Some Things I Don’t Know

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slides available on DBW preprint page

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- Eriksson–Eriksson-Karlander–Svensson–Wástlund [2001] $\le \left\lfloor \frac{2}{3}n - \frac{2}{3} \right\rfloor$ for sorting by block transpositions, via longer proof.

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Ques. How many $(r+1)$-cliques must occur?

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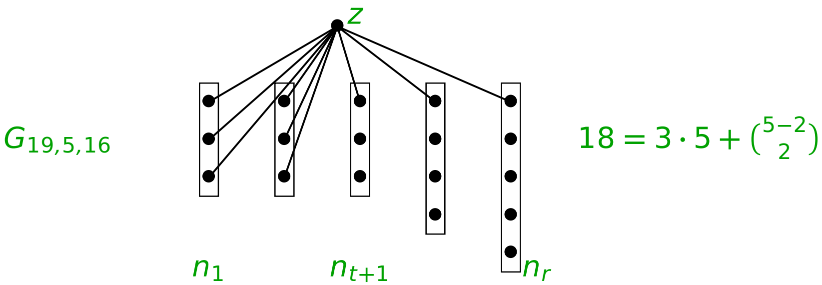
Sharp: $G_{n,r,D} - z$ is r-partite: $t+1$ parts of size $n-D$, then strict increasing. All $(r+1)$-cliques use z, which neighbors all in the first t parts and one in the others.



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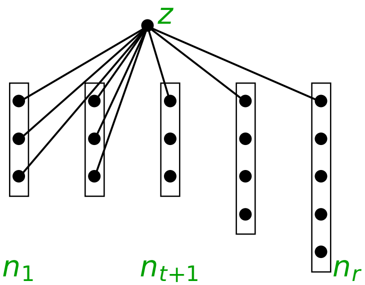
$G_{19,5,16}$



$18 = 3 \cdot 5 + \binom{5-2}{2}$

True: for $r = 2$, for $t = 0$, and for $(r, n, D) = (3, 7, 5)$.

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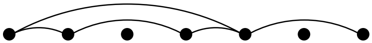

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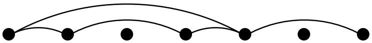

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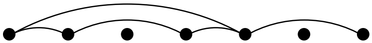

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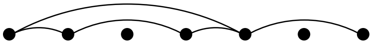

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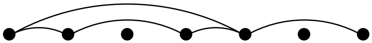

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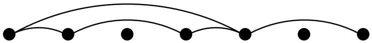

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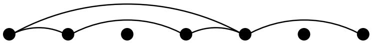

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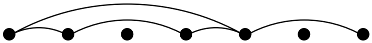

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Ques. (Leighton) What is $p(K_n \square K_n)$?

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No: Turán graph $T_{n,r}$ when $r \mid n$ (Chang–Lin–Tong ['09]).

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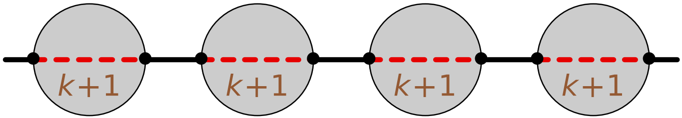

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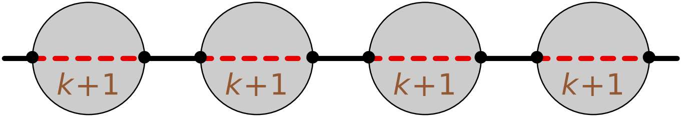

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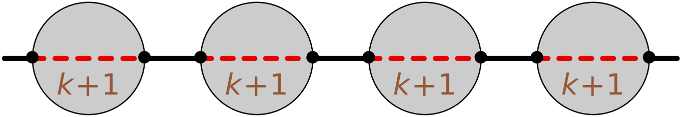

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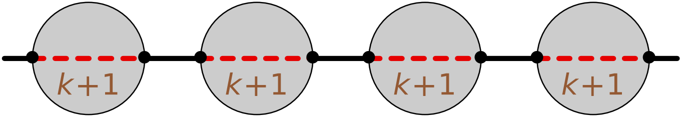

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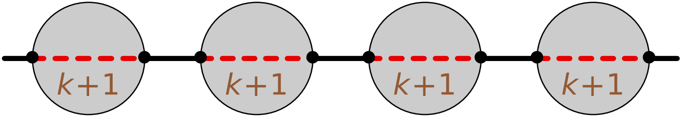

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Ques. How does $\frac{l(n,k)}{n}$ decline from $\frac{k-2}{k+1}$ to $\frac{k-\ln(k+1)}{k+1}$?

