CLASSES OF INTERVAL DIGRAPHS AND 0,1-MATRICES

In-Jen Lin[1]

National Ocean University, Taipei, Taiwan, ijlin@tiger.cs.nthu.edu.tw

Malay K. Sen[2]

North Bengal University, Darjeeling, India

Douglas B. West[3]

University of Illinois, Urbana, IL 61801-2975, west@math.uiuc.edu

Abstract. We consider a hierarchy of four classes of interval digraphs, or equivalently four classes of 0,1-matrics. We provide forbidden submatrix characterizations separating the successive classes. The largest class is the set of (adjacency matrices of) interval digraphs; the smallest is the set of (adjacency matrices of) unit interval digraphs.

1. INTRODUCTION

An undirected graph G is an *interval graph* if it is the intersection graph of a collection of intervals on the real line; each vertex is assigned an interval, and vertices are adjacent when the corresponding intervals intersect. If some such representation uses intervals of the same length, then G is a *unit interval graph.* Roberts [9] proved that an interval graph is a unit interval graph if and only if it does not have $K_{1,3}$ as an induced subgraph.

We prove several such forbidden substructure characterizations for classes of digraphs related to interval graphs. An *intersection representation* of a digraph D assigns an ordered pair (S_v, T_v) to each vertex $v \in D$ so that $uv \in E(D)$ if and only if $S_u \cap T_v \neq \emptyset$. This model was first described by Beineke and Zamfirescu [1] under the name *connection digraph.* A digraph D is an *interval digraph* if and only if it has an intersection representation assigning each vertex an ordered pair of real intervals. If this can be done with all intervals having the same length, then D is a *unit interval digraph.*

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Since we ignore intersections within $\{S_v\}$ or within $\{T_v\}$, permutations of the rows and columns of the adjacency matrix do not affect the existence of such intersection representations. Membership in these classes is thus a property of the adjacency matrix, and we consider a hierarchy of matrix conditions. We include non-square 0,1-matrices, which corresponds to the model of *interval bigraphs* introduced by Harary, Kabell, and McMorris [6].

A 0,1-matrix is *zero-partitionable* if its rows and columns can be permuted so that every 0 can be labeled by R or C with every position to the right of an R being a 0 labeled R and every position below a C being a 0 labeled C. Such a permutation and labeling is a *zero-partition.* A *monotone consecutive arrangement* (MCA) of a 0,1-matrix is a permutation of its rows and a permutation of its columns and a labeling of its 0's by R or C with every position to the right or above an R being a 0 labeled R and every position to the left or below a C being a 0 labeled C. Every MCA is a zero-partition.

Sen, Das, Roy, and West [10] proved that a digraph is an interval digraph if and only if its adjacency matrix has a zero-partition. Sen and Sanyal [11] proved that a digraph is a unit interval digraph if and only if its adjacency matrix has an MCA. Lin and West [7] proved that a zero-partitionable matrix has an MCA if and only if it does not contain one of the three forbidden submatrices shown below or their transposes. In this paper, we refine that result by inserting intermediate classes of matrices and obtaining a forbidden submatrix characterization for each consecutive pair of classes by a simpler overall proof.

F_1	F_2	F_3
1 1 1 0	1 1 0 0	1 1 0 0
0 1 1 1	1 1 1 1	1 0 1 0
0 0 1 0	0 1 1 0	1 0 0 1

A 0,1-matrix has the *consecutive ones property for columns* if and only if its rows can be permuted so that the ones in each column appear consecutively. Gilmore and Hoffman [5] and Fulkerson and Gross [4] proved that a graph is an interval graph if and only if the incidence matrix between its maximal cliques and vertices has the consecutive ones property for columns. The matrix F_3 above is the incidence matrix for the graph $K_{1,3}$. It has the consecutive ones property for columns but not the analogous consecutive ones property for rows. This explains why the matrix cannot arise in an MCA and partially why $K_{1,3}$ is not a unit interval graph.

