Circular-Arc Digraphs: A Characterization

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ABSTRACT

Given a family **F** of ordered pairs of sets $\{(S_u, T_u): u \in V\}$, the *intersection digraph* of **F** is the digraph with vertices V defined by $uv \in E(D)$ if $S_u \cap T_v \neq \emptyset$. *Interval digraphs* are intersection digraphs of families in which every set is an interval, and were characterized in a previous paper by conditions on the adjacency matrix. In this paper, another such condition leads to an adjacency matrix characterization of *circular-arc digraphs*, the intersection digraphs of families in which every set is an arc of a circle.

1. INTRODUCTION

The subject of intersection graphs and interval graphs has been an important area of study for the last two decades. Given a family $\mathbf{F} = \{S_v : v \in V\}$ of sets, the graph with vertices V formed by $uv \in E(G)$ if and only if $S_u \cap S_v \neq \emptyset$ is called the *intersection graph* of F. A graph G is an *interval graph* or *circular-arc graph* if it is the intersection graph of a family of intervals on the real line or a family of arcs on a circle, respectively. Interval graphs have become sufficiently important that a special issue of *Discrete Mathematics* [1] has been devoted exclusively to this topic. Extensive work on circular-arc graphs was

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sion, but there are digraphs of Ferrers dimension 2 and complementary digraphs of Ferrers dimension 3 that are not circular-arc digraphs.

2. ELEMENTARY CHARACTERIZATIONS

In [5], we gave an elementary sufficient condition for interval digraphs in terms of the adjacency matrix. In fact, D has an intersection representation by intervals in which all terminal sets are single points if and only if its adjacency matrix has the consecutive ones property for rows. Analogously, we have a sufficient condition for circular-arc digraphs that characterizes those with similarly restricted representations:

Theorem 1. A digraph D has an intersection representation by circular arcs in which all terminal arcs are single points if and only if its adjacency matrix has the circular ones property for rows.

Proof. D has such a representation if and only if there are arcs S_v and points t_v in $[0,1]$ such that $uv \in E$ if and only if $t_v \in S_v$, where we may assume the t_v's are distinct. These exist if and only if numbering the vertices in circular order of $\{t_v\}$ exhibits the circular ones property for rows of the adjacency matrix. ∎

This condition is not necessary in general, as shown by the following example:

Example 1. Consider the digraph consisting of a two-directional path along vertices 1, 2, 3, 4, and 5 plus a loop at 3; its adjacency matrix appears below on the left. From the representation on the right, we see that this is not only a circular-arc digraph, but in fact an interval digraph. Nevertheless, the matrix does not have the circular ones property for rows or columns.

01000						
10100	i:	1	2	3	4	5
01110	S_i:	$\{-2\}$	$\{1\}$	$[-2,0]$	$\{2\}$	$\{-1\}$
00101	T_i:	$\{1\}$	$\{-2\}$	$[0,2]$	$\{-1\}$	$\{2\}$
00010						

To obtain an elementary characterization of interval digraphs, we defined a *generalized complete bipartite subdigraph* (GBS) of a digraph to be a subdigraph with vertices $X \cup Y$, where X and Y are not necessarily disjoint, and all edges xy such that $x \in X$, $y \in Y$. If $\mathbf{B} = \{(X_k, Y_k)\}$ is a collection of GBSs whose union is D, then the *vertex-source incidence matrix* ("V,X-matrix") for $\mathbf{B}$ is the incidence matrix between the vertices (indexing the rows) and the sets of sources $\{X_k\}$ of the GBSs in $\mathbf{B}$ (indexing the columns). The *vertix-sink incidence matrix* ("V,Y-matrix") for $\mathbf{B}$ is defined analogously. Our first characterization of circular-arc digraphs uses these incidence matrices. Its proof is analogous to the proof of the corresponding characterization of interval digraphs in [5], but we include it here for completeness.

Theorem 2. *D is a circularlike-arc digraph if and only if there is a numbering of the GBSs in some covering* **B** *of D that exhibits the circular ones property for rows for both the V,X-matrix and V,Y-matrix of* **B**.

Proof. For sufficiency, consider such a **B** whose union is D, and let (X_k, Y_k) be a common numbering of the columns of the V,X- and V,Y-matrices that exhibits the circular ones property for both. Assign $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$, where a_v, b_v, c_v, and d_v are defined by $v \in X_k$ if and only if $a_v \leq k \leq b_v$ (circularly) and $v \in Y_k$ if and only if $c_v \leq k \leq d_v$ (circularly). Then $S_u \cap T_v \neq \emptyset$ if and only if $u \in X_k$ and $v \in Y_k$ for some k.

