3-Regular Graphs Are 2-Reconstructible

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Dedicated to Prof. Xuding Zhu on his 60th Birthday

Abstract

A graph is ℓ-reconstructible if it is determined by its multiset of induced subgraphs obtained by deleting ℓ vertices. We prove that 3-regular graphs are 2-reconstructible.

1 Introduction

The *k-deck* of an n-vertex graph is the multiset of its $\binom{n}{k}$ induced subgraphs with k vertices. The famous Reconstruction Conjecture of Ulam [4, 12] asserts that when $n \geq 3$, every n-vertex graph is determined by its $(n-1)$-deck. In 1957, Kelly [5] extended the conjecture, considering deletion of more than one vertex. A graph or graph property is *ℓ-reconstructible* if it is determined by the deck obtained by deleting ℓ vertices. Kelly conjectured that for each ℓ there is a threshold M_ℓ such that every graph with at least M_ℓ vertices is ℓ-reconstructible.

It is thought that perhaps $M_2 = 6$ (McMullen and Radziszowski [8] conjectured $M_\ell \leq 3\ell$). Since the graph $C_4 + K_1$ and the tree $K'_{1,3}$ obtained by subdividing one edge of $K_{1,3}$ have the same 3-deck, $M_2 \geq 6$. Spinoza and West [10] showed that $P_{2\ell}$ and $C_{\ell+1} + P_{\ell-1}$ have the same ℓ-deck [10], and hence $M_\ell \geq 2\ell + 1$.

Let $\mathcal{D}_k(G)$ denote the k-deck of a graph G. The elements of $\mathcal{D}_k(G)$ are called *k-cards* or just *cards*. Fix n to be the number of vertices of the graph G whose k-deck we are given, so that $|\mathcal{D}_k(G)| = \binom{n}{k}$. Since every $(k-1)$-card appears in exactly $n-k+1$ of the k-cards, always $\mathcal{D}_k(G)$ determines $\mathcal{D}_{k-1}(G)$. It is thus sensible to define the *reconstructibility* of a graph G to be the maximum ℓ such that G is ℓ-reconstructible. Spinoza and West [10] determined the reconstructibility of all graphs with maximum degree at most 2. They also

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showed that almost all graphs are $(1-o(1))n/2$-reconstructible, extending the observations in [1, 3, 9] that almost all graphs are 1-reconstructible.

Much research in graph reconstruction has focused on finding classes or properties of graphs that are 1-reconstructible. In the spirit of Kelly's Conjecture, we ask what can be shown to be ℓ-reconstructible for larger ℓ. Initial attention has considered the degree list, which trivially is 1-reconstructible because the 2-deck already determines the number of edges. Chernyak [2] showed that the degree list is 2-reconstructible when $n \geq 6$ (sharp by $\{C_4+K_1, K'_{1,3}\}$). The present authors [6] showed that the degree list is 3-reconstructible when $n \geq 7$ (sharp by $\{C_5+K_1, K''_{1,3}\}$, where $K''_{1,3}$ is the tree obtained from $K_{1,3}$ by subdividing two edges). For ℓ in general, Taylor [11] showed that the degree list is ℓ-reconstructible when $n \geq \mathrm{e}\ell + O(\log \ell)$, where e is the base of the natural logarithm.

In one of the first results on reconstruction, Kelly [4] proved that disconnected graphs are 1-reconstructible. Manvel [7] proved that disconnected graphs having no component with $n-1$ vertices are 2-reconstructible. He also observed that 2-reconstructibility of disconnected graphs consisting of one isolated vertex and a connected graph is equivalent to the original Reconstruction Conjecture that all graphs are 1-reconstructible when $n \geq 3$.

In addition, Manvel [7] showed that whether an n-vertex graph is connected can be determined from its $(n-2)$-deck when $n \geq 6$. That is, connectedness is 2-reconstructible when $n \geq 6$ (sharp by $\{C_4+K_1, K'_{1,3}\}$). The present authors [6] showed that connectedness is 3-reconstructible when $n \geq 7$ (sharp by $\{C_5+K_1, K''_{1,3}\}$). Spinoza and West [10] showed that connectedness is ℓ-reconstructible when $n > 2\ell^{(\ell+1)^2}$ (this is not sharp).

Since the degree list is 1-reconstructible, regular graphs are 1-reconstructible (after determining that the deck arises only from r-regular graphs, make the missing vertex adjacent to the r vertices of degree $r-1$ in any card). At a meeting in Sanya in 2019, Bojan Mohar asked whether regular graphs are 2-reconstructible. This is not immediate, even though the degree list is 2-reconstructible, because we must determine which of the deficient vertices is adjacent to which of the two missing vertices. In this paper, we prove the following result.