When a 0,1-matrix has the consecutive ones property for rows, we can list the rows in nondecreasing order of the position of their leftmost ones to obtain a zero-partition. Hence every 0,1-matrix with the consecutive ones property for rows (or for columns) also has a zero-partition. An MCA permutes the rows and columns so that the ones appear consecutively in both rows and columns. This suggests five classes of 0,1-matrices:

I	zero-partitionable
R	having the consecutive ones property for rows
C	having the consecutive ones property for columns
B	having the consecutive ones property for rows AND the consecutive ones property for columns
U	having an MCA

The names **I** and **U** remind us that the square matrices in these classes are the adjacency matrices of interval digraphs and unit interval digraphs, respectively. We have observed the inclusions

$$\mathbf{I} \supseteq \mathbf{R}, \mathbf{C} \supseteq \mathbf{B} \supseteq \mathbf{U}.$$

These containments are proper. We will characterize the separations, obtaining the result of [7] as a corollary. When we say that a matrix F is forbidden from M, we mean that all matrices obtained by permuting rows and/or columns of F are forbidden as submatrices.

A digraph is a *Ferrers digraph* if its successor sets (equivalently, predecessor sets) form a chain under inclusion. Equivalently, the rows and columns of the adjacency matrix can be permuted independently so the ones appear in the positions of a Ferrers diagram. Equivalently, the adjacency matrix has no 2 by 2 permutation submatrix. The positions of the ones in such a matrix constitute a *Ferrers relation.* In a zero-partition the R's and C's each constitute Ferrers relations, and conversely any *partition* of the zeros into two Ferrers relations yields a zero-partition. Hence a digraph D is an interval digraph if and only if its complement is the union of two disjoint Ferrers digraphs; equivalently, D is the intersection of two Ferrers digraphs whose union is complete.

A digraph D has *Ferrers dimension* k if it is the intersection of k Ferrers digraphs (or if $\bar{D}$ is the union of k Ferrers digraphs). Thus interval digraphs form a special class of digraphs with Ferrers dimension 2. Ferrers dimension 2 has a good characterization. Let $H(D)$ be a graph whose vertices are the zeros in the adjacency matrix of D, with two zeros adjacent if and only if they are the zeros of a 2 by 2 permutation submatrix. Zeros in the matrix that are adjacent in H cannot belong to the same Ferrers digraph in any union forming $\bar{D}$. The Ferrers dimension of D is 2 if and only if $H(D)$ is bipartite (Cogis [2], Doignon-Ducamp-Falmagne [3]). Müller [8] obtained a polynomial-time algorithm for recognizing interval digraphs. The digraph with adjacency matrix below is the smallest digraph of Ferrers dimension 2 that is not an interval digraph.

$$\begin{matrix} 1&1&0&0&0&0 \\ 1&1&1&1&0&0 \\ 0&1&1&1&1&0 \\ 0&1&1&1&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&1 \end{matrix}$$

We also need Tucker's forbidden submatrix characterization of the consecutive ones property using three infinite classes and two isolated examples. Each of the matrices A_n, B_n, C_n for $n \geq 3$ contains in its upper left corner the n by $n-1$ matrix P_n of 0's and

1's that has a 1 in position (i, j) if and only if $i \in \{j, j+1\}$. The matrix P_n has two 1's in each column; this can be viewed as the "bipartite adjacency matrix" of a $2n-1$-vertex path. In the illustration, we record precisely the 1's in the these three classes, with ellipses indicating vertical or double-diagonal strings of 1's. The matrices A_n, B_n, C_n are n by n, $n+1$ by $n+1$, and $n+1$ by n, respectively (A_n is the bipartite adjacency matrix of a $2n$-vertex cycle).

THEOREM A (Tucker [12]) A 0,1-matrix M has the consecutive ones property for columns if and only if M does not have a submatrix in the three infinite classes and two isolated examples listed below. □

1 1	1 1	1	1 0 0 0	
1	1 1 1	1 1	1 0 0 1	1 1 0 1
⋱	⋱ ⋮ ⋮	⋱ ⋮	0 1 0 0	1 1 0 0
1	1 1 1	1 1	0 1 0 1	0 1 1 0
1 1	1 1	1	0 0 1 0	0 1 1 1
	1 1	1	0 0 1 1	0 0 0 1
A_n $(n \geq 3)$	B_n $(n \geq 3)$	C_n $(n \geq 3)$	D	E

We prove the following results. First, a zero-partitionable matrix has the consecutive ones property for columns if and only if it has no submatrix in $\{B_n\} \cup \{C_n\}$. Second, a matrix having the consecutive ones property for columns also has the consecutive ones property for rows if and only if it has no submatrix in $\{B_3^T, C_3^T, C_4^T, C_5^T\}$. Third, let $\mathbf{F} = \{F_1, F_2, F_1^T, F_2^T\}$, where F_1, F_2, F_3 are the matrices from [7] defined earlier. Our main result is that a matrix having the consecutive ones property for columns and the consecutive ones property for rows also has an MCA if and only if it has no submatrix in $\mathbf{F}$. Finally, we use these results to prove the Lin-West [7] result that an interval digraph is a unit interval digraph if and only if its adjacency matrix has no submatrix in $\mathbf{F} \cup \{F_3, F_3^T\}$.