For necessity, consider a representation of D by a family $\{(S_v, T_v)\}$ of ordered pairs of arcs. We may assume they are closed and have endpoints indexed circularly by integers, with $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$. For each integer k belonging to any of these intervals, define a GBS $B_k = (X_k, Y_k)$ of the intersection digraph of the interval pairs, by setting $X_k = \{v\colon k \in S_v\}$ and $Y_k = \{v\colon k \in T_v\}$. Then $S_u \cap T_v \neq \emptyset$ if and only if $u \in X_k$ and $v \in Y_k$ for some k, so the intersection digraph of the interval pairs is in fact the union of the specified GBSs. Furthermore, by construction the resulting V,X- and V,Y-matrices have the simultaneous circular ones property for rows. ∎

3. ADJACENCY MATRIX CHARACTERIZATIONS

In order to phrase our adjacency matrix characterization for interval digraphs in a fashion useful for circular-arc digraphs, we prove the following lemma:

Lemma 1. *The partitionable zeros property is equivalent to the generalized linear ones property.*

Proof. The generalized linear ones property implies the partitionable zeros property by changing each 0 in the upper part of the stair partition to R and each 0 in the lower part to C. For the converse, consider an assignment of R's and C's to 0's that exhibits the partitionable zeros property. Note that no C appears above an R and no R appears to the left of a C. Therefore, if we let U' consist of all R's and positions above them, and L' consist of all C's and positions to their left, then U' and L' are disjoint and can be extended arbitrarily to a stair-partition U,L with $U' \subseteq U$ and $L' \subseteq L$ that demonstrates the generalized linear ones property. ∎

Theorem 3. *A digraph D is an interval digraph if and only if the rows and columns of its adjacency matrix can be permuted independently so that the resulting matrix has the generalized linear ones property.*

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tionable zeros property can be replaced to obtain a short proof of the following characterization.

Theorem 5. The following are equivalent:

A: D is an interval digraph.
B: The adjacency matrix $A(D)$ has independent row and column permutations that exhibit the partitionable zeros property.
C: The adjacency matrix $A(D)$ has independent row and column permutations that exhibit the generalized linear ones property.

Proof. A implies B using row and column orderings constructed as in the preceding proof. Indeed, the method of proof of Theorem 4 shows that A and C are equivalent, and Lemma 1 is the equivalence of B and C.

4. CIRCULAR-ARC DIGRAPHS AND FERRERS DIMENSION

In [5], we considered the relationship between interval digraphs and Ferrers digraphs. In the terminology we have been using in this paper, a Ferrers digraph is a digraph whose adjacency matrix has independent row and column permutations such that the resulting matrix has a stair partition (L,U) in which L consists entirely of 1's and U consists entirely of 0's. Equivalently, the successor sets (sets of the form $\{v\colon uv \in E\}$ for some u) are linearly ordered by inclusion, as are the predecessor sets (sets of the form $\{v\colon vu \in E\}$ for some u). Equivalently, the adjacency matrix has no 2 by 2 submatrix that is a permutation matrix.

The *Ferrers dimension* of a digraph D is the minimum number of Ferrers digraphs whose intersection is D. By the last characterization above, Ferrers dimension of D equaling k is equivalent to being able to k-color the 0's of $A(D)$ so that the 0's in any 2 by 2 permutation submatrix get different colors. The edges missing from the ith Ferrers digraph then correspond to the positions with the ith color. In general, let $H(D)$ be the graph whose vertices correspond to the 0 positions in $A(D)$, with vertices adjacent when they are the 0's of a 2 by 2 permutation submatrix. Then $H(D)$ is invariant under row or column permutations of $A(D)$, and the Ferrers dimension of D is $\chi(H(D))$.

The characterization of interval digraphs using the generalized linear ones property or partionable zeros property implies that all interval digraphs have Ferrers dimension at most 2. However, there is no such bound for circular-arc digraphs, as we show next.

Theorem 6. There are n-vertex circular-arc digraphs with Ferrers dimension n; i.e., Ferrers dimension of circular-arc digraphs is unbounded.

Proof. Let D be an n-vertex digraph whose adjacency matrix A has exactly one 0 in each row and column. The Ferrers dimension of D is n, because $H(D)$

is the complete graph on n vertices (no pair of these positions can be omitted by a single Ferrers digraph used in a collection intersecting to form D). Nevertheless, A has the circular ones property for rows, so Theorem 1 implies that D is a circular-arc digraph having a representation in which each terminal set is represented by a single point. Indeed, with n distinct points for the terminal sets, the source set for v can be the entire circle except the terminal point for its nonsuccessor. ∎

Given this result, it is natural to ask for the smallest Ferrers dimension of a digraph that is not a circular-arc digraph. The answer is 2, which follows readily from the existence of digraphs of Ferrers dimension 2 that are not interval digraphs and the next lemma.

Lemma 2. Given a digraph D, let D' be the digraph obtained from D by adding a vertex w and a loop at w (no other edge incident to w). Then D' is a circular-arc digraph if and only if D is an interval graph.

