# Extending Precolorings to Circular Colorings

Michael O. Albertson* and Douglas B. West†

September, 2005

**Abstract**

Fix positive integers $k', d', k, d$ such that $k'/d' > k/d \geq 2$. If $P$ is a set of vertices in a $(k,d)$-colorable graph $G$, and any two vertices of $P$ are separated by distance at least $2\left\lceil \frac{kk'}{2(k'd-kd')} \right\rceil$, then every coloring of $P$ with colors in $\mathbb{Z}_{k'}$ extends to a $(k',d')$-coloring of $G$. If $k'd - kd' = 1$ and $\lfloor k'/d' \rfloor = \lfloor k/d \rfloor$, then this distance threshold is nearly sharp. The proof of this includes showing that up to symmetry, there is only one $(k',d')$-coloring of the canonical $(k,d)$-colorable graph $G_{k,d}$ in this case.

## 1 Introduction

A precoloring extension problem asks whether a coloring of a specified vertex subset $P$ in a graph $G$ extends to a "good" coloring of all of $G$. Such problems are usually NP-complete [11]. One might hope that if the vertices of $P$ were far enough apart, then there would be enough flexibility to extend a given precoloring to an optimal coloring of $G$ (that is, a proper coloring with the minimum possible number $\chi(G)$ of colors).

More generally, let $G[P]$ denote the subgraph of $G$ induced by a vertex set $P$, and define $d(P)$ to be the minimum distance between components of $G[P]$. A precoloring extension theorem for a family $\mathcal{G}$ of graphs might take the following form: there is a constant $d^*$ such that if $G$ is a $k$-chromatic graph in $\mathcal{G}$, and $P \subseteq V(G)$ with $d(P) \geq d^*$, then every proper $k$-coloring of $G[P]$ extends to a proper $k$-coloring of $G$. We call this an *optimal extension theorem*, because the resulting coloring is an optimal coloring of $G$.

*Smith College, Northampton MA 01063: albertson@math.smith.edu.

†University of Illinois, Urbana, IL 61801-2975: west@math.uiuc.edu. Work supported in part by the NSA under Award No. MDA904-03-1-0037.

Optimal extension theorems seem to be rare. Albertson and Moore [6] proved an optimal extension theorem with $d^* = 12$ for the class of graphs having an optimal coloring with some color class having no two vertices with a common neighbor. "Locally planar" graphs on a given surface are 5-colorable; Albertson and Hutchinson [3] proved an optimal extension theorem for 5-chromatic graphs satisfying a local planarity condition that depends on the particular surface. Their theorem requires that $G[P]$ is 2-colorable, each component of $G[P]$ is precolored with two colors (not necessarily the same two colors), and $d^* = 18$. There are also optimal precoloring extensions of Brooks' Theorem, which we describe below.

For most commonly studied classes of $k$-chromatic graphs there is no optimal extension theorem. Example 1 shows that there is no such theorem for 4-chromatic planar graphs.

**Example 1** Let $P_n^3$ denote the graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_iv_j \colon |i-j| \leq 3\}$. The distance between $v_i$ and $v_j$ is $\lceil |j-i|/3 \rceil$. In every proper 4-coloring of $P_n^3$, the color assigned to $v_i$ depends only on the congruence class of $i$ modulo 4. Therefore, when $n \equiv 1 \pmod 4$, a precoloring of $\{v_1, v_n\}$ extends to a proper coloring of $P_n^3$ if and only if the colors assigned to $v_1$ and $v_n$ are the same. □

Precoloring extension became an active topic when Thomassen [10] asked the following: if $G$ is a planar graph and $P$ is an independent set in $G$ such that $d(P) \geq 100$, does it follow that every 5-coloring of $P$ extends to a 5-coloring of $G$? Albertson [1] showed that the flexibility of allowing an extra color beyond $\chi(G)$ permits a precoloring extension theorem without any additional hypotheses. He proved that if $P$ is an independent set in an arbitrary $k$-colorable graph $G$ and $d(P) \geq 4$, then every coloring of $P$ from a set of $k+1$ colors extends to a proper $(k+1)$-coloring of $G$.

More generally, suppose that each component of $G[P]$ is a complete graph of order $j$. Albertson [1] proved that if $d(P) \geq 6j - 2$, then every proper $(k+1)$-coloring of $G[P]$ extends to a proper $(k+1)$-coloring of $G$. Kostochka (see [5]) improved this by weakening the distance constraint to $d(P) \geq 4j$, and Albertson and Moore [5] showed that $d(P) \geq 3j$ suffices when $j = k$.

Albertson and Hutchinson [2] showed that if $G$ is $k$-colorable, $G[P]$ is $s$-colorable, and $d(P) \geq 4$, then every $(k+s)$-coloring of $G[P]$ in which each component is properly $s$-colored (not necessarily with the same $s$ colors) extends to a proper $(k+s)$-coloring of $G$. Here both the distance constraint and the number of colors required are best possible.