2. PRELIMINARY SEPARATIONS

We have observed that $\mathbf{I} \supseteq \mathbf{R}, \mathbf{C}$; now we characterize the separation.

THEOREM 1. A zero-partitionable matrix M has the consecutive ones property for columns if and only if M has no submatrix in $\{B_n\} \cup \{C_n\}$.

Proof: Assume that $M \in \mathbf{I}$ but $M \notin \mathbf{C}$, so M must contain one of Tucker's matrices. Since $M \in \mathbf{I}$ implies that M has Ferrers dimension at most 2, M does not contain any of Tucker's matrices that have Ferrers dimension 3. For A_n, D, and E, we present odd cycles in the corresponding graphs $H(A_n)$, $H(E)$, and $H(D)$, which by the result of Cogis implies that these have Ferrers dimension 3.

In A_n, observe that the 0's in positions $(1, \lceil n/2 \rceil)$ and $(\lceil (n+1)/2 \rceil, n)$ are adjacent in $H(A_n)$; the path between them completing an odd cycle works its way down two diagonals, visiting the 0's in positions $(1, \lceil n/2 \rceil)$, $(1 + \lceil n/2 \rceil, 1)$, $(2, 1 + \lceil n/2 \rceil)$, $(2 + \lceil n/2 \rceil, 2)$ $, \ldots,$ $(n, \lfloor n/2 \rfloor)$, $(\lceil (n+1)/2 \rceil, n)$. In $H(D)$, we obtain a 9-cycle passing through the 0's in positions (54, 23, 61, 14, 41, 22, 34, 62, 43). In $H(E)$, we obtain a 5-cycle through positions (52, 24, 41, 13, 34).

To complete the proof, we show that B_n and C_n are interval digraphs. For B_n, a zero-partition is obtained by interchanging the last two rows and then moving the last two columns between the first and second column. For C_n, we interchange the last two rows and then move the last column between the first and second column. □

The separation of **I** and **R** now follows by taking transposes.

COROLLARY 1. A zero-partitionable matrix has the consecutive ones property for rows if and only if it has no submatrix in $\{B_n^T\} \cup \{C_n^T\}$. □

If a 0,1-matrix has both the consecutive ones property for rows and the consecutive ones property for columns, then the two properties can be obtained simultaneously, since the first property is exhibited by permuting only rows and the second property is exhibited by permuting only columns. Hence $\mathbf{B} = \mathbf{R} \cap \mathbf{C}$. This makes it easy to characterize the separation between **B** and one of **R**, **C**.

THEOREM 2. A matrix M having the consecutive ones property for columns also has the consecutive ones property for rows if and only if M has no submatrix in $\{B_3^T, C_3^T, C_4^T, C_5^T\}$.

Proof: Suppose $M \in \mathbf{C}$. We know that M does not contain any of $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, D, E, where $n \geq 3$. We also know that $M \in \mathbf{R}$ if and only if M does not contain any of $\{A_n^T\}$, $\{B_n^T\}$, $\{C_n^T\}$, D^T, E^T, where again $n \geq 3$. However, almost all matrices in the second list are already forbidden by $M \in \mathbf{C}$. In particular, the first four columns of E^T form B_3 when the first column is moved between the third and fourth. Also, A_n^T becomes A_n when the last row is moved before the first.

For the remaining matrices to be eliminated, we find a single row permutation of C_3 as a submatrix. If we move row two of C_3 to the bottom, we obtain a 3 by 3 identity matrix plus a row of three 1's at the bottom. This is the submatrix of D^T consisting of columns $2, 4, 6$. For $n \geq 4$, this is the submatrix of B_n^T consisting of rows $1, 2, 4, n$ and columns $1, 3, 4$. For $n \geq 6$, this is the submatrix of C_n^T consisting of rows $1, 3, 4, n$ and columns $2, 3, 5$. Hence all these matrices are already forbidden.

