COMPATIBLE MATCHINGS IN BIPARTITE GRAPHS

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Abstract

In a bipartite graph with vertices X,Y indexed as $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$, a *compatible matching* is a matching with no pairs of edges $x_i y_j$ and $x_k y_l$ having $i<k$ and $j>l$. We show that any graph with N edges must contain a compatible matching with at least $\lceil \sigma-\sqrt{\sigma^2-N} \rceil$ edges, where $\sigma=(m+n)/2$ is the average of the part-sizes. This is best possible; we exhibit graphs where equality holds. When $\sigma-\sqrt{\sigma^2-N}$ is an integer, we characterize and enumerate the extremal graphs. We also obtain extremal results for the number of compatible matchings needed to cover the edges, and integral min-max relations for both parameters. We describe correspondences with problems in perfect graphs, partially ordered sets, and Young tableaux.

Keywords: bipartite, matching, perfect graph, permutation graph, edge-coloring, integral min-max relation, symmetric chain order.

1. Introduction

Matching in bipartite graphs is a well-studied subject in graph theory; we consider a variant, called *compatible matching,* first studied by Borodin and Hopcroft [BH]. Given a fixed ordering $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ of the vertices in the two parts, a *compatible matching* or *c-matching* is a matching in which crossing edges do not appear. Having a matching but no crossings forbids pairs of edges $x_i y_j$ and $x_k y_l$ with $i \leq k$ and $j \geq l$. Let α_c be the size of the maximum c-matching.

Compatible matchings can be motivated by layout problems. The vertices on the two sides of the graph can represent fixed terminals, and non-

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crossing edges represent wires that can appear on the same layer. Let χ_c be the minimum number of c-matchings needed to cover the edges, called the *compatible chromatic index*. If wires must keep to a single layer, then χ_c is the number of layers needed to solve the wiring problem.

The extremal problem for c-matchings arose from the computational problem of merging. Borodin and Hopcroft [BH] proved a special case of the lower bound on α_c using the pigeonhole principle, and used it to prove a lower bound on the complexity of parallel merging. They showed that n processors require (at least) $\frac{1}{2}\log_2\log_2 n + \Omega(1)$ comparison steps to merge two sorted lists of n elements each into one sorted list. Using the general result here, Kruskal [Kr] improves this by showing that $\log_2\log_2 n + \Omega(1)$ comparison steps are required. This result is tight, since [Kr] also presents an algorithm that uses only $\log_2\log_2 n + O(1)$ comparison steps.

We consider extremal values of α_c and χ_c in terms of the number of edges in the graph. Let **G** be the class of bipartite graphs with N edges and parts of size m and n, and let $\sigma=(m+n)/2$. We show that any graph in **G** must contain a c-matching with at least $\lceil \sigma-\sqrt{\sigma^2-N} \rceil$ edges, and we exhibit graphs where equality holds. The extremal graphs are far from unique; we describe and enumerate them when $\sigma-\sqrt{\sigma^2-N}$ is an integer (§6). We obtain similar extremal results for χ_c (§5).

To understand the bound, consider the case $m=n$. Then $\sigma-\sqrt{\sigma^2-N}=n-n\sqrt{\beta}$, where $\beta=1-N/n^2$ is the fraction of edges omitted. The proportion of vertices unmatchable can be as high as the square root of the proportion of edges missing. The extremum can be attained by using all the edges $x_i y_j$ with $\min\{i,j\}=1$, then all those with $\min\{i,j\}=2$, and so on, until N edges have been included (see §2).

We give two proofs that graphs in **G** have $\alpha_c \geq \lceil \sigma-\sqrt{\sigma^2-N} \rceil$. In §3 we partition the edges of $K_{m,n}$ into classes of compatible edges, where the edges $x_i y_j$ with a fixed value of $i-j$ belong to the same class. A careful application of the pigeonhole principle yields the result. This generalizes the argument given by Borodin and Hopcroft [BH] for the case $m=n$, $N=\frac{3}{4}n^2$.

In §4 we take a different approach by considering a dual covering problem. This is suggested by the corresponding extremal problem for arbitrary matchings, which is a standard exercise in elementary graph theory. By the Konig-Egervary Theorem, the size of the largest matching in a bipartite graph equals the smallest number of vertices needed to cover all the edges. To forbid a large matching, concentrate the edges at few vertices. N edges can be concentrated at a set of $\lceil N/\max\{m,n\} \rceil$ vertices and no fewer, so any graph with N edges has a matching of size at least $\lceil N/\max\{m,n\} \rceil$.