Theorem 1.1. *Every 3-regular graph is 2-reconstructible.*

A useful property of 3-regular graphs not shared by regular graphs of higher degree is that any two cycles through a vertex have a common edge. Lacking this property, it seems difficult to extend our approach to regular graphs of higher degree.

2 Preliminaries

Let $\mathcal{D}$ be the $(n-2)$-deck of a 3-regular graph with n vertices (henceforth we simply say *deck* for the $(n-2)$-deck). A *reconstruction* (from $\mathcal{D}$) is an n-vertex graph whose deck is $\mathcal{D}$. Since K_4 is determined by its 2-deck and n must be even, we may assume $n \geq 6$. Now the 2-reconstructibility of the degree list implies that every reconstruction is 3-regular.

To prove that such a deck has only one reconstruction, up to isomorphism, we will restrict the properties of a reconstruction from a deck that has nonisomorphic reconstructions. We use *alternative reconstruction* to mean a graph having the same deck as G that is not isomorphic to G. We repeatedly (often implicitly) use the following observation.

Observation 2.1. *If H is an alternative reconstruction from a card in the deck of a 3-regular graph G, then H satisfies all properties that have been shown to hold for every reconstruction from a deck that has more than one 3-regular reconstruction.*

Here H witnesses that G is a counterexample to Theorem 1.1. Our first restriction on the properties of such a graph G arose in discussion with Martin Merker, Bojan Mohar, and Hehui Wu. Let a *j-vertex* be a vertex of degree j.

Lemma 2.2. *Given a card obtained by deleting adjacent vertices of G, in every reconstruction the missing vertices are adjacent. Given a card obtained by deleting vertices with a common neighbor, in every reconstruction the missing vertices are both adjacent to that common neighbor. Finally, every reconstruction has girth at least 5.*

Proof. The first two remarks hold because every reconstruction from the deck is 3-regular.

If G has a triangle T, then a card obtained by deleting two vertices of T has two 1-vertices or has one 1-vertex and two 2-vertices. In a 3-regular reconstruction, the two missing vertices must be adjacent and must both be adjacent to any 1-vertex. If the card has two 2-vertices, then the 2-vertices must each be adjacent to one missing vertex. The two reconstructions are isomorphic, preventing $H \ncong G$.

If G has a 4-cycle (and no triangle), then a card obtained by deleting two nonadjacent vertices on a 4-cycle has two 1-vertices and two 2-vertices. In a 3-regular reconstruction, the two missing vertices must both be adjacent to both 1-vertices, and each must be adjacent to one of the 2-vertices. The two reconstructions are isomorphic, preventing $H \ncong G$. □

With girth at least 5, we have $n \geq 10$. Next we note the analogue of Kelly's Lemma [5].

Lemma 2.3. *For each graph F with at most $n - \ell$ vertices, the number of subgraphs of G isomorphic to F is ℓ-reconstructible. In particular, the number of cycles of any length at most $n - 2$ is 2-reconstructible.*

Proof. Each copy of F appears in exactly $\binom{n-|V(F)|}{\ell}$ cards. □

A elementary exercise states that every n-vertex graph with at least $n+1$ edges has girth at most $\lceil 2n/3 \rceil$. With $\lceil 2n/3 \rceil \leq n - 3$ when $n \geq 9$, Lemma 2.3 yields the following.

Corollary 2.4. *Every reconstruction has the same girth g, the same number of g-cycles, and the same number of $(g + 1)$-cycles.* □

Let $d_G(x,y)$ denote the distance between x and y in G.

Lemma 2.5. *Let $\mathcal{D}$ be the deck of a 3-regular non-2-reconstructible graph G. Fix $F = G - \{x,y\} \in \mathcal{D}$. If $d_G(x,y) = 1$, then F has four 2-vertices. If $d_G(x,y) = 2$, then F has one 1-vertex and four 2-vertices. Also, we can recognize when $d_G(x,y)$ is 1 or 2 or larger.*

Proof. The degree claims follow from G being 3-regular with girth at least 5. Since F has six 2-vertices when $d_G(x,y) > 2$, we can recognize $d_G(x,y)$ being 1 or 2 or greater than 2.□

We will usually consider cards in which the deleted vertices are at distance at most 2 and lie together on a shortest cycle. We say that two 2-vertices in a card F are *paired* in a reconstruction from F when they have one of the missing vertices as a common neighbor.