There remain the matrices listed in the theorem statement. If $M \in \mathbf{C} - \mathbf{R}$, then M must contain one of these. To prove that these are the minimal forbidden submatrices for **R** in **C**, it remains only verify that these matrices belongs to **C**. For C_n^T with $3 \leq n \leq 5$, it suffices to move row 1 after row $\lceil n/2 \rceil$, though actually no change at all is needed for C_3^T. For B_3^T, it suffices to move row 2 to the bottom. □

COROLLARY 2. A matrix having the consecutive ones property for rows also has the consecutive ones property for columns if and only if it has no submatrix in $\{B_3, C_3, C_4, C_5\}$. □

3. SEPARATION BETWEEN B AND U

We have observed that $\mathbf{B} \supseteq \mathbf{U}$; we now characterize the separation. We use "column-consecutivity" to indicate an invocation of the consecutive ones property for columns and "row-consecutivity" to indicate an invocation of the consecutive ones property for rows, with "consecutivity" indicating the use of both.

We call a matrix *nontrivial* if no permutation of it has a block decomposition of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. In particular, a nontrivial matrix has no row or column of zeros. We begin with a property of nontrivial matrices in $\mathbf{B}$.

LEMMA 1 ("Fill-in") Suppose that $M \in \mathbf{B}$ is a nontrivial matrix permuted to exhibit the consecutive ones property for both rows and columns. If $M_{a,b} = M_{i,j'} = M_{i',j} = 1$ with $a \leq i \leq i'$ and $b \leq j \leq j'$, then $M_{i,j} = 1$.

Proof: If for fixed a, b there is such a configuration with $M_{i,j} = 0$, we consider such a configuration with a, b fixed and i, j minimal; we cannot reduce i or j to obtain another instance. Consecutivity implies that $M_{p,j} = 0$ and $M_{i,q} = 0$ for all $p < i$ and $q < j$. This yields $a < i < i'$ and $b < j < j'$. The minimality of i implies that $M_{p,s} = 0$ whenever $p < i$ and $s > j$, and the minimality of j implies that $M_{r,q} = 0$ whenever $q < j$ and $r > i$. Finally, $M_{a,b} = 1$ implies that $i > a \geq 1$ and $j > b \geq 1$. Now we have a block decomposition of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, where A is has $i-1$ rows and $j-1$ columns. We conclude that $M_{i,j} = 1$ for each such configuration. □

We call this phenomenon "fill-in". If $M_{1,1} = 1$, fill-in implies that the column indices of the leftmost ones form a non-decreasing sequence.

To specify a submatrix of a matrix M, we list its column indices and row indices, separated by a colon, as in $M[r, q, p; j, i, k, l]$ or $M[p, q, r;]$. The order within each list may indicate a permutation of interest.

THEOREM 3. A matrix M having the consecutive ones property for both columns and rows has a monotone consecutive arrangement if and only if it has no submatrix in $\{F_1, F_2, F_1^T, F_2^T\}$, where

$$F_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Proof: The necessity of the condition follows from a short case analysis. Every MCA contains an MCA of each submatrix. Let F' be the matrix $F_1[;2,3,4]$; this also equals $F_2[;1,2,4]$. Every MCA of F' permutes the rows and columns so that the zeros form a triangle in the upper right or in the lower left (exhibiting Ferrers dimension 1). After permuting F' in this way, it is not possible to insert the remaining column needed to form an MCA of F_1 or F_2.

For sufficiency, a minimal counterexample must be a nontrivial matrix with no duplicate rows or columns. Thus we consider such an m by n matrix $M \in \mathbf{B}$, permuted to exhibit the consecutive ones property for both rows and columns. Let p be the maximum number of initial rows forming an MCA in M. If $p = m$, then $M \in \mathbf{U}$. If $p < m$, then we either obtain a forbidden submatrix or reorder M to preserve consecutivity and obtain a matrix in which the first $p+1$ rows form an MCA. We call the latter outcome "increasing the MCA".

Case 1: $M_{1,1} = 1$. Suppose that the 1's in row p extend from column j to column l and that in row $p+1$ they extend from column j' to column l'. Fill-in implies that $j' \geq j$. If $l' \geq l$, then row $p+1$ can be included in the MCA, contradicting the maximality of p. Hence $M_{p+1,l} = 0$. By column-consecutivity and the lack of a block decomposition, the ones in rows below $p+1$ are confined to columns $j', \ldots, l'$; call this "confinement".