Proof. In a circular-arc representation of D', S_w and T_w must intersect. Since neither is required to intersect another arc, we may trim them to assume $S_w = T_w$. Now no other arc can intersect either of these without generating an unwanted edge. This means D must be an interval digraph. Conversely, if D is an interval digraph, we can augment an interval representation by adding intersecting S_w and T_w in an unused portion of the line, so that D' is also an interval digraph and hence a circular-arc digraph. ∎

Example. 2. A digraph of Ferrers dimension 2 that is not a circular-arc digraph. In [5], we proved that Ferrers dimension 2 is equivalent to the existence of independent row and column permutations of the adjacency matrix so that the resulting matrix has no 0 with a 1 both below it and to its right. Let D be the digraph with adjacency matrix on the left below. It has Ferrers dimension 2, but in [5] we showed that it is not an interval digraph. If we add a row at the top and a column at the left of this matrix, all 0 except for a 1 in the upper left corner, then we obtain the adjacency matrix of the digraph D' obtained by adding a single vertex with a loop. The matrix illustrates that D' has Ferrers dimension 2, but by Lemma 2 it is not a circular-arc digraph.

1110000	1111110
1111100	1110110
1111110	1000100
0111111	1110000
0111101	1101111
0011000	1100111
0001101	0000111

Note that although D itself is not an interval digraph, it is a circular-arc digraph. On the right above, we have written the rows of the adjacency matrix of D in the order 3261457 and the columns in the order 3216457. There are several stair partitions that exhibit the generalized circular ones property for this matrix; the simplest to describe puts the top four rows in U and the bottom three in L. Then the sets W_5 and W_6 bend around to absorb the 1's on the top that do not belong to the left-justified sets of 1's. By Theorem 4, D is a circular-arc digraph. There does not seem to be a simultaneous row and column permutation allowing a suitable stair partition. ∎

Although the digraph D' above is not a circular-arc digraph, its complement is a circular-arc digraph. More generally, we have

Theorem 7. The complement of any digraph with Ferrers dimension at most 2 is a circular-arc digraph.

Proof. Begin with a Ferrers digraph D of dimension at most 2. Permute the rows and columns of its adjacency matrix so that no 0 has a 1 both below and to its right. Take the complement and reverse the order of the rows and the order of the columns. In the new matrix, no 1 has a 0 both above and to its left. Let (L,U) be the stair partition of this matrix in which U is the entire matrix. If a 1 in position i,j has no 0 to its left [above it], then this position belongs to $V_i[W_j]$, where V_i and W_j are defined for (L,U) as before. Hence (L,U) exhibits the generalized circular ones property for this matrix $A(\overline{D})$, and $\overline{D}$ is a circular-arc digraph. ∎

Although the complement of any Ferrers digraph is a Ferrers digraph, the digraphs of Ferrers dimension 2 are not closed under complementation. For example, the digraph consisting of a loop at each vertex or any union of disjoint cycles has a permutation matrix as its adjacency matrix and is the intersection of two Ferrers digraphs, but its complement is not. Indeed, its complement digraph D has $H(D) = K_n$, so D has Ferrers dimension n and is not an interval digraph.

That a trivial stair partition suffices to prove Theorem 7 suggests that some stronger result may hold. Nevertheless, beyond Ferrers dimension 2, nothing can be guaranteed, as we show next.

Example 3. The two complementary digraphs on four vertices whose adjacency matrices appear below both have Ferrers dimension 3, but neither is a circular-arc digraph.

D_1	D_2
0110	1001
1010	0101
1100	0011
0001	1110

For each of these digraphs, $H(D_i)$ contains a triangle but is easy to 3-color; hence their Ferrers dimension is 3. As noted above, the complement of a collection of loops obtained by striking the last vertex from D_1 is not an interval digraph, and Lemma 2 then implies that D_1 itself is not a circular-arc digraph.

For D_2, let $\{x, y, z, w\}$ denote the vertices in row and column order, and suppose D_2 has a circular-arc representation. The intervals S_w and T_w must be disjoint. However, for $i \in \{x, y, z\}$, S_i and T_i intersect; select $a_i \in S_i \cap T_i$ from their intersection. We also know that S_i meets T_w and T_i meets S_w. Thus $S_i \cup T_i$ is an interval meeting both S_w and T_w, and it must cover a gap between them. There are only two such gaps. Assume by symmetry that $S_x \cup T_x$ and $S_y \cup T_y$ cover one gap between S_w and T_w. In moving from S_w to T_w along this gap, we may assume by symmetry that we reach a_x at least as early as a_y. Since T_y must meet S_w and S_x must meet T_w, this means that S_x and T_y both contain the arc from a_x to a_y, introducing the forbidden edge xy. ∎

Although Example 2 and Theorem 7 show that the complement of a circular-arc digraph need not be a circular-arc digraph, we can make the following weaker remark: If H_1 and H_2 are the subdigraphs of D generated by a stair-partition of $A(D)$ yielding a circular-arc representation of D, i.e., $D = H_1 \cup H_2$ as in the proof of Theorem 4, then the union of the complements of H_1 and H_2 is a circular-arc digraph. The appropriate stair-partition can be obtained by reversing the order of the rows and columns in $A(D)$ and rotating the stair-partition by π.

We close the paper with two natural complexity questions suggested by working with examples: Given a 0,1-matrix, what is the complexity of testing whether is has the generalized circular ones property (or generalized linear ones property)? Given a digraph, what is the complexity of testing whether it is a circular-arc digraph (or interval digraph)?

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