Lemma 2.6. *If $d_G(x,y) \le 2$ and x and y both lie on a shortest cycle C in G (with $g = |V(C)|$), then $G - \{x,y\}$ has only one alternative reconstruction, H. In H, the missing vertices x' and y' complete a copy C' of C obtained by substituting x' for x and y' for y. If also $xy \in E(C)$, then the number of g-cycles using one or both of $\{x,', y'\}$ in H is the same as the number of g-cycles using one or both of $\{x,y\}$ in G, respectively.*

Proof. When $d_G(x,y) \le 2$, the four 2-vertices in $G - \{x,y\}$ must form two pairs in any reconstruction: two neighbors of x' and two neighbors of y'. There are three ways to pair four vertices. However, two of those 2-vertices lie on C, with the path joining them through x and y having length 3 or 4. Pairing them as neighbors of one of $\{x',y'\}$ creates a shorter cycle. Since Lemma 2.3 provides the girth of G, this alternative pairing is forbidden, leaving only G and one alternative.

If $xy \in E(G)$, then in any reconstruction the two vertices that were adjacent to x and y on any shortest C must each be adjacent to one of $\{x',y'\}$ (and not the same one); otherwise a shorter cycle is formed (see Figure 1). Hence in every reconstruction from $G - \{x,y\}$ the number of shortest cycles that use both missing vertices is the same.

Since we know the number of g-cycles in $G - \{x,y\}$, we know the number of shortest cycles that were destroyed. Hence we now also know the number of g-cycles that use exactly one of the two missing vertices. □

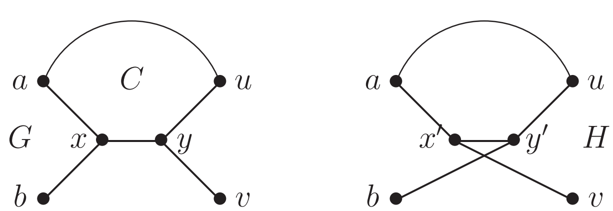

Figure 1: An edge on a shortest cycle.

3 Configurations of Short Cycles

Our approach to prohibiting 3-regular graphs with alternative reconstructions is to prohibit short cycles with common or adjacent vertices in such graphs. With g being the girth of G (already $g \geq 5$ by Lemma 2.2), we will eventually forbid having two g-cycles sharing an edge or connected by an edge, and we will forbid having a g-cycle and a g'-cycle sharing an edge, where henceforth $g' = g + 1$. These exclusions lead to a final contradiction, because we will also show that a g-cycle must share an edge with some g'-cycle.

Thus we consider a 3-regular n-vertex graph G whose deck ($(n-2)$-deck) $\mathcal{D}$ is also the deck of some 3-regular graph H not isomorphic to G. The statements we prove restrict the structure of an arbitrary reconstruction G from $\mathcal{D}$, but once proved they hold also for an alternative reconstruction H in all subsequent steps, as formalized in Lemma 2.1. Hence we do not mention G in the statements of the lemmas. Also, when some reconstruction has the assumed property, we can always find a card as described by looking at all reconstructions from each card in the given deck.

Lemma 3.1. *Two g-cycles cannot share two consecutive edges. A g-cycle and a g'-cycle cannot share three consecutive edges.*

Proof. Let C and D be a g-cycle and a cycle of length at most g' in G such that $C \cap D$ has a component P with at least two edges. Let x be an endpoint of P, with $xy \in E(P)$. Let a and b be the other neighbors of x on C and D, respectively. Let u and v be the neighbors of y other than x, with $yu \in E(C) \cap E(D)$. To avoid being G, the alternative reconstruction H from $G - \{x, y\}$ must not pair a and b; to avoid having a shorter cycle, it must not pair a and u. Hence it pairs a and v, and we may assume $ax', vx', y'u, y'b \in E(H)$. Now H has a g-cycle C' obtained from C by substituting x' for x and y' for y (see Figure 2).

Also H has a cycle D' consisting of the path $\langle b, y', u \rangle$ and the u, b-path along D that does not use x. This cycle D' is shorter than D. If D has length g, this is a contradiction. If D has length g' and P has a third edge, then C' and D' are two g-cycles with two consecutive common edges, which the first case already forbids for all reconstructions. $\square$

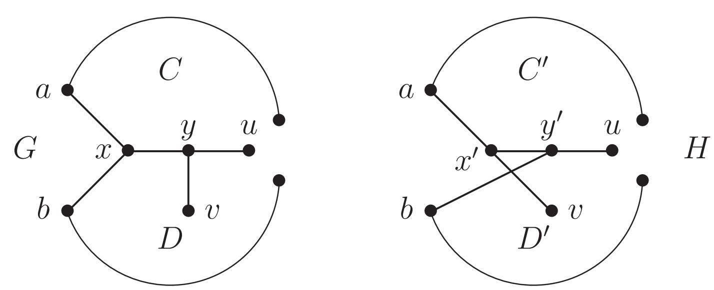

Figure 2: Consecutive edges shared by short cycles.

We refer to two cycles sharing two consecutive edges as *spliced* cycles. We have now forbidden spliced g-cycles from 3-regular graphs that are not 2-reconstructible (a spliced g-cycle and g'-cycle remain allowed, but they can't share three consecutive edges).