Let $\Delta(G)$ denote the maximum vertex degree in $G$. When $\Delta(G)+1$ colors are allowed, every proper coloring of $G[P]$ trivially extends to a proper coloring of $G$. Axenovich [7] proved that if $G$ is not a complete graph and $\Delta(G) \geq 3$, then every proper $\Delta(G)$-coloring of an independent set $P$ with $d(P) \geq 8$ extends to a proper $\Delta(G)$-coloring of $G$, and this distance threshold is best possible. Albertson, Kostochka, and West [4] generalized this, showing that the same conclusion holds when the components of $G[P]$ are complete graphs, with the same distance threshold $d(P) \geq 8$ if $\Delta(G) \geq 4$, but only with $d(P) \geq 10$ if $\Delta(G) = 3$. They also strengthened the result for precolored independent sets to the context of list colorings.

Here we initiate the study of precoloring extensions for the more general model of "circular" coloring. For integers $k$ and $d$ with $k/d \geq 2$, a $(k,d)$-*coloring* of a graph $G$ is a map $f\colon V(G) \to \mathbb{Z}_k$ such that if vertices $u$ and $v$ are adjacent, then $f(u) - f(v)$ is congruent modulo $k$ to a member of $\{d, \ldots, k-d\}$. A graph having a $(k,d)$-coloring is $(k,d)$-*colorable*. For a graph $G$, the *circular chromatic number* $\chi_c(G)$ is $\inf\{k/d\colon G \text{ is } (k,d)\text{-colorable}\}$. A $(k,1)$-coloring is simply an ordinary proper $k$-coloring; thus $\chi_c(G) \leq \chi(G)$.

Vince [12] introduced $(k,d)$-colorings, calling them "star colorings". Bondy and Hell [8] observed that the infimum in the definition of $\chi_c(G)$ can be replaced by the minimum; that is, the infimum is achieved. Vince [12] proved that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, and hence $\lceil \chi_c(G) \rceil = \chi_c(G)$. Thus the circular chromatic number is a refinement of the (ordinary) chromatic number.

Other equivalent models for circular coloring are based on representation by circular arcs (Zhu [14]), homomorphisms (Bondy and Hell [8]), and duals of flow problems (Goddyn, Tarsi, and Zhang [9]). Zhu [13] presents a thorough survey of results on the circular chromatic number. Our circular precoloring extension theorem was originally proved using the $(k,d)$-coloring formulation. We are grateful to an anonymous referee who observed that a variation on the standard circular arc formulation provides a cleaner context for our proof. We present the extension theorem and this simpler proof in Section 2.

Say that $(k,d,k',d')$ is *tight* if $k'd - kd' = 1$ and $\lfloor k/d \rfloor = \lfloor k'/d' \rfloor$. In Section 3 we use the graph homomorphism formulation to establish lower bounds for the distance threshold, especially for the tight case. The proof needs a characterization of the $(k',d')$-colorings of $G_{k,d}$, the graph that plays the role for $(k,d)$-coloring that the complete graph $K_k$ plays for proper $k$-coloring. We prove that up to symmetry, in the tight case there is only one $(k',d')$-coloring of $G_{k,d}$ (this result is of independent interest). Using this theorem and a construction involving a special type of graph product, we show that the distance constraint in our extension result is nearly sharp in the tight case. When $k/d = 2$, we prove the near-sharpness for more general choices of $(k',d')$. In the last section, we discuss the possible generalization of other results mentioned here to the setting of circular coloring.

## 2 The Extension Theorem

Let $S^r$ denote the circle with circumference $r$. For $u, v \in S^r$, let the *circular distance* $|u-v|_r$ between $u$ and $v$ be the minimum length of an arc of $S^r$ with endpoints $u$ and $v$. An *$r$-coloring* of a graph $G$ is a map $f\colon V(G) \to S^r$ such that $|f(x)-f(y)|_r \ge 1$ whenever $xy \in E(G)$.

An $r$-coloring $f$ can be viewed as assigning each vertex $x$ a unit-length arc on $S^r$ centered at $f(x)$; the arcs for adjacent vertices must be disjoint. Letting $\chi^*(G) = \inf\{r\colon G$ has an $r$-coloring$\}$, the result of Bondy and Hell [8] is that $\chi^*(G) = \chi_c(G)$, so we use only the notation $\chi_c$ for both models.

For ease of computation (including multiplication of color values by constants), we name the points of $S^r$ by the real numbers modulo $r$ in the natural order, using $[0, r)$ as canonical representatives of the classes. With this notation, a $(k, d)$-coloring $\phi$ becomes a $k/d$-coloring $f$ by letting $f(x) = \phi(x)/d$; thus $\chi^*(G) \le \chi_c(G)$.

For the opposite inequality, note that once a cyclic order of the images in an $r$-coloring is given, the constraints for edges become linear inequalities with coefficients $\pm 1$. Minimizing $r$ is then a linear program with a rational solution. Hence $\chi^*(G)$ is the minimum of $(|V(G)|-1)!$ rational values, we obtain $(k, d)$-colorings showing that $\chi_c(G) \le \chi^*(G)$, and $\chi_c(G)$ is rational.

