Some Things I Don’t Know

Douglas B. West

Department of Mathematics
Zhejiang Normal University and
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

slides available on DBW preprint page
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Pancake Variations

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  \( \leq \left\lfloor \frac{2}{3} n - \frac{2}{3} \right\rfloor \) for sorting by block transpositions, via longer proof.
Number of \((r + 1)\)-cliques [1982]

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For further study of \(\min k_{r+1}(G)\) given \(n\) and \(m\), see Bollobás Extremal GT [1978; reprinted 2004]
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**Ques.** How many \((r+1)\)-cliques must occur?
A Structural Variation

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Sharp: $G_{n,r,D-z}$ is $r$-partite: $t+1$ parts of size $n-D$, then strict increasing. All $(r+1)$-cliques use $z$, which neighbors all in the first $t$ parts and one in the others.

$G_{19,5,16} \quad 18 = 3 \cdot 5 + \binom{5-2}{2}$
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**True:** for $r = 2$, for $t = 0$, and for $(r, n, D) = (3, 7, 5)$.

\[ G_{19,5,16} \]

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**Def.** (Bernhart–Kainen [1979]) book embedding: Order the vertices along the spine of a book, embed edges on pages. Each edge is on one page; edges on a page do not cross. $\text{pagenumber} = \min \#\text{pages}$. 
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Ex. $\rho(K_n) = \lfloor n/2 \rfloor$. 
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**Ques.** (Leighton) What is \( p(K_n \boxtimes K_n) \)?
Acyclic Orientations [1995]

**Def.** An edge in an acyclic orientation is dependent if reversing it creates a cycle. Let $d_{\text{min}}(G)$ and $d_{\text{max}}(G)$ be the min & max #dependent edges in orientations of $G$. 
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- $d_{\text{min}}(G) = 0 \iff G$ is the cover graph of a poset.
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**Ques.** Which graphs are fully orientable? Bipartite?
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**Yes:** Complete bipartite graphs (West [1995]), cover graphs (Fisher–Fraughnaugh–Langley–West [1997]), graphs with \( d_{\text{min}}(G) \leq 1 \) (Lai–Lih–Tong [2009]), outerplanar graphs (Lih–Lin–Tong [2006]), 2-degenerate graphs (Lai–Chang–Lih [2008]), etc.
Acyclic Orientations [1995]

**Def.** An edge in an acyclic orientation is dependent if reversing it creates a cycle. Let $d_{\text{min}}(G)$ and $d_{\text{max}}(G)$ be the min & max #dependent edges in orientations of $G$.

- $d_{\text{min}}(G) = 0 \iff G$ is the cover graph of a poset.
- $d_{\text{max}}(G) = |E(G)| - |V(G)| + \#\text{components}$ (Edelmann)

**Def.** $G$ is fully orientable if $\exists$ acyclic orientation with $k$ dependent edges whenever $d_{\text{min}}(G) \leq k \leq d_{\text{max}}(G)$.

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**No:** Turán graph $T_{n,r}$ when $r \mid n$ (Chang–Lin–Tong [’09]).
Spanning Trees with Many Leaves [2000]

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- $l(n, k) \leq \frac{k-2}{k+1} n + 2$: 

```
  k+1 ——— k+1 ——— k+1 ——— k+1
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**Thm.** (Caro–West–Yuster [2000]) \( l(n, k) \sim n^{\frac{k-\ln(k+1)}{k+1}} \).
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**Thm.** (Caro–West–Yuster [2000]) \( l(n, k) \sim n \frac{k-\ln(k+1)}{k+1} \).

**Ques.** How does \( \frac{l(n,k)}{n} \) decline from \( \frac{k-2}{k+1} \) to \( \frac{k-\ln(k+1)}{k+1} \)?
Parity Edge-Coloring [2008]

**Def.** A parity edge-coloring (pec) assigns colors to edges so no path has an even number of each color.
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Ex. $p(P_n) = \lceil \lg n \rceil$.

Conj. $\hat{p}(G) = p(G)$ for every bipartite $G$. 
$p(G)$ when $G$ is dense

**Ex.** Give the vertices of $K_{2^k}$ distinct $k$-tuple binary codes. Color $E(K_{2^k})$ by giving $uv$ the color $u \oplus v$.

![Graph with vertices and edges colored according to the given rule]
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\begin{align*}
01 & \quad 11 \\
00 & \quad 10
\end{align*}

\begin{align*}
\text{Purple} & \quad = 01 \\
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\text{Red} & \quad = 10
\end{align*}

**Thm.** (Bunde–Milans–Wu–West [2008]) $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$. 
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![Diagram of $K_{2^k}$ vertices and edges colored with binary codes]

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$00 \oplus 10 = 10$

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\end{array}
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A more detailed conjecture for \( \hat{p}(K_{r,s}) \) would strengthen "Yuzvinsky’s Theorem" on sums of subsets of \( \mathbb{F}_2^k \).
The Reconstruction Problem

**Def.** The deck of a graph $G$ is the multiset of cards of the form $G - v$ for $v \in V(G)$. 
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![Graphs](image)

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![Graph example](image)

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**Obs.** $|E(G)| = \frac{\sum_v |E(G-v)|}{n-2}$ when $G$ has $n$ vertices.

This info is lost when keeping only some cards.
Degree-Associated Reconstruction [2010]

**Def.** (Ramachandran [1981]) the dacards are the pairs 
\((G - v, d_G(v))\) for \(v \in V(G)\). The degree-associated reconstruction number \(\text{drn}(G)\) is the minimum 
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**Ques.** Must equality hold when $G$ has no “twins”? 
More on $\text{drn}(G)$

- $\text{drn}(tK_m) = 3$ (Ramachandran [2006]) but $\text{rn}(tK_m) = m + 2$ (Myrvold [1989]).
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[Diagram of trees $H_1$ and $H_2$]
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![Trees](image)

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![Tree Diagrams](image)

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- Hannah Spinoza has extended the upper bound to “subdivided caterpillars with toes”.
Nine Dragon Tree Conjecture [2010]

**Aim:** Common generalization of Nash-Williams’ Formula and decomposition results for planar graphs.
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(Payan [1986])
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**Def.** $G$ is $d$-bounded if $\Delta(G) \leq d$.

Nine Dragon Tree (NDT) Conjecture:
(Montassier, Ossona de Mendez, Raspaud, Zhu [2010]) $\text{Arb}(G) \leq k + \frac{d}{k+d+1} \Rightarrow G$ decomposes into $k+1$ forests, with the last being $d$-bounded.