Remark 3.2. Henceforth, when $\langle a, x, y, u\rangle$ is a path along a g-cycle C, and the third neighbors of x and y are b and v, respectively, the arguments we have made imply that any alternative reconstruction H from $G - \{x, y\}$ is obtained by adding the vertices x' and y' with $N_H(x') = \{y', a, v\}$ and $N_H(y') = \{x', u, b\}$. □

When xy is an edge on a g-cycle, Lemma 3.1 implies that x and y each lie in at most one cycle not containing the other, since any such cycle uses both incident edges other than xy.

Lemma 3.3. *If xy is an edge in two g-cycles, then x and y cannot each lie in a g-cycle not containing the other element of $\{x, y\}$.*

Proof. Let C and D be g-cycles containing xy, with $\langle a, x, y, u\rangle$ along C and $\langle b, x, y, v\rangle$ along D. By Lemma 3.1, these six vertices are distinct. Let H be the alternative reconstruction from $G - \{x, y\}$ as in Remark 3.2. Note that H has two g-cycles through xy (see Figure 3).

Suppose that each of x and y lies in a g-cycle not containing the other element of $\{x, y\}$. Since G has no spliced g-cycles, these two g-cycles Q and R pass through $\langle a', a, x, b, b'\rangle$ and $\langle u', u, y, v, v'\rangle$, respectively, where w' for $w \in \{a, b, u, v\}$ is the neighbor of w not in $C \cup D$. By Lemma 2.6, in H each of x' and y' lies in a g-cycle not containing the other. To avoid spliced g-cycles in H, these g-cycles Q' and R' must pass through $\langle a', a, x', v, v'\rangle$ and $\langle u', u, y', b, b'\rangle$, respectively. In particular, H and G contain an a', v'-path P of length $g - 4$.

Now consider $G - \{a, x\}$. Since ax lies in the g-cycle C, an alternative reconstruction H' replacing $\{a, x\}$ with $\{a'', x''\}$ can be assumed to have the g-cycle C'' through $\langle z, a'', x'', y\rangle$, where z is the neighbor of a on C other than x, and the edges $a'x''$ and $a''b$ (see Remark 3.2). In H', the path $\langle a', x'', y, v, v'\rangle$ combines with P to form a g-cycle. However, this g-cycle shares consecutive edges yv and vv' with R, creating spliced g-cycles, which is forbidden. □

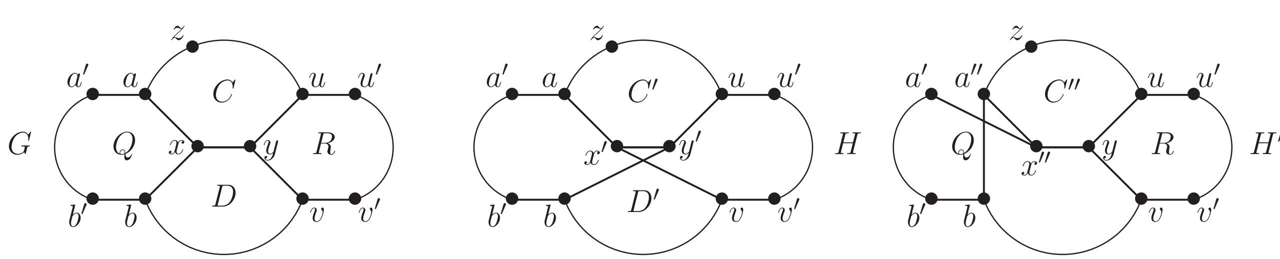

Figure 3: Many g-cycles through $\{x, y\}$.

Lemma 3.4. *No vertex lies in three g-cycles.*

Proof. Suppose that y with neighborhood $\{x, u, v\}$ lies in three g-cycles in G. Each of these g-cycles uses two edges at y; any two of them have one common edge. With $a, b \in N_G(x)$, $c, d \in N_G(u)$, and $e, f \in N_G(v)$, label the vertices so that the three cycles C, D, and R contain the paths $\langle a, x, y, u, d\rangle$, $\langle c, u, y, v, f\rangle$, and $\langle e, v, y, x, b\rangle$, respectively (see Figure 4). Since $g \geq 5$, these 10 vertices are distinct.

Since xy lies in the g-cycle C containing $\langle a, x, y, u, d\rangle$, the card $G - \{x, y\}$ has only one alternative reconstruction H. As in Remark 3.2, we may obtain H from the card by adding x' and y' with $N_H(x') = \{y', a, v\}$ and $N_H(y') = \{x', u, b\}$.