By $M_{1,1} = 1$ and fill-in, there are no $r < p$ and $i > j$ such that $M_{r,j} = 0$ and $M_{r,i} = 1$. If each row above p is constant from column j' through column l, confinement now allows us to interchange $M[;j', \ldots, l']$ with $M[;l'+1, \ldots, l]$ to increase the MCA. Thus we may assume that there exists $r < p$ such that $M_{r,j'} = 1$ and $M_{r,l} = 0$.

Subcase 1a: $j' > j$, so that $M_{p+1,j} = 0$. If $M_{r,l'} = 0$, then $M[r,p,p+1;j,j',l',l] = F_2$. If there exists s such that $M_{s,j} = 1$ and $M_{s,j'} = 0$, then $M[s,r,p,p+1;j,j',l] = F_1^T$. Hence the rows above p are constant from column j to column l'. If $M_{s,j-1} = M_{s,j} = 1$ for some $s < p$, then $M[s,p,p+1;j-1,j,j',l] = F_1$. With $M_{1,1} = 1$ and no row having 1's in columns $j-1$ and j, avoiding a block decomposition (since M is nontrivial) requires $j = 1$. Now we interchange $M[;j', \ldots, l']$ with $M[;j, \ldots, j'-1]$ while preserving consecutivity, after which we move row $p+1$ to the top to increase the MCA.

Subcase 1b: $j' = j$, so that $M_{p+1,j} = 1$. If $j = 1$, then again we move row $p+1$ to the top to increase the MCA. Hence we may let q be the largest index such that $M_{q,j-1} = 1$; fill-in yields $q < p$ and $M_{r,j-1} = 1$. If $M_{q,l} = 1$, then $q \neq r$ and $M[r,q,p,p+1;j-1,j,l] = F_2^T$. Thus $M_{q,l} = 0$. If $M_{q,l'+1} = 0$, then row $p+1$ can follow the last row with 0 in column $l'+1$ to increase the MCA. If $M_{q,l'+1} = 1$, then $M[q,p,p+1;j-1,l'+1,j,l] = F_1$.

Case 2: $M_{1,1} = 0$. In this case $p = 0$. If some permutation of M that preserves consecutivity puts a 1 in the upper left, then it increases the MCA; we suppose not. Let C_i denote the set of columns having 1's in row i. Select as M the permutation preserving consecutivity that has the longest chain of inclusions $C_1 \subset C_2 \subset \cdots \subset C_q$. If $q = m$, then we have a Ferrers relation and permute columns to obtain an MCA. Hence $q < m$.

Let $C_q = \{j, \ldots, l\}$. Since $C_q \not\subseteq C_{q+1}$, row-consecutivity implies that $M_{q+1,j} = 0$ or

$M_{q+1,l} = 0$. Avoiding a column of zeros then requires $j = 1$ or $l = n$, respectively. By reversing the column order if necessary, we may assume that $M_{q,1} = 1$ and $M_{q+1,1} = 0$. Maintaining the notation of Case 1, we let $C_{q+1} = \{j', \ldots, l'\}$.

Let r be the minimum index such that $M_{r,1} = 1$; we have $1 < r \leq q$. If $C_{r-1} \cap C_{q+1} = \emptyset$, then we permute columns $1, \ldots, l$ so that those of C_{r-1} precede those of C_{q+1} and $M_{1,1} = 1$, increasing the MCA. Hence we may select $h \in C_{r-1} \cap C_{q+1}$. We now consider two subcases.

Subcase 2a: $C_r \supset C_{q+1}$. In this case, we can move row $q + 1$ higher to increase the MCA unless there exists $s < r$ such that C_s and C_{q+1} are not ordered by inclusion. Choose $i \in C_s - C_{q+1}$ and $i' \in C_{q+1} - C_s$. If there exists $h' \in C_s \cap C_{q+1}$, then $M[s, r, q + 1; i, h, i', 1] = F_2$. If $C_s \cap C_{q+1} = \emptyset$, then $s < r - 1$, since $h \in C_{r-1} \cap C_{q+1}$. Now $M[s, r - 1, r, q + 1; i, h, 1] = F_1^T$.