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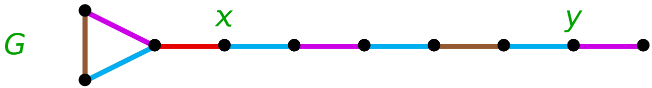

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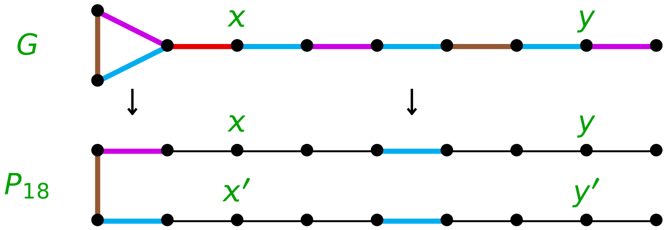

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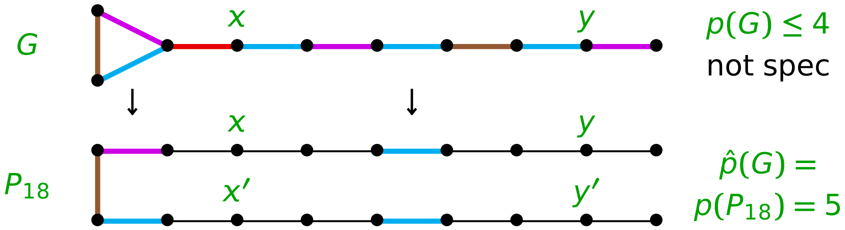

Conj. $\hat{p}(G) = p(G)$ for every bipartite G.

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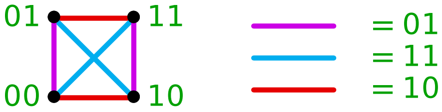
Ex. Give the vertices of K_{2^k} distinct k-tuple binary codes. Color $E(K_{2^k})$ by giving uv the color $u \oplus v$.



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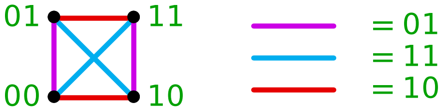

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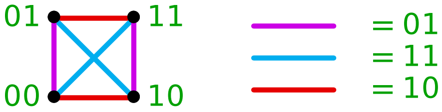

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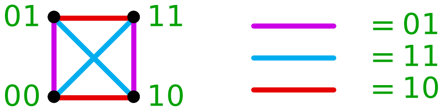

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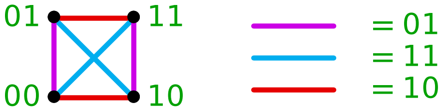

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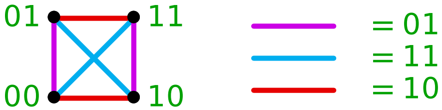

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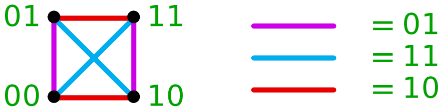

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A more detailed conjecture for $\hat{p}(K_{r,s})$ would strengthen "Yuzvinsky's Theorem" on sums of subsets of $\mathbb{F}_2^k$.

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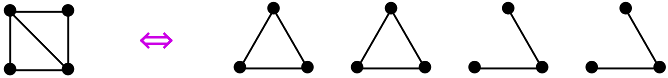

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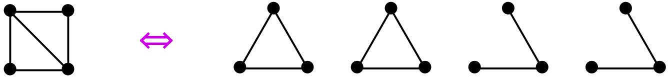

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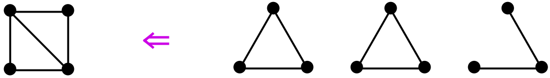

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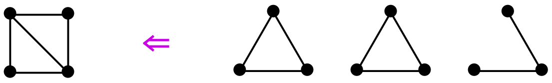

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This info is lost when keeping only some cards.

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Ques. Must equality hold when G has no "twins"?

More on $\mathrm{drn}(G)$

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Thm. For caterpillars, $\mathrm{drn}(T)$=2 unless T is a star or H_1.

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Ex. The trees below satisfy drn(H_1) $=$ drn(H_2) $= 3$.

Thm. For caterpillars, drn(T)=2 unless T is a star or H_1.

Conj. drn(T) ≤ 2 except for finitely many trees (maybe only H_1 and H_2).

More on $\mathrm{drn}(G)$

- $\mathrm{drn}(tK_m) = 3$ (Ramachandran [2006]) but $\mathrm{rn}(tK_m) = m + 2$ (Myrvold [1989].

Ques. What other graphs satisfy $\mathrm{rn}(G) - \mathrm{drn}(G) > 1$?

Thm. (Myrvold [1990]) If T is a tree with at least five vertices, then $\mathrm{rn}(T) = 3$.

Ex. The trees below satisfy $\mathrm{drn}(H_1) = \mathrm{drn}(H_2) = 3$.

Thm. For caterpillars, $\mathrm{drn}(T)$=2 unless T is a star or H_1.

Conj. $\mathrm{drn}(T) \leq 2$ except for finitely many trees (maybe only H_1 and H_2).

- Hannah Spinoza has extended the upper bound to "subdivided caterpillars with toes".

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Nine Dragon Tree (NDT) Conjecture:
(Montassier, Ossona de Mendez, Raspaud, Zhu [2010])
$\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$ $\Rightarrow$ G decomposes into $k+1$ forests, with the last being d-bounded.