By Lemma 3.3, G has no g-cycle through $\langle a, x, b\rangle$. Hence G has exactly one g-cycle containing exactly one of $\{x, y\}$. By Lemma 2.6, in H exactly one g-cycle Q contains exactly one of $\{x', y'\}$. Hence Q contains $\langle a, x', v\rangle$ or $\langle b, y', u\rangle$. Avoiding spliced g-cycles in H implies that Q contains vf in the first case and uc in the second case. Hence G contains an a, f-path or a b, c-path of length $g - 3$, and not both.

Applying the symmetric argument to $G-\{u, y\}$ and $G-\{v, y\}$ yields paths with length $g-3$ in G whose endpoints are exactly one pair in each of the following three sets: $\{(a, f), (b, c)\}$, $\{(c, b), (d, e)\}$, $\{(f, a), (e, d)\}$. This is impossible: as soon as one pair of endpoints is picked, it satisfies one other set, which then prevents the third set from contributing a pair. □

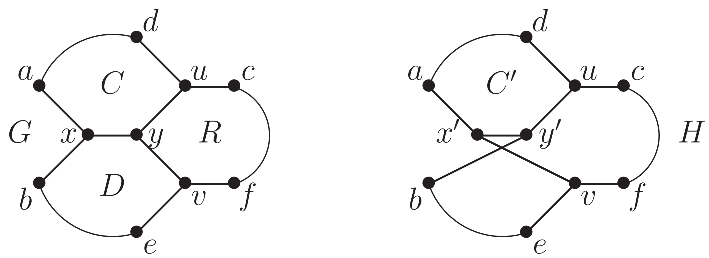

Figure 4: A vertex in three g-cycles.

Lemma 3.5. *No vertex lies in two g-cycles.*

Proof. Since G is 3-regular, a vertex in g-cycles C and D requires an edge xy in C and D (only one common edge, since there are no spliced g-cycles). Label vertices as in Remark 3.2, with $\langle z, a, x, y, u\rangle$ lying along a g-cycle C. Since G has girth g, the neighbor w of a that is not on C is not on the other g-cycle through xy. Note that ax is not in another g-cycle, since that would put x on three g-cycles.

Since ax lies in a g-cycle, $G - \{a, x\}$ has only one alternative reconstruction, H. We may label H so the g-cycle C' through the missing vertices a' and x' arises from C by replacing a with a' and x with x', and so $N_H(a') = \{x', z, b\}$ and $N_H(x') = \{a', y, w\}$ (see Figure 5).

Since G has a g-cycle D through exactly one of $\{a, x\}$, also H has a g-cycle D' through exactly one of $\{a', x'\}$. This cycle cannot use $a'x'$, so it uses $\langle w, x', y\rangle$ or $\langle b, a', z\rangle$. In each case, we will obtain an alternative reconstruction from the deck that has three g-cycles containing a single vertex, which by Lemma 3.4 is forbidden.

In the first case, D' cannot use yu, since H has no spliced g-cycles. Hence D' uses yv, and there is a v,w-path P of length $g-3$ in H and G (not using y). Note that P completes a g'-cycle Q in G with $\langle w,a,x,y,v\rangle$. Since xy lies on a g-cycle, $G-\{x,y\}$ has only one alternative reconstruction, H' (see Figure 5). We may label the missing vertices x'' and y'' so that $ax'', y''u \in E(H')$, which forces $vx'', y''b \in E(H')$. Now replacing $\langle w,a,x,y,v\rangle$ in Q with $\langle w,a,x'',v\rangle$ yields three g-cycles in H' containing x''.

In the second case, D' must continue after $\langle b,a',z\rangle$ to the neighbor z' of z not on C', since H has no spliced g-cycles. Replacing $a'b$ in D' with $\langle a,x,b\rangle$ yields a g'-cycle R in G containing the path $\langle z',z,a,x,b\rangle$. Since az lies on a g-cycle, there is a unique alternative reconstruction from $G-\{a,z\}$; call it H''. We may label H'' so its missing vertices a'' and z'' replace a and z in C to form a g-cycle C'', and then the remaining edges incident to $\{a'',z''\}$ must be $z'a''$ and wz'', as in Figure 5. Now replacing $\langle z',z,a,x,b\rangle$ in R with $\langle z',a'',x,b\rangle$ yields three g-cycles in H'' containing x. □

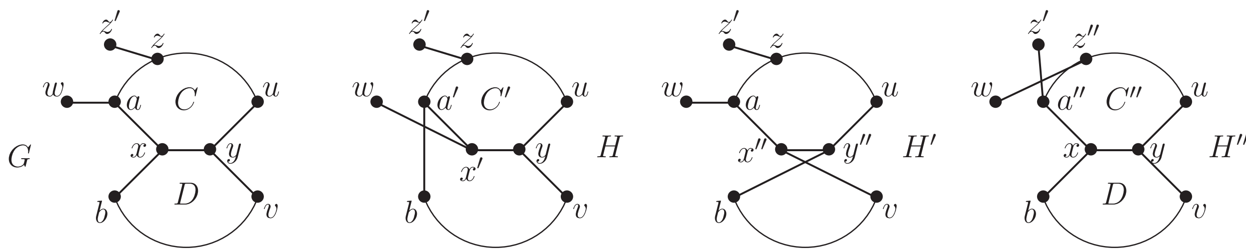

Figure 5: Two g-cycles C and D with a common vertex.

