\lceil \frac{n-m}{3} \right\rceil + 1$ kings, Min makes a king in Y. With Max to move, the Min-strategy Lemma applies with $q = \left\lceil \frac{n-m}{3} \right\rceil + 1$, $t = 1$, and $s-p \le (n-m)-(q-2)$.

Thus Min ensures at most $\max\{q, t+(n-m)-2q+3\}$ live vertices, i.e., at most $\left\lfloor \frac{n-m}{3} \right\rfloor + 2$. ■

Min-Start: Upper Bound for Large $n - m$

Idea: Min introduces a temporary king into Y to absorb some kings from X, making sure it does not get heavy.

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Def. $\hat{x}$ and $\hat{y}$ are currently heaviest vertices in X and Y.
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Algorithm: (until Y has no king and X has no pawn)
(1) If Max makes a king in Y **or** $w(\hat{x}) < 2(w(\hat{y}) + 2)$,
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(2) If Max doesn't make king in Y **and** $w(\hat{x}) \geq 2(w(\hat{y})+2)$,
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Step 3: Each Type 1 move increases $w(\hat{x})$ by a factor of at least $3/2$, and $w(\hat{x})$ cannot exceed $6m$.

Min-Start: Lower Bound

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Def. At a given time, let
$X^{-} = \{x \in X\colon\ 0 < w(x) \leq \frac{1}{2} w(\hat{x})\}$,
x' = a heaviest in X with $w(x') \leq w(\hat{y})$,
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While the vertices x', y', x^* exist, Max plays as follows:

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(1) If $w(x') + w(\hat{y}) \le w(\hat{x})$, then Max absorbs y' into x^*.

(2) If $w(x') + w(\hat{y}) > w(\hat{x})$, then Max absorbs x' into $\hat{y}$.

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The analysis is difficult!

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