The vertex cover is, equivalently, a set of "stars" covering the edges; no two edges in a matching can belong to a single star. To forbid large c-matchings, we need to describe the appropriate dual covering object, and put as many edges as possible in the union of few of these. Let a *twist* be a set of edges such that each edge intersects every other (possibly at an endpoint). Twists and c-matchings intersect in at most one edge, a fact that yields two natural dual pairs of packing and covering problems using the additional parameters θ_c and ω_c. In particular, the largest c-matching size (α_c) is bounded by the minimum number of twists needed to cover the edges (θ_c), and the largest twist size (ω_c) is bounded by the minimum number of c-matchings required to cover the edges (χ_c).

In fact, equality holds for both pairs of problems in every bipartite graph; this follows from translating these problems into vertex packing and covering problems in a class of perfect graphs called permutation graphs. These can be defined in the context of the channel routing problem defined earlier. If each terminal gets exactly one wire, then the terminal labels on one side form a permutation of those on the other according to the connections. Vertices are adjacent in the corresponding graph if and only if the labels appear in opposite order on the two sides, i.e. if the line segments between the two pairs of terminals intersect. A graph is a *permutation graph* if it has such a representation. Permutation graphs are the cocomparability graphs of two-dimensional partial orders, and are well-known to be perfect. (Permutation graphs and perfect graphs are discussed in great detail in [Go].)

A bipartite graph $G\in\mathbf{G}$ can be transformed into a permutation graph $P(G)$ on N vertices so that the compatible matchings of G correspond to independent sets in $P(G)$ and the twists of G correspond to cliques in $P(G)$ (see details in §4). The fact that $\alpha=\theta$ and $\chi=\omega$ in perfect graphs implies $\alpha_c(G)=\theta_c(G)$ and $\chi_c(G)=\omega_c(G)$.

§5 contains extremal results about the compatible chromatic index χ_c, again using the dual problem, which is the size of the maximum twist. For both parameters, the lower bounds are relevant to a problem in partially ordered sets, as noted in §5. The extremal results can be summarized as follows, where G is an arbitrary graph in $\mathbf{G}$.

THEOREM 1. Best possible bounds on $\alpha_c(G)$ and $\theta_c(G)$ are

$$\lceil\sigma-\sqrt{\sigma^2-N}\,\rceil\leq\alpha_c(G)=\theta_c(G)\leq\min\{N,m,n\}.$$

THEOREM 2. Best possible bounds on $\chi_c(G)$ and $\omega_c(G)$ are

$$\max\{\lceil N/n\rceil,\lceil m+n-2\sqrt{mn-N}\,\rceil\}\leq\chi_c(G)=\omega_c(G)\leq\min\{N,m+n-1\}.$$

In both cases, the upper bounds are trivial. As soon as there are $\min\{m,n\}$ edges available, there are graphs with c-matchings of size $\min\{m,n\}$. With fewer edges, all edges can be placed compatibly. No c-matching exceeds size $\min\{m,n\}$. Similarly, for the upper bound in Theorem 2, there are twists with up to $m+n-1$ edges, but no more, since there are only $m+n-1$ possible $j-i$ differences for x_iy_j, and no pair of edges with the

same difference appear in a single twist.

Finally, we mention a related parameter with more flexibility, which has been considered from a purely graph-theoretic viewpoint. Given a graph, place the vertices on a circle in some order, and fill in the edges of the graph as chords in noncrossing classes. The minimum number of classes required, over all vertex orderings, is the *book thickness* [BK] or *pagenumber* [BS] of the graph. Book thickness has been determined for complete graphs and some complete bipartite graphs [BK], and is at most 7 for planar graphs [BS,He], which may not be best possible. The present context is obtained by considering a single fixed ordering of the vertices.

2. The smallest maximum c-matching

We construct a graph G_N on N edges whose largest c-matching has size $\lceil\sigma-\sqrt{\sigma^2-N}\,\rceil$. Order the edges $\{x_iy_j\}$ of $K_{m,n}$ lexicographically on $(\min\{i,j\},\max\{i,j\})$, and in addition put $x_iy_j\leq x_jy_i$. This yields a total order L on the edges. Let the edges of G_N be the first N edges in this ordering. The fact that $\alpha_c(G_N)=\lceil\sigma-\sqrt{\sigma^2-N}\,\rceil$ follows most easily from the dual twist-covering problem.