Lemma 3.6. *No two cycles of length at most g' are spliced.*

Proof. Already from Lemma 3.1 no two g-cycles are spliced.

Next consider a spliced g-cycle C and g'-cycle D sharing the path $\langle x,y,u\rangle$ such that the other neighbors of x are a on C and b on D. Since xy lies on a g-cycle, $G-\{x,y\}$ has only one alternative reconstruction H, expressible so that the g-cycle C' through the two missing vertices x' and y' is obtained from C by replacing x with x' and y with y'. In H we then also have the edge $y'b$. Now replacing $\langle b,x,y\rangle$ with by' in D yields a second g-cycle in H containing $y'u$; this contradicts Lemma 3.5. Hence a g-cycle and g'-cycle cannot be spliced.

Now let C and D be two g'-cycles sharing $\langle x,y,u\rangle$, defining a and b as above. Let v be the neighbor of y not in $C\cup D$. By Lemma 3.1, C and D cannot share three consecutive edges, so we may let c and d be the neighbors of u other than y on C and D, respectively.

Consider $G-\{x,y\}$, and let H be an alternative reconstruction whose vertices deleted to form $G-\{x,y\}$ are x' and y'. We know $x'y' \in E(H)$, and we can label x' and y' so that

$y'u, x'v \in E(H)$. The remaining neighbor of x' may be a or b, but since C and D are both g'-cycles these choices are symmetric. Hence we may assume $ax', by' \in E(H)$ (see Figure 2).

Replacing $\langle b, x, y, u\rangle$ in D with $\langle b, y', u\rangle$ yields a g-cycle D' in H containing exactly one of $\{x', y'\}$. Hence G must have a g-cycle Q containing exactly one of $\{x, y\}$. Since Q cannot contain xy, it contains $\langle v, y, u\rangle$ or $\langle a, x, b\rangle$. In the first case, continuing Q along the edge leaving u in either C or D yields two spliced g-cycles (Q with C or D), which is forbidden. Hence Q contains $\langle a, x, b\rangle$.

Applying the symmetric argument to $G - \{u, y\}$ allows us to conclude that G also contains a g-cycle through $\langle c, u, d\rangle$. This g-cycle R also appears in H. Now R and D' are g-cycles in H that both contain the edge ud. By Lemma 3.5, this is forbidden. □

Lemma 3.7. *There is no edge whose endpoints lie in distinct g-cycles.*

Proof. Let xy be such an edge in G, joining g-cycles C through x and D through y. In any alternative reconstruction H from $G - \{x, y\}$, the missing vertices x' and y' are adjacent, and to avoid recreating G each of x' and y' must have one neighbor in $V(C)$ and one neighbor in $V(D)$. Each possible assignment yields in H two g'-cycles containing $x'y'$. By symmetry, we may label $C - x$ as an a, b-path and $D - y$ as a u, v-path so that $a, u \in N_H(x')$ and $b, v \in N_H(y')$. See Figure 6, where we have not yet established the dashed edges.

Since G had two g-cycles each containing exactly one of $\{x, y\}$, also H must have two g-cycles each containing exactly one of $\{x', y'\}$. One must use $\langle a, x', u\rangle$, and the other must use $\langle b, y', v\rangle$; let these be Q' and R', respectively. Replacing $\langle a, x', u\rangle$ in Q' with $\langle a, x, y, u\rangle$ and $\langle b, y', v\rangle$ in R' with $\langle b, x, y, v\rangle$ yields g'-cycles Q and R in G, respectively.

Now let z and w be the neighbors of a other than x in Q and C, respectively, and let t be the neighbor of z on Q other than a. Consider an alternative reconstruction H' from $G - \{a, z\}$, with a' and z' being the missing vertices. We have $a'z' \in E(H')$. By symmetry we may assume $a'x \in E(H')$, and hence $wz' \in E(H')$ to avoid recreating G. Still t may be ajacent to z' or to a'. The edge $z't$ would complete a g'-cycle Q'' in H' (shown below) that is spliced with C', forbidden by Lemma 3.6. The edge $a't$ would complete a g-cycle in H' sharing the edge uy with the g-cycle R, forbidden by Lemma 3.5. □