We use the $r$-coloring model for our proof, even though the theorem is stated using $(k, d)$-colorings. In the language of $r$-coloring, it is immediate that if $\chi_c(G) \le r$, then $G$ has an $r$-coloring, because inserting extra length in the circle does not violate any constraint.

**Theorem 1** Given positive integers $k, d, k', d'$ with $k'/d' > k/d \ge 2$, let $l = \lceil \frac{kk'}{2(k'd-kd')} \rceil$. If $\chi_c(G) \le k/d$, and $P \subset V(G)$ is an independent set such that $d(P) \ge 2l$, then every precoloring of $P$ from $\mathbb{Z}_{k'}$ extends to a $(k', d')$-coloring of $G$.

**Proof** Set $r = k/d$ and $r' = k'/d'$, so $l = \lceil \frac{rr'}{2(r'-r)} \rceil$. Since $l \ge \frac{rr'}{2(r'-r)}$, also $\frac{r'}{2l} \le \frac{r'-r}{r}$. For $x \in P$, let $c(x) = c'(x)/d'$, where $c'(x)$ is the color preassigned to $x$.

Since $\chi_c(G) \le r$, there is an $r$-coloring $f$ of $G$. Define an $r'$-coloring $f'$ of $G$ by letting $f'(x) = \frac{r'}{r} f(x)$ for $x \in V(G)$. Since $r' > r$, the multiplication spreads images apart, so edge constraints still hold under $f'$. In fact, if $uv \in E(G)$, then $|f'(u) - f'(v)|_{r'} \ge \frac{r'}{r} = 1 + \frac{r'-r}{r}$.

Fix a vertex $x$ in $P$. We know that $|f'(x) - c(x)|_{r'} \le r'/2$. By symmetry, we may assume that $0 \le f'(x) - c(x) = t \le r'/2$. We modify $f'$ to make the color on $x$ become $c(x)$, changing colors only on vertices whose distance from $x$ in $G$ is at most $l-1$. Since $d(P) \ge 2l$, changes for distinct vertices in $P$ are independent.

For each vertex $y$ with distance $i$ from $x$, where $0 \le i \le l-1$, let $f^*(y) = f'(y) - t\frac{l-i}{l}$.

Since $f^*(x) = c(x)$, it suffices to show that $f^*$ is an $r'$-coloring of $G$. If $uv \in E(G)$, then the distances from $x$ to $u$ and $v$ differ by at most 1. Hence the amounts by which $f'(u)$ and $f'(v)$ shift to obtain $f^*(u)$ and $f^*(v)$ differ by at most $t/l$. Labeling $u$ and $v$ in order so that $|f'(u) - f'(v)|_{r'} = f'(u) - f'(v)$, we have

$$|f^*(u) - f^*(v)|_{r'} \geq f'(u) - f'(v) - \frac{t}{l} \geq 1 + \frac{r' - r}{r} - \frac{r'}{2l} \geq 1. \qquad \square$$

## 3 Sharpness of the Distance Threshold

A *homomorphism* from a graph $G$ to a graph $H$ is a function $f\colon V(G) \to V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. Let $G_{k,d}$ be the graph with vertex set $\{v_0, v_1, \ldots, v_{k-1}\}$ such that vertices $v_i$ and $v_j$ form an edge if and only if $d \leq |i-j| \leq k-d$. A homomorphism $f$ from $G$ to $G_{k,d}$ becomes a $(k, d)$-coloring of $G_{k,d}$ by viewing the subscripts of the image vertices as congruence classes modulo $k$. Similarly, a $(k, d)$-coloring describes a homomorphism to $G_{k,d}$; the concepts are equivalent, as observed in [8, 13]. The graph $G_{k,d}$ is a generalization of a complete graph, since $G_{k,1} = K_k$. The graph $G_{k,d}$ is unchanged under cyclic rotations or reflections; hence we call two vertices *consecutive* if their subscripts differ by 1 modulo $k$.

We begin with a natural family of examples to show that for every pair $(k, d)$, there is no sufficient distance for precolored vertices in a $(k, d)$-colorable graphto guarantee an extension to a $(k, d)$-coloring. Our constructions use a special graph product.

**Definition 1** Given two graphs $G$ and $H$, the *extension product* $G \bowtie H$ is the graph with vertex set $V(G) \times V(H)$ such that vertices $(u, v)$ and $(u', v')$ are adjacent if and only if $uu' \in E(G)$ and the vertices $v$ and $v'$ are either equal or adjacent in $H$.

In terms of more familiar products, $G \bowtie H$ consists of the tensor (also called "weak" or "categorical") product of $G$ and $H$ plus copies of $G$ for the vertices of $H$.