Subcase 2b: $C_r \not\supset C_{q+1}$. Let k be the column index of the rightmost 1 in C_r. If $r \neq q$, then we must have $M_{q,k+1} = 1$ to avoid duplicate rows; now $M[q + 1, q, r, r - 1; k + 1, h, 1] = F_2^T$. Hence we have $r = q$. If $C_1 \supseteq C_q \cap C_{q+1}$, then we can reverse the first q rows to increase the MCA. Otherwise, we select $i \in C_q \cap C_{q+1} - C_1$. Since $C_1 \neq \emptyset$, we may also select $i' \in C_1 \subset C_q$. If $i' \in C_{q+1}$, then $M[q, q + 1, 1; 1, i, i', k + 1] = F_1$. If $C_1 \cap C_{q+1} = \emptyset$ and no such i' exists, then $C_{r-1} \cap C_{q+1} \neq \emptyset$ implies that $1 < r - 1$. Now $M[q + 1, q, r - 1, 1; h, i', 1] = F_1^T$. □

This proof, like the earlier proofs, is constructive. In places where the argument uses nontriviality to eliminate a case, an implementation would instead recursively analyze the blocks of a decomposition. Testing for the existence of an MCA for an $m \times n$ matrix that exhibits the consecutive ones property for for both rows and columns runs in time linear in mn.

Finally, we characterize the interval digraphs that are unit interval digraphs.

COROLLARY 3 (Lin-West [7]). A zero-partitionable matrix has an MCA if and only if it has no submatrix in $\mathbf{F} = \{F_1, F_2, F_3, F_1^T, F_2^T, F_3^T\}$.

Proof: Since F_3 does not have the consecutive ones property for rows, the condition is necessary. For sufficiency, we show that if $M \in \mathbf{I}$ has no submatrix in $\mathbf{F}$, then M cannot contain B_n or C_n for $n \geq 3$. This places M in $\mathbf{B}$ by Theorems 1 and 2, and then Theorem 3 yields $M \in \mathbf{U}$. The claim holds because $B_n[n, n - 1, n + 1; n - 1, n + 1, n, n - 2] = F_2$ when $n \geq 3$, and $C_n[2, 3, n - 1; 1, 2, n, 3] = F_1$ when $n \geq 4$. When $n = 3$, C_n itself is a row permutation of F_3^T (this is the only statement in the sufficiency proofs that uses F_3). □

References

[1] L.W. Beineke and C.M. Zamfirescu, Connection digraphs and second order line graphs.

Discrete Math. 39(1982), 237–254.

[2] O. Cogis, A characterization of digraphs with Ferrers dimension 2. CNRS Research Report 19(1979).

[3] J.P. Doignon, A. Ducamp, and J.-C. Falmagne, On realizable biorders and the biorder dimension of a relation. *J. Math. Psych.* 28(1984), 73-109.

[4] D.R. Fulkerson and O.A. Gross, Incidence matrices and interval graphs. *Pac. J. Math.* 15(1965), 835-855.

[5] P.C. Gilmore and A.J. Hoffman, A characterization of comparability graphs and of interval graphs. *Canad. J. Math.* 16(1964), 539-548.

[6] F. Harary, J.A. Kabell, and F.R. McMorris, Bipartite intersection graphs. *Comm. Math. Univ. Carolinae* 23(1982), 739–745.

[7] I.-J. Lin and D.B. West, Interval digraphs that are indifference digraphs, *Graph theory, Combinatorics, and Algorithms* (Proc. 7th Intl. Conf. Graph Th. - Kalamazoo 1992) (Y. Alavi and A. Schwenk, eds.) (Wiley 1995), 751–765.

[8] H. Müller, Recognizing interval digraphs and bi-interval graphs in polynomial time, preprint.

[9] F.S. Roberts, Indifference graphs, *Proof Techniques in Graph Theory* (F. Harary, ed.). Academic Press (1969), 139–146.

[10] M. Sen, S. Das, A.B. Roy, and D.B. West, Interval digraphs: an analogue of interval graphs. *J. Graph Theory* 13(1989), 189-202.

[11] M. Sen and B.K. Sanyal, Indifference digraphs: a generalization of indifference graphs. *SIAM J. Discr. Math.* 7(1994), 157–165.

[12] A.C. Tucker, A structure theorem for the consecutive 1's property. *J. Comb. Theory* 12(1972), 153-162.