LEMMA 1. $\alpha_c(G_N)=\lceil\sigma-\sqrt{\sigma^2-N}\,\rceil$.

Proof. It suffices to show G_N a twist-covering of size $r=\lceil\sigma-\sqrt{\sigma^2-N}\,\rceil$ and a c-matching of size r. If $\min\{i,j\}\leq r$ for every edge x_iy_j, the former is easy: let twist T_i consist of the edges $x_iy_n,\ldots,x_iy_{i+1},x_iy_i,x_{i+1}y_i,\ldots,x_my_i$. The edges x_iy_i form the desired c-matching if r is the smallest value such that $\min\{i,j\}\leq r$ for every edge.

To show this, it suffices that $f(r-1)<N\leq f(r)$, where $f(r)=rm+rn-r^2=2r\sigma-r^2$ is the number of pairs i,j with $\min\{i,j\}\leq r$. By the quadratic formula, $-r^2+2r\sigma-N\geq 0$ for positive r if and only if $r\geq\sigma-\sqrt{\sigma^2-N}$, and similarly $f(r-1)<N$ if and only if $r-1<\sigma-\sqrt{\sigma^2-N}$. □

3. Pigeonhole proof of lower bound

By the computation above, $f(r-1)<N\leq f(r)$ if and only if $r=\lceil\sigma-\sqrt{\sigma^2-N}\,\rceil$. Hence the lower bound is equivalent to:

LEMMA 2. Any subgraph of $K_{m,n}$ with more than $(r-1)(2\sigma-r+1)=f(r-1)$ edges has a c-matching of size r.

Proof. Let G be an arbitrary graph in $\mathbf{G}$. We partition the edges of G into $m+n-1$ compatible classes; all the edges in a given class can appear together in a single c-matching. We show that a graph with N edges must have at least $\lceil \sigma - \sqrt{\sigma^2 - N} \rceil$ edges in some single class.

Partition the edges into classes C_j, where C_j consists of all edges of the form $x_i y_{i+j}$; any subset of a single class constitutes a c-matching. The extreme classes C_{-m+1} and C_{n-1} consist of single edges $x_m y_1$ and $x_1 y_n$. Each successive class with smaller displacement has one more edge, until size $\min\{m,n\}$ is reached. Discard all classes with less than r edges; these contain a total of $2 \sum_{i=1}^{r-1} i = r(r-1)$ edges. This make sense because the formula $\sigma - \sqrt{\sigma^2 - N}$ always yields $r \le \min\{m,n\}$.

Discarding these $2r-2$ classes leaves $2\sigma - 2r + 1$ classes of size at least r. Since G has at most $r(r-1)$ edges in the discarded classes, it has more than $f(r-1) - r(r-1) = (r-1)(2\sigma - 2r + 1)$ edges in the remaining classes. By the pigeonhole principle, some class must contain r edges of G. □

4. Lower bounds using duality

As discussed in the introduction, we can bound the size of a c-matching by covering the edges of G with fewest twists. To show this gives $\alpha_c(G)$ exactly, we must interpret c-matchings and twist coverings as independent sets and clique coverings in perfect graphs.

To transform G into a permutation graph, expand each vertex v into $d(v)$ vertices of degree 1, each getting one of the edges, and index these vertices in opposite order from their neighbors in the other part. This ensures that all the edges incident to a single vertex in G cross in the new graph. A set of edges is compatible if and only if it corresponds to an independent set in the intersection graph of the edges in the new drawing. This intersection graph $P(G)$ is a permutation graph, as described in the introduction. Permutation graphs are perfect, which means that $\alpha(H) = \theta(H)$ for all induced subgraphs H (see [Go] for an extensive exposition on permutation graphs). Hence we have $\alpha_c = \theta_c$. This leads to another proof of the main result:

Alternate proof of lower bound. The minimum α_c is a nondecreasing function of N. Hence it suffices to determine the largest number of edges in a graph with $\alpha_c = r$. As noted above, this is the same as finding the maximum size union of r twists in $K_{m,n}$. We need only consider maximal twists. Every maximal twist has $m+n-1$ edges, beginning with $x_1 y_n$ and successively increasing the subscript on x by 1 or decreasing the subscript on y by 1 until reaching the edge $x_m y_1$. This yields a bijection between maximal twists and lattice paths from $(1,n)$ to $(m,1)$ that always move one step down or to the right; call this the *lattice correspondence*. We must maximize the number of lattice points in the union of r such paths.