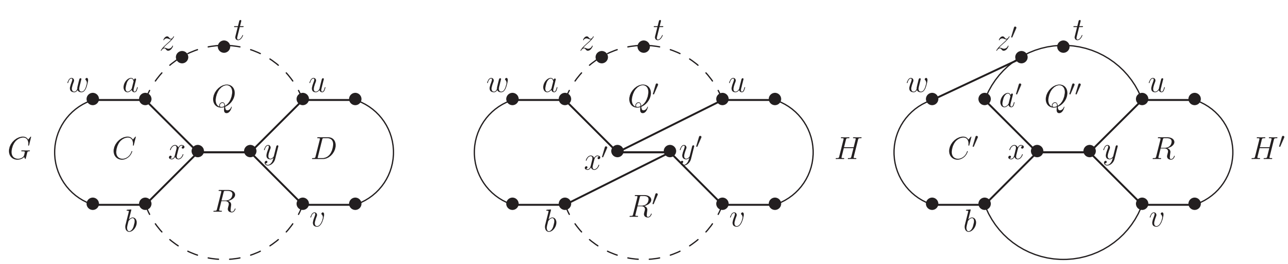

Figure 6: Adjacent vertices in g-cycles.

Lemma 3.8. *No vertex lies in a g-cycle and two g'-cycles.*

Proof. Suppose that x is such a vertex in the 3-regular graph G. Each of the three cycles uses two edges incident to x. Since there are no spliced cycles of length at most g', each remaining edge incident to the neighbors of x lies in exactly one of these cycles. Let $N_G(x) = \{y, a, b\}$ and $N_G(y) = \{x, u, v\}$, with xy shared by the g'-cycles Q and R, and with $a, u \in V(Q)$ and $b, v \in V(R)$ (see Figure 7). The g-cycle C through x contains $\langle a, x, b\rangle$.

Let H be an alternative reconstruction from $G - \{x, y\}$, with missing vertices x' and y'. We have $x'y' \in E(H)$, and by symmetry we may assume $ax' \in E(H)$ and hence also $by' \in E(H)$. Now there are two cases, shown in Figure 7. If $ux', vy' \in E(H)$, then replacing $\langle a, x, y, u\rangle$ in Q with $\langle a, x', u\rangle$ and replacing $\langle b, x, y, v\rangle$ in R with $\langle b, y', v\rangle$ yields g-cycles Q' and R' in H through the endpoints of the edge $x'y'$, which is forbidden by Lemma 3.7. If $uy', vx' \in E(H)$, then replacing $\langle a, x, b\rangle$ in C with $\langle a, x', y', b\rangle$ yields a g'-cycle C' in H that is spliced with the g'-cycles Q'' through $\langle a, x', y', u\rangle$ and R'' through $\langle b, y', x', v\rangle$, which is forbidden by Lemma 3.6. □

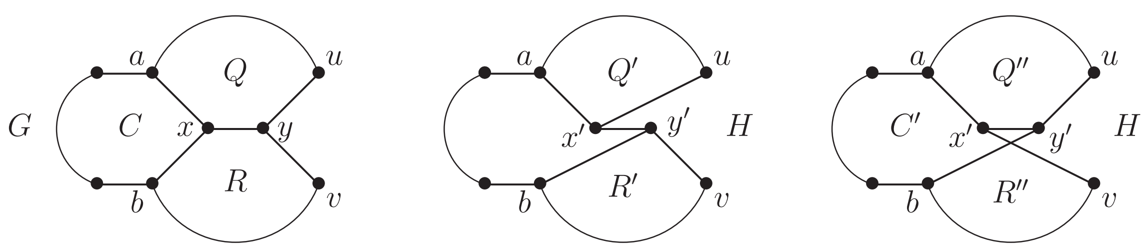

Figure 7: Three short cycles at a vertex.

Lemma 3.9. *If $\langle w, x, y, z\rangle$ is a path in a g-cycle, and wx lies in a g'-cycle, then yz also lies in a g'-cycle.*

Proof. Let C be the g-cycle in G containing $\langle w, x, y, z\rangle$, and let D be the g'-cycle containing wx. Let a be the neighbor of y outside C, and let $N_G(a) = \{y, b, c\}$. Let H be an alternative reconstruction from $G - \{y, a\}$, with y' and a' being the missing vertices. We have $y'a' \in E(H)$, and we may label y' and a' so that $y'z, a'x \in E(H)$. Also, label b and c so that $a'b, y'c \in E(H)$ (see Figure 8). Note that replacing $\langle x, y, z\rangle$ in C with $\langle x, a', y', z\rangle$ yields a g'-cycle C' in H.