It is useful to describe $G \bowtie H$ as the union of copies of $G \bowtie K_2$ corresponding to the edges of $H$. The graph $G \bowtie K_2$ consists of disjoint copies $G^1$ and $G^2$ of $G$, using vertices $v_i^1$ in $G^1$ and $v_i^2$ in $G^2$ for each $v_i \in V(G)$, plus edges $v_i^1 v_j^2$ and $v_j^1 v_i^2$ for each $v_i v_j \in E(G)$. In the notation "$\bowtie$", the vertical segments suggest $v_i^1 v_j^1$ and $v_i^2 v_j^2$, while the diagonal segments suggest the added edges $v_i^1 v_j^2$ and $v_j^1 v_i^2$. We call the result the "extension product" because we are "extending" $G$ along the edges of $H$. Our constructions have the form $G_{k,d} \bowtie H$, using $G_{k,d} \bowtie K_2$ as the basic building block.

**Example 2** Consider the graph $G_{k,d} \bowtie P_n$, where $P_n$ is the path with $n$ vertices. We write the vertex set as $\{v_i^t\colon 0 \le i \le k-1 \text{ and } 1 \le t \le n\}$. For each $t$, a copy of $G_{k,d}$ is induced by $\{v_i^t\colon 0 \le i \le k-1\}$. The extension edges are $\{v_i^t v_j^{t+1}\colon d \le |i-j| \le k-d \text{ and } 1 \le t \le n-1\}$.

A fundamental result in circular coloring is that all $(k,d)$-colorings of $G_{k,d}$ arise by rotation and reflection from the "standard" coloring $f^*$ given by $f(v_i^*) = i$. This can be proved using $r$-coloring or using a simpler version of the proof of Theorem 2 below. In such a coloring, the colors on the neighbors of $v_i$ leave no flexibility for $f^*(v_i)$. Since all $(k,d)$-colorings of $G_{k,d}$ have this form, and substituting $v_i^2$ for $v_i^1$ yields another copy of $G_{k,d}$, we conclude that $f(v_i^2) = f(v_i^1)$ in a $(k,d)$-coloring $f$ of $G_{k,d} \bowtie P_n$.

Continuing down the path, we find that a precoloring of $\{v_i^1, v_i^n\}$ extends to a $(k,d)$-coloring of the entire graph if and only if these two vertices are assigned the same color. The distance between $v_i^1$ and $v_i^n$ is at least $n-1$, so no distance threshold suffices for an optimal extension theorem. □

For our lower bound on the distance threshold guaranteeing extensions when $k'd - kd' = 1$ and $\lfloor k'/d' \rfloor = \lfloor k/d \rfloor$, we need to prove that in this case $(k',d')$-colorings of $G_{k,d}$ also have very little flexibility.

**Definition 2** The *canonical* $(k',d')$-coloring $c^*$ of $G_{k,d}$ is defined by $c^*(v_i) = \left\lfloor i\frac{k'}{k} \right\rfloor$ for $0 \le i \le k-1$. Replacing $c^*(v_i)$ with $c^*(v_{i+r})$ for all $i$ yields a *rotation* of $c^*$. Replacing $c^*(v_i)$ with $c^*(v_i)+t$ for all $i$ yields a *translation* of $c$. Replacing $c^*(v_i)$ with $c^*(v_{-i})$ for all $i$ or with $-c^*(v_i)$ for all $i$ (viewing indices modulo $k$ and colors modulo $k'$) yields a *reflection* of $c^*$.

We now describe all $(k',d')$-colorings of $G_{k,d}$ in the tight case. No assumption is made about which of $d$ and $d'$ is larger. Note that the condition $k'd - kd' = 1$ cannot be written simply in terms of the ratios $k/d$ and $k'/d'$. Thus, although Theorem 1 is easy to prove in the context of $r$-coloring, it seems that we cannot avoid using $(k,d)$-coloring here.

**Theorem 2** In the tight case ($k'd - kd' = 1$ and $\lfloor k'/d' \rfloor = \lfloor k/d \rfloor \ge 2$), every $(k',d')$-coloring of $G_{k,d}$ is obtained from the canonical coloring by rotation, translation, and/or reflection, except that when $d = 1$ also the vertices can be permuted arbitrarily.

**Proof** Let $c^*$ be the canonical coloring. We first obtain $c^*(v_{i+d}) - c^*(v_i) \in \{d', d'+1\}$ for all $i$, with exactly one of these differences being $d'+1$ and the rest being $d'$. Since $k'd - kd' = 1$,

$$c^*(v_{i+d}) - c^*(v_i) = \left\lfloor (i+d)\frac{k'}{k} \right\rfloor - \left\lfloor i\frac{k'}{k} \right\rfloor = d' + \left\lfloor \frac{ik'+1}{k} \right\rfloor - \left\lfloor \frac{ik'}{k} \right\rfloor .$$

The value is $d'$ except when $ik' \equiv -1 \bmod k$. Since $k$ and $k'$ are relatively prime, this occurs exactly once, which proves the claim.

Every coloring with this multiset of color differences (or its negatives) arises from the canonical coloring by rotation, translation, and/or reflection. It thus suffices to prove for an arbitrary $(k', d')$-coloring $c$ that the differences $c(v_{i+d}) - c(v_i)$ for $0 \le i \le k-1$ are all $d'$ except for one $d'+1$, or all $-d'$ except for one $-d'-1$.