Assume $r \le \min\{m,n\}$. Starting from $1,n$, there are $i+1$ lattice points on the diagonal reached after i steps, for $i < r$. Similarly for the last r diagonals near $m,1$. The other $m+n+1-2r$ diagonals have at least r points each. Since each path hits each diagonal exactly once, the maximum number of

points in the union is $2 \sum_{i=0}^{r-1} i + r(m+n+1-2r) = 2r\sigma - r^2$. This is achieved, for example, by the twists in §2. □

5. Compatible edge-colorings

Having determined the largest and smallest α_c for graphs with N edges, we can also ask for the extremal values of χ_c.

Proof of Theorem 2. Since $P(G)$ is perfect, we have $\chi_c = \omega_c$. The upper bound on ω_c was given in §1. Achieving the lower bound is almost as easy; as before, a canonical edge-ordering works. The minimum value of χ_c is nondecreasing in N, so maximizing the union of r compatible classes is an equivalent problem. Let $d = j - i - (|m-n|/2)$. Intuitively, we want to choose edges $x_i y_j$ in increasing order of $|d|$, since this fills up the largest compatible classes first. These classes correspond to the diagonals in the lattice correspondence of §4.

To show that this greedy ordering gives the largest union of r compatible classes, partition $K_{m,n}$ into the twists T_i of §2. In the lattice correspondence, each consists of a one vertical and one horizontal segment. Since twists and compatible classes intersect in at most one edge, T_i contributes at most $\min\{r, |T_i|\}$ to the union of r compatible classes. Each T_i intersects the $|T_i|$ displacement classes with minimal $|d|$, so the union of the r displacement classes with minimal $|d|$ achieves the bound.

Let $g(r)$ be the maximum size of the union of r compatible classes. To compute $g(r)$, we must determine the size of these displacement classes. Suppose $m \ge n$, and let $\rho = (m-n)/2$. If $r \le m-n+1$, then $g(r) = rn$. When r is larger, we successively add classes of size $n-1, n-1, n-2, n-2, \cdots$. If $r-(m-n+1) = t$ is even, this yields $g(r) = rn - 2\sum_{i=1}^{t/2} i = rn - (t/2+1)(t/2)$. Since $t/2 = (r-1)/2 - \rho$, this can be written symmetrically in m and n as $r\sigma - (r^2-1)/4 - \rho^2$. To get $g(r)$ when t is odd, evaluate this for $r-1$ and add $n - [(r-1)-(m-n+1)]/2 - 1 = n - r/2 + \rho$. This yields $g(r) = r\sigma - r^2/4 - \rho^2$. To summarize, let $\epsilon(r) = 0$ for r odd, $\epsilon(r) = 1$ for r even. Then

$$g(r) = \begin{cases} rn & r \le m-n+1 \\ r\dfrac{m+n}{2} - \dfrac{r^2 - \epsilon(r)}{4} - \left(\dfrac{m-n}{2}\right)^2 & r > m-n+1 \end{cases}$$

To get the lower bound on χ_c, we invert $g(r)$ to express r in terms of N. In particular, $\min_{G \in \mathcal{G}} \chi_c(G) = r$ if and only if $g(r-1) < N \le g(r)$. Using the quadratic formula again yields the pleasant formula $\chi_c \ge \lceil m+n-2\sqrt{mn-N} \rceil$ for $N > n(m-n+1)$, $\chi_c \ge \lceil N/n \rceil$ for $N < n(m-n+1)$. □

The functions f and g are relevant to the theory of chains and antichains in posets. A compatible class translates via the lattice correspondence into a collection of incomparable elements (an *antichain*) in the ordering on lattice points that puts $(i,j) \leq (k,l)$ if $i \geq k$ and $j \leq l$. Twists translate into totally ordered sets *(chains)*. The lattice points in $[1,m] \times [1,n]$, under this ordering, form the direct product of two chains (totally ordered sets) $C_m \times C_n$. Maximum unions of r twists or c-matchings become maximum unions of r chains or antichains. The twist decomposition $\{T_i\}$ of §2 becomes the standard symmetric chain decomposition of $C_m \times C_n$ found in [dBTK], [GK], and elsewhere. We do not know whether the simple formulas f and g given here for the maximum size union of r chains or antichains in $C_m \times C_n$ have been recorded before.