Since C is a g-cycle in G containing exactly one of $\{y, a\}$, in H there must be a g-cycle D' containing exactly one of $\{y', a'\}$. It must contain $\langle x, a', b\rangle$ or $\langle z, y', c\rangle$. In the first case, x in H lies in the g-cycle D' and g'-cycles C' and D, forbidden by Lemma 3.8. In the second case, replacing $\langle z, y', c\rangle$ in D' with $\langle z, y, a, c\rangle$ yields the desired g'-cycle in G through yz. □

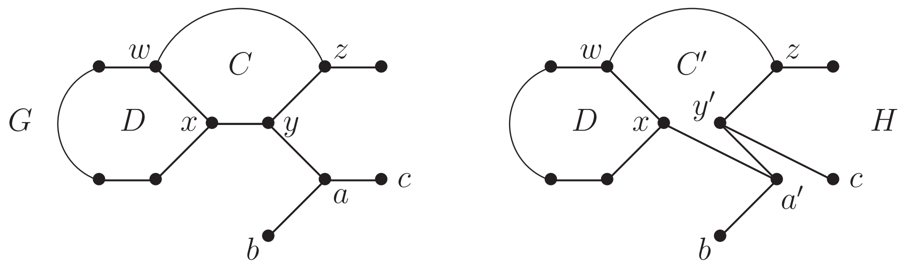

Figure 8: A g-cycle and a g'-cycle with a common edge.

Lemma 3.10. *No g-cycle and g'-cycle share an edge.*

Proof. Let wx be such an edge in G, shared by the g'-cycle D and the g-cycle C containing the path $\langle w, x, y, z, c\rangle$. By Lemma 3.9, yz lies in a g'-cycle B in G.

Since w and y lie on a g-cycle, from $G-\{w, y\}$ there is only one alternative reconstruction H, in which by Lemma 2.6 the g-cycle C' through the missing vertices w' and y' is obtained from C by replacing w with w' and y with y'. Thus also $w'a, y'b \in E(H)$, where a and b are the neighbors outside C of y and w in G, respectively (see Figure 9).

Replacing $\langle b, w, x\rangle$ in D with $\langle b, y', x\rangle$ yields a g'-cycle D' in H. Since C' and D' share the edge xy', by Lemma 3.9 the edge zc lies in a g'-cycle Q in H. Since shorter cycles and spliced cycles must be avoided, Q avoids y' and w'. Hence Q appears also in G. Now in G the vertex z appears in the g-cycle C and g'-cycles B and Q, which is forbidden by Lemma 3.8.□

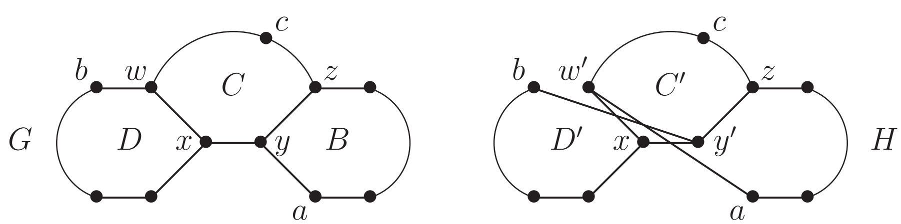

Figure 9: Another g-cycle and a g'-cycle with a common edge.

Lemma 3.11. *Every g-cycle shares an edge with some g'-cycle.*

Proof. Let $\langle a, x, b\rangle$ be a path along a g-cycle C in G. Let y be the neighbor of x outside C, with $N_G(y) = \{x, u, v\}$. Let H be an alternative reconstruction from $G-\{x, y\}$, with missing vertices x' and y'. As usual, $x'y' \in E(H)$, and by symmetry we may assume $x'a, y'b \in E(H)$. We may also choose the labels u and v so that $x'v, y'u \in E(H)$ (see Figure 10).

Since C is a g-cycle in G containing exactly one of x and y, in H there must be a g-cycle C' containing exactly one of x' and y'. Such a cycle must contain $\langle a, x', v\rangle$ or $\langle b, y', u\rangle$.

Replacing this path in C' with $\langle a, x, y, v\rangle$ or $\langle b, x, y, u\rangle$, respectively, yields a g'-cycle in G that shares an edge with C. □

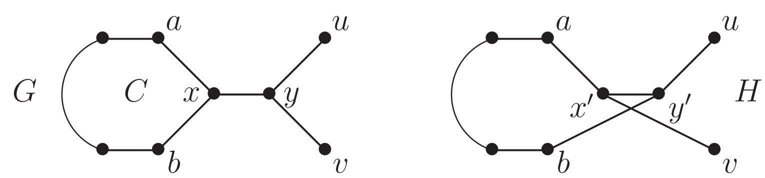

Figure 10: Forcing a common edge.

Lemmas 3.10 and 3.11 are contradictory. Hence no 3-regular graph G has an alternative reconstruction from its $(n-2)$-deck, which proves Theorem 1.1.

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