If $d = 1$, then $G_{k,d} \cong K_k$. Now the symmetries of $K_k$ allow the vertices to be permuted arbitrarily. The colors used must pairwise differ by at least $d'$. Since $k' = kd' + 1$, the $k$ used colors in cyclic order must differ successively by $d'$, except for one $d'+1$. Permuting the vertices to agree with the cyclic order of the colors now completes the proof.

We may therefore assume that $k/d$ is not an integer. Let $q = \lfloor k/d \rfloor = \lfloor k'/d' \rfloor$. Define $t$ and $s$ by $k = qd + t$ and $k' = qd' + s$. Note that $0 \le t < d$ and $0 \le s < d'$. Since $k/d$ is not an integer, $t > 0$. Eliminating $q$ and using $kd' = k'd - 1$ yields $s = (td'+1)/d$, so also $s > 0$.

**Claim 1:** *The colors on any $t+1$ consecutive vertices lie in a set of $s+1$ consecutive values in $\mathbb{Z}_{k'}$.* Let $T = \{v_{i-t}, \ldots, v_i\}$, for some $i$ (computations with indices are modulo $k$; computations with colors are modulo $k'$). Note that $v_{i-t} = v_{i+qd}$. The $q-1$ vertices $\{v_{i+jd}\colon 1 \le j \le q-1\}$ form a clique $Q$. Each vertex in $T$ is adjacent to each vertex in $Q$.

For $v \in T$, the set $Q \cup \{v\}$ is a clique, so the colors on it differ pairwise by at least $d'$. When we view them in cyclic order, each gap from one color to the next is at least $d'$. Since $k' = qd' + s$ and $|Q \cup \{v\}| = q$, no gap exceeds $d' + s$. Deleting the color on $v$ leaves a gap of length at least $2d'$ and at most $2d' + s$ between the colors closest to it. Since no other gap is as large as $2d'$, the colors on all of $T$ lie in this gap. They are at least $d'$ away from the colors used on the vertices of $Q$ at the top and bottom of the gap, so the colors available for use on $T$ form a set of at most $(2d' + s - 1) - 2(d' - 1)$ consecutive values in $\mathbb{Z}_{k'}$.

**Claim 2:** *$d' \le c(v_{i+d}) - c(v_i) \le d' + s$ for all $i$ (after reflection of $c$, if necessary).* By Claim 1, moving by at most $t$ vertices in the indexing of $G_{k,d}$ changes the color under $c$ by at most $s$. Also, $s = (td'+1)/d$ yields $sd/t = d' + 1/t$. Hence

$$|c(v_{i+d}) - c(v_i)| \le s \left\lceil \frac{d}{t} \right\rceil \le s \left( \frac{d}{t} + \frac{t-1}{t} \right) = d' + \frac{1}{t} + s\frac{t-1}{t} \le d' + s,$$

as desired. Since $s < d'$ and the colors on adjacent vertices differ by at least $d'$, the colors on any clique of $q$ vertices spaced by steps of size $d$ now must occur in cyclic order.

For $q > 2$, successive such cliques overlap by at least two vertices, implying that as we successively take steps of size $d$ along the vertices of $G_{k,d}$, the changes in color by between $d'$ and $d' + s$ are always upward or always downward. For $q = 2$, the successive cliques overlap by only one vertex; nevertheless, the statement remains true, because a downward change by between $d'$ and $d'+s$ is also an upward change by between $d'$ and $d'+s$, since $k' = d'+d'+s$. Since the changes are always upward or always downward, we have the desired inequalities for $c$ or its reflection; we may assume that they hold for $c$.

**Claim 3.** *Fix $i$; for $1 \le j \le t$, the difference $c(v_{i+jd}) - c(v_{i+(j-1)d})$ lies in $\{d', d'+1\}$, and at most one of these differences is $d'+1$.* By Claim 2, $d' \le c(v_{i+d}) - c(v_i) \le d'+s$. In traversing $t$ steps of length $d$ along the vertex set, the sum of these changes is at least $td'$, counting upward each time. Moving by $dt$ vertices in $G_{k,d}$ can also be viewed as traversing $d$ steps of length $t$, and Claim 1 implies that the color changes by at most $s$ along each such step. Since $s < d' \le k'/2$, we can treat each such change in a well-defined way as an amount of change between $-s$ and $+s$ in the direction that color increases when steps of length $d$ are taken. The total of the $d$ differences in this direction along steps of length $t$ is at most $ds$. Since $ds = td'+1$, we conclude that $t-1$ of the changes along steps of length $d$ equal $d'$, and the remaining change is $d'$ or $d'+1$.

**Conclusion.** *$c(v_{i+d}) - c(v_i) = d'$ for all $i$, except for one instance of $d'+1$.* We have observed that this statement suffices. To prove it, sum all $k$ such differences, successively moving $d$ vertices per step upward along the indexing of $V(G_{k,d})$. Since $k'd - d'k = 1$, the numbers $k$ and $d$ are relatively prime, so we return to the original vertex and hence also the original color. By Claim 3, each step increases the color by $d'$ or $d'+1$. Let $j$ be the number of steps along which the color increases by $d'+1$. Since the total color change is a multiple of $k'$ and all steps are positive, $kd'+j = mk'$ for some positive integer $m$.