6. The number of extremal graphs

In Section 4 we showed that the maximum number of edges in a graph with $\alpha_c(G)=r$ is $f(r)$. This is achieved precisely when G is a maximum union of r twists. In the lattice correspondence, such graphs correspond to using all points on the r diagonals nearest $(1,n)$, all points on the r diagonals nearest $(m,1)$, and taking r distinct points from each intermediate diagonal. Geometrically, this means these graphs are in one-to-one correspondence with choices of r disjoint monotone lattice paths from $\{(1,n-r+1), \ldots, (r,n)\}$ to $\{(m-r+1,1), \ldots, (m,r)\}$. Let the bottom-most and top-most paths be the "first" and "last" paths.

Each path in such a configuration takes $m-r$ steps to the right and $n-r$ steps down, and can be encoded by recording the step numbers on which down steps occur. Record these numbers column by column in a rectangular array; note that the columns are strictly increasing. Each of the paths reaches the same diagonal after j steps. The paths are disjoint if and only if after j steps the ith path has taken at least as many down steps as the $i+1$st path, for all i and j. In terms of the corresponding array, this means that the step on which path $i+1$ goes down for the kth time must be at least as late as the step on which path i goes down for the kth time. Hence row k of the array must be nondecreasing, for all k. Conversely, any array of numbers satisfying these conditions corresponds to an acceptable set of paths. Hence the maximum unions of r twists (or maximum graphs with $\alpha_c=r$) are in one-to-one correspondence with column-strict tableaux with r columns, $n-r$ rows, and entries at most $m+n-2r$.

These tableaux have been enumerated. More generally, column-strict tableaux are positive integer arrays with nondecreasing rows and strictly increasing columns, having λ_i numbers in the ith row, with $\lambda_1 \geq \cdots \geq \lambda_m$. For the number of such tableaux having a bound on the entry size, Littlewood [Li] originally obtained a generating function via algebraic methods (see also [St]); Remmel and Whitney [RW] give a bijective proof of this. Bender and Knuth [BK] obtained an explicit formula. The computation now appears

as a nontrivial homework exercise in [Kn].

We suspend our notational conventions temporarily to describe the enumerative result in standard tableau notation. Suppose the number of parts (rows) is m, and the upper limit on entry values is N. The trick is to add zeros to the shape λ so the shape has $m=N$ parts. Then the number of column-strict tableaux is the determinant of a matrix whose ijth entry is $\binom{\lambda_j+m-j}{m-i}$. This can be converted to $V(\lambda_1+m-1, \lambda_2+m-2, \ldots, \lambda_m)/(m-1)! \cdots 0!$ by row operations, where $V(x_1, \ldots, x_n)$ denotes the Vandermonde determinant, which is the determinant of the matrix whose ijth entry is x_j^i. It is well-known that $V(x_1, \ldots, x_n) = \prod_{i<j}(x_j - x_i)$. The formulas for the number of tableaux are proved by induction on $\Sigma\lambda_i$ and the number of rows.

Consider the special case of a rectangular shape with m rows, r columns (i.e. $\lambda_i=r$), and upper limit N on tableau entries, where $N \geq m$. The formula becomes

$$V(r+N-1, \ldots, r+N-m, N-m-1, \ldots, 0)/(N-1)! \cdots 0!.$$

Using the expression for V as a product of differences, the differences between the first m terms yield $(m-1)! \cdots 0!$, the differences between the last $N-m$ terms yield $(N-m-1)! \cdots 0!$, and the differences between the first m and last $N-m$ terms yield $\frac{(r+N-1)!}{(r+m-1)!} \cdots \frac{(r+N-m)!}{r!}$. Collecting all this together, canceling $(N-m-1)! \cdots 0!$ from the numerator and denominator, and introducing m factors of $(N-m)!$ in the numerator and denominator, we can rewrite the answer in terms of binomial coefficients as

$$\prod_{i=1}^{m} \frac{\binom{r+N-i}{N-m}}{\binom{N-i}{N-m}} .$$

In the context of counting the extremal graphs, which is the same as counting the maximum-sized unions of r chains in the product of an m-chain with an n-chain, we want $n-r$ rows, r columns, and entries at most $m+n-2r$. Substituting these values for m, r and N in the expression above yields

$$\prod_{i=1}^{n-r} \frac{\binom{m+n-r-i}{m-r}}{\binom{m+n-2r-i}{m-r}} .$$

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