These $k$ successive steps of size $d$ wrap $d$ times around the vertices of $G_{k,d}$. Each change between two successive vertices is captured in exactly $d$ of these steps. Hence $m = dl$, where $l$ is the number of times that the changes in colors wrap around $\mathbb{Z}_{k'}$ when we step one-by-one around $V(G_{k,d})$. We compute $l$ by the method used in proving Claim 3. By Claim 1, the difference between colors on successive vertices is at most $s$, and $s < k'/2$ allows us to write the change as between $-s$ and $+s$ in the upward direction through $\mathbb{Z}_{k'}$.

Through any set of $d$ unit steps, Claim 3 implies that the total color change is $d'$ or $d'+1$. Since $k = qd+s$, a single trip through the vertices consists of $q$ steps of length $d$ and a final step of length $t$. Summing the color changes now yields $qd' - s \le lk' \le q(d'+1)+s$.

With $qd'+s = k'$, we rewrite this as $k'-2s \le lk' \le k'+q+s$. The conditions $k'/d' > k/d$ and $\lfloor k'/d \rfloor = \lfloor k/d \rfloor$ require $d' > 1$, and therefore $q+s < k'$. Also $2s < k'$, so $0 < lk' < 2k'$. Since $l$ is an integer, $l = 1$ and $m = d$. Thus $j = k'd - kd' = 1$, and the proof is complete. $\square$

**Example 3** It is worth noting that the conclusion of Theorem 2 need not completely hold when the condition $\lfloor k/d \rfloor = \lfloor k'/d' \rfloor$ is dropped. If $k/d < n < k'/d'$ for some integer $n$, then $kd'/dd' < ndd'/dd' < k'd/dd'$, and comparing numerators yields $k'd - kd' >= 2$. Therefore, we may assume that $k'/d'$ is an integer. Since $k'd - kd' = 1$ puts the fractions in lowest terms, $d' = 1$. Now $k = k'd - 1$. Echoing our earlier notation, let $q = k'$.

We seek a characterization of the proper $q$-colorings of $G_{qd-1,d}$. The independence number of this graph is $d$, and all independent $d$-sets consist of $d$ consecutive vertices in $\mathbb{Z}_{qd-1}$. With $qd-1$ vertices, a proper $q$-coloring must have $q-1$ color classes of size $d$ and one class of size $d-1$. Since an independent set of size $d-1$ must lie within $d$ consecutive vertices (we may assume that $d > 1$), all the color classes appear consecutively. This is precisely what the canonical coloring does. However, since $d' = 1$, we can permute the names of the colors arbitrarily. When $q \geq 4$, the resulting coloring cannot always be obtained from the canonical coloring by the operations allowed in the conclusion of Theorem 2. □

We next use Theorem 2 to construct examples showing that in the tight case ($k'd - kd' = 1$ and $\lfloor k'/d' \rfloor = \lfloor k/d \rfloor$), the distance threshold of Theorem 1 is nearly the best possible. Within the graph $G_{k,d} \bowtie P_n$ constructed in Example 2, let $H_t$ be the copy of $G_{k,d}$ induced by $\{v_i^t\colon 0 \leq i \leq k-1\}$. Pick a $(k', d')$-coloring $c$ of $H_1$. (If $d = 1$, then we reindex the vertices to agree with the cyclic order of the colors on $H_1$.) By Theorem 2, there is essentially only one way to choose $c$, up to symmetry. It yields exactly one *special edge*, which consists of two vertices whose subscripts differ by $d$ and whose colors differ by $d' + 1$.

As observed in Example 2, there is no flexibility in extending a $(k, d)$-coloring from $H_1$ to $H_2$. There is a slight bit of flexibility in extending a $(k', d')$-coloring $c$. There are exactly two vertices in $H_2$ whose colors are not forced by $c$ on $H_1$: the vertices $v_i^2$ and $v_{i+d}^2$ such that $v_i^1 v_{i+d}^1$ is the special edge under $c$. Instead of agreeing with $\{c(v_i^1), c(v_{i+d}^1)\}$ (and repeating the coloring), either of these (but not both) can have its color moved by 1 toward the other. The special edge for the resulting $(k', d')$-coloring of $H_2$ is $v_{i-d}^2 v_i^2$ or $v_{i+d}^2 v_{i+2d}^2$, whichever is incident to the vertex whose color "moved".

From $H_2$ to $H_3$ we can again shift color by 1 on one vertex to move farther from a copy of $c$, but only at the vertex $v_j^3$ such that $v_j^2$ is in the special edge in the coloring of $H_2$, but $v_j^1$ is not in the special edge in the coloring of $H_1$. Repeating this argument shows that if $c(v_i^2) \neq c(v_i^1)$, then we cannot change the color again for copies of $v_i$ until we move from $H_{k+1}$ to $H_{k+2}$ (since $k$ and $d$ are relatively prime). To shift the color by the amount $\lfloor k'/2 \rfloor$ for some copy of $v_i$ (to an opposite or nearly opposite color), we need $n \geq 2 + k(\lfloor k'/2 \rfloor - 1)$.

This idea leads to a graph and a precolored independent set $P$ with the property that $d(P)$ is large, yet the precoloring does not extend. The construction is reminiscent of the *koosh ball* examples in [5].

**Example 4** Form $G$ by identifying the labeled subgraphs $H_1$ in each of $kk'$ copies of $G_{k,d} \bowtie P_n$. Call the copies of $G_{k,d} \bowtie P_n$ *branches.* For the copy of $H_n$ in each branch, choose an index $i \in \mathbb{Z}_k$ and a color $j \in \mathbb{Z}_{k'}$ (done in all $kk'$ possible ways in the $kk'$ branches), and preassign color $j$ to the vertex representing $v_i^n$ in this copy of $H_n$.

In this precoloring, $d(P) = 2(n-1)$. For any $(k', d')$-coloring of the subgraph $H_1$, there exists $i$ such that the color on copies of $v_i$ cannot change until copies of $H_{\lceil k/2 \rceil}$. In some branch, the color on copies of $v_i$ must change $\lfloor k'/2 \rfloor$ times to agree with the precolored copy of $v_i^n$ in that branch. This takes $k(\lfloor k'/2 \rfloor - 1)$ more steps. Thus when $n < k(\lfloor k'/2 \rfloor - 1) + \lceil k/2 \rceil$, no $(k', d')$-coloring of $G$ agrees with the precoloring on all of $P$. □

The construction of Example 4 and its analysis proves the following result.

**Theorem 3** In the tight case ($k'd - kd' = 1$ and $\lfloor k'/d' \rfloor = \lfloor k/d \rfloor$), there exists a graph $G$, an independent set $P$ in $G$, and a precoloring from $\mathbb{Z}_{k'}$ for $P$ such that
(i) $\chi_c(G) = k/d$;
(ii) $d(P) = 2k(\lfloor k'/2 \rfloor - 1) + \lceil k/2 \rceil - 2$; and
(iii) the precoloring does not extend to a $(k', d')$-coloring of $G$. □

When $k'd - kd' = 1$, Theorem 1 implies that $d(P) = 2\lceil kk'/2 \rceil$ suffices for $(k', d')$-coloring extension. This is close to the lower bound from Theorem 3. Indeed, if $k'd - kd' = 1$ with $k'$ even and $k$ odd, then $kk' - k \leq \rho \leq kk'$, where $\rho$ is the minimum $d(P)$ that suffices.

For $k'd - kd' > 1$, the construction of Example 4 provides a graph and a precoloring that extends only when $d(P)$ is large, but the distance needed is harder to compute due to the increased flexibility in $(k', d')$-colorings of $G_{k,d}$.

A different construction shows that the threshold from Theorem 1 is nearly sharp when $k/d = 2$ and $2 < k'/d' \leq 3$.

**Example 5** The path $P_n$ with vertices $v^1, \ldots, v^n$ in order is $(k, d)$-colorable whenever $k/d \geq 2$. Let $c$ be a $(k', d')$-coloring of $P_n$, where $k'd - kd' = r$. If $(k, d) = (2, 1)$, then $r = k'd - kd' =$

$k' - 2d'$. Given $c(v^1)$, there are $r+1$ possible values for $c(v^2)$. In general, the range of possible values widens by $r$ with each step. Therefore, when $n - 1 < \frac{k'-1}{r}$, some combinations of colors on $v^1$ and $v^n$ are unachieveable.

Let $G$ be the tree consisting of $k'$ copies of $P_n$ with a common endpoint. If $n \le \left\lceil \frac{k'-1}{r} \right\rceil$, then the precoloring that assigns all the colors in $\mathbb{Z}_{k'}$ to the leaves does not extend to a $(k', d')$-coloring of $G$. Letting $P$ be the set of leaves, we have $d(P) = 2(\left\lceil \frac{k'-1}{r} \right\rceil - 1) = 2 \left\lfloor \frac{k'-2}{r} \right\rfloor$.

The upper bound from Theorem 1 is $2 \left\lceil \frac{kk'}{2r} \right\rceil$, which equals $2 \left\lceil \frac{k'}{k'-2d'} \right\rceil$ when $k/d = 2$. The lower bound provide by this example is $2 \left\lfloor \frac{k'-2}{k'-2d'} \right\rfloor + 1$. The difference between the two bounds is 1 or 3. □

A curious aspect of Example 5 is that when $k/d$ is fixed, the threshold $d(P)$ for a $(k', d')$-extension theorem can be non-monotone. When $k/d = 2$, the quantity $\frac{k'}{k'-2d'}$ decreases as $k'/d'$ increases, but the integer part matters. In particular, [1] yields a $(3, 1)$-extension theorem for bipartite graphs when $d(P) \ge 4$ by [1], but for a $(10, 3)$-extension theorem the threshold is 5 or 6.

Example 5 is weak when $k/d > 2$, because then $k' - 2d' > r$, and the range of accessible colors widens by more than $r$ values with each step.

# 4 Larger Precolored Components

As mentioned in the introduction, there are precoloring extension theorems for precolored cliques when $d(P)$ is large enough [1, 5]. Suppose that $\chi_c(G) = k/d$ and that $P$ is a precolored set such that the components of $G[P]$ are all isomorphic to $G_{k,d}$. For $k'/d' > k/d$, it is natural to ask whether every $(k', d')$-precoloring of $P$ extends to a $(k', d')$-coloring of $G$ when $d(P)$ is sufficiently large. The next example shows that this may fail.

**Example 6** Consider $G_{5,2} \bowtie P_n$. Let $H_1$ be the copy of $G_{5,2}$ induced by $\{v_i^1 \colon 0 \le i \le 4\}$. Consider a 3-coloring of $H_1$; thus $(k, d) = (5, 2)$ and $(k', d') = (3, 1)$.

Note that $H_1 \cong C_5$. Every proper coloring of $C_5$ with colors $\{1, 2, 3\}$ is specified by choosing the vertex whose color appears nowhere else and then placing a permutation of 123 on the 3-vertex path centered on that vertex. The rest of the coloring is then forced: ?123? forces 31231.

The central permutation is an even or an odd permutation of $\{1, 2, 3\}$. In extending the proper 3-coloring to $H_2$, there is flexibility to change only the color of a vertex whose neighbors in $H_1$ have the same color. These are the two vertices at distance 2 from the central vertex in the description above. The coloring on $H_2$ can fail to copy the coloring on $H_1$ only if the copy of one of these two vertices takes on the former singleton color.

Hence if the coloring on $H_1$ is 31231 around the cycle and the coloring on $H_2$ is different, then the coloring on $H_2$ is 31232 or 21231. The new central permutation on the cycle is $312xx$ or $xx231$. Thus the sign of the central permutation cannot change! Every extension of a proper 3-coloring of $H_1$ to all of $G_{5,2} \bowtie P_n$ can produce at most half of the proper 3-colorings on $H_n$ no matter how large $n$ is (exactly half of the colorings can appear). □

Perhaps the difficulty in Example 6 arises only because $k'/d'$ and $k/d$ are too close (less than 1 apart). We offer the following conjecture.

**Conjecture 1** There exists a constant $d^*$ such that if $\chi_c(G) = k/d$, and $P$ is a precolored set with $d(P) \geq d^*$ such that the components of $G[P]$ are all isomorphic to $G_{k,d}$, and $k'/d' - k/d \geq 1$, then every $(k', d')$-precoloring of $G[P]$ extends to a $(k', d')$-coloring of $G$.

One could even hope for a generalization of the result of Albertson and Hutchinson [2] mentioned in the introduction, in which $k$ and $s$ are generalized to $k/d$ and $k_s/d_s$, and $k_s/d_s \geq 1$ is required.

# References


[1] Michael O. Albertson, You can't paint yourself into a corner, *J. Combinatorial Theory Ser B*, 78 (1998), 189-194.

[2] Michael O. Albertson and Joan P. Hutchinson, Graph color extensions: when Hadwiger's conjecture and embedding helps, *Electronic J. Comb* 9 R37 (2002) (10 pages).

[3] Michael O. Albertson and Joan P. Hutchinson, Extending precolorings of subgraphs of locally planar graphs, *European J. Combinatorics* 25 (2004), 863-871.

[4] Michael O. Albertson, Alexandr V. Kostochka, and Douglas B. West, Precoloring extendsions of Brooks' Theorem *SIAM J. Discrete Math.* 18(3) (2005), 542-553.

[5] Michael O. Albertson and Emily H. Moore, Extending graph colorings, *J. Combinatorial Theory Ser B* 77 (1999), 83-95.

[6] Michael O. Albertson and Emily H. Moore, Extending graph colorings using no extra colors, *Discrete Math.* 234 (2001), 125-132.

[7] Maria Axenovich, A note on graph coloring extensions and list-colorings, *Electronic J. Comb.* 10 (2003), #N1 (5 pages).

[8] J.Adrian Bondy and Pavol Hell, A note on the star chromatic number, *J. Graph Theory* 14 (1990), 479-482.

[9] Luis A. Goddyn, Michael Tarsi, and C.-Q. Zhang, On $(k, d)$-colorings and fractional nowhere-zero flows, *J. Graph Theory* 28 (1998), 155-161.

[10] Carsten Thomassen, Color-critical graphs on a fixed surface, preprint.

[11] Zsolt Tuza, Graph colorings with local constraints - a survey, *Discuss. Math. Graph Theory* 17 (1997), 161-228.

[12] Andrew Vince, Star chromatic number, *J. Graph Theory* 12 (1988), 551-559.

[13] Xuding Zhu, Circular chromatic number: a survey, in Combinatorics, graph theory, algorithms and applications, *Discrete Math.* 229 (2001), 371-410.

[14] Xuding Zhu, Star chromatic number and products of graphs, *J. Graph Theory* 16 (1992), 557-569.