A Hike Around Circular Coloring

Douglas B. West
Univ. of Illinois

Happy Birthday, Joan!
Circular Coloring

Petersen graph

How to capture "just barely non-bipartite"?

Ex: Traffic flow

Conflict graph

"green light" must be independent set

3 colors needed

More efficient schedule:

circular coloring: Zhu [1992]

each vertex gets open unit arc

adjacent vertices get disjoint arcs

\[ \chi_c(G) = \inf \{ \text{circumference} \} \]

Ex: \[ \chi_c(C_{2k+1}) = 2 + \frac{1}{k} \]
Elementary Properties

Prop: \( \chi_c(G) \geq \frac{n(G)}{\alpha(G)} \)

PF: \( \alpha(G) \cdot r \geq n(G) \) (total length)

Cor: \( \chi_c(C_{2k+1}) = 2 + \frac{1}{k} \quad \chi(\text{Pete}) \geq \frac{5}{2} \)

Prop: \( \chi_c(G) \) is rational

PF: For each cyclic ordering of starting points, ordering and non-adjacency constraints are linear, minimizing circumference is linear program.

\[ \therefore \chi_c(G) \text{ is minimum of } (n-1)! \text{ rational numbers.} \]

Remarks:  
- If \( H \subseteq G \), then \( \chi_c(H) \leq \chi_c(G) \).
- If \( r = \chi_c(G) \), then \( G \) has an \( r \)-circular coloring.
Background

Def [Vince 1988] A \((k,d)\)-coloring of \(G\) maps \(V(G)\) to \(\mathbb{Z}_k\) s.t. \(u,v \in E(G) \Rightarrow \phi(u), \phi(v)\) circularly at least \(d\) apart.

Ex: \((5,2)\)-coloring

\[(k,1)\)-coloring \(\iff\) ordinary proper \(k\)-coloring

Def. star chromatic number
\[
\chi^*(G) = \inf \left\{ \frac{k}{d} : G \text{ has a } (k,d)\text{-coloring} \right\}
\]

survey Zhu, Discrete Math 2001, 39 pages

Fundamental Fact: \(\chi(G)-1 < \chi_c(G) \leq \chi(G)\)

"refinement" of \(\chi(G)\)
Elementary Properties

Prop: $\chi^*(G) = \chi_c(G)$

Pf: $\chi^*(G) \leq \chi_c(G)$:

Given rational $r$-circular coloring $f$, let $d = \text{lcm}$ of denominators, $k = rd \in \mathbb{Z}$.

Let $\phi(v) = d \cdot \text{[start of } f(v)] \Rightarrow \phi$ is $(k,d)$-coloring with $k/d = r$.

$\chi_c(G) \leq \chi^*(G)$:

Given $(k,d)$-coloring $\phi$,

let $f(v) = \left( \frac{\phi(v)}{d}, \frac{\phi(v) + d \mod k}{d} \right) \Rightarrow f$ is $k/d$-circular coloring.

---

Prop: $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$

Pf: Given $(k,d)$-coloring $f$,

let $\hat{f}(v) = \left\lfloor \frac{f(v)}{d} \right\rfloor$

If $|f(v) - f(w)| \geq d$, then $|\hat{f}(v) - \hat{f}(w)| \geq 1$

Also $\hat{f}(v) \in \{0, \ldots, \left\lfloor \frac{k-1}{d} \right\rfloor\}$

$\therefore \chi(G) \leq 1 + \frac{k}{d}$

Cor: $\lceil \chi_c(G) \rceil = \chi(G)$
What Values are Attained?

Def: homomorphism from $G$ to $H$ is a fcn $f: V(G) \rightarrow V(H)$ such that $uv \in E(G) \Rightarrow f(u)f(v) \in E(H)$

Remark: A $(k,d)$-coloring of $G$ is a homomorphism from $G$ to $G_{k,d}$

$V(G_{k,d}) = \mathbb{Z}_k$

$ij \in E(G_{k,d}) \iff |i-j| \geq d$

Remark: $\chi_c(G_{k,d}) = \frac{k}{d}$ (since $\alpha(G_{k,d}) = d$) "canonical"

Thm Among planar graphs, $\chi_c$ attains every rational in $[2,3]$ (Moser 1997) and in $[3,4]$ (Zhu 1999)

Thm Among graphs with no $K_n$ minor, for $n \geq 5$, $\chi_c$ attains every rational in $[2, n-1]$. (Liaw-Pan-Zhu 2003)
Finding $\chi_c(G)$

Upper Bound: Exhibit $(k,d)$-coloring

Lower Bound: Forbid "next lower" rational

Thm (Zhou 1997) For $G$ not a tree, $\chi_c(G) = \frac{p}{q}$ with $p \leq$ circumference and $q \leq \alpha(G)$.

Lemma: Given $r$-circular coloring $f$ of $G$, let $D_f$ be digraph on $V(G)$ with $x \rightarrow y$ iff $xy \in E(G)$ and $f(x)$ ends where $f(y)$ starts. If $D_f$ is acyclic, then $\chi_c(G) < r$.

PF: Let $v$ be sink in $D_f$.

Modify $f$ to $f'$ so $v$ isolated in $D_f'$.
Continue to $f^*$ so $D_{f^*}$ has no edges.
Expand arcs. Shrink circle.

for Thm:
Optimality $\Rightarrow$ cycle $C$ in $D_f$ (length $p$).
$r = \frac{p}{q}$, where $C$ runs $q$ times around circle.
The $q$ vertices "at" a point must be indep.
Ex: $\chi(\text{Pete}) = 3$

No fraction in $(\frac{8}{3}, 3)$ has numerator $< 9$.

Consider $(8,3)$-coloring; non-adjacent $x, y$ get same color.

Path $x, u, v, y$ fails

Remark: $\chi_c(G) = \frac{k}{d}$ in lowest terms

$\Rightarrow G$ has $k$-cycle with distinct colors in $(k,d)$-coloring.

Fundamental Questions

1) When is $\chi_c(G) = \chi(G)$?

2) When is $\chi_c(G)$ close to $\chi(G) - 1$?

3) How do results on $\chi$ generalize to $\chi_c$?
\( \chi_c(G) = \chi(G) \)

Testing \( \chi_c(G) = \chi(G) \) is NP-complete (Guichard 1993).

For fixed \( k \geq 2 \) and \( n \geq 3 \), among \( \chi(G) = n \) testing \( \chi_c(G) = n \) and \( \chi_c \leq n - \frac{1}{k} \) are both NP-e (Hatami 2009, Tusserkani).

Thm (Abbott-Zhou 1993) \( \overline{G} \) disconnected \( \Rightarrow \chi_c(G) = \chi(G) \).

Thm (Steffen-Zhu 1996) If \( V(G) \) has a partition \( S, \overline{S} \) that cuts no color class in any optimal coloring, then \( \chi_c(G) = \chi(G) \).

Thm (Fan 2004) If \( \chi_c(G) < \chi(G) \), then \( \overline{G} \) is Hamiltonian.

Pf. For \( \chi_c(G) = \frac{k}{d} \), let \( X_0, \ldots, X_{k-1} \) be the color classes in a \((k,d)\)-coloring. All are nonempty.

\( \chi_c(G) < \chi(G) \Rightarrow d \geq 2 \Rightarrow X_i \cup X_{i+1} \) is independent.

\( X_i \cup X_{i+1} \) is a clique in \( \overline{G} \).

\( \therefore \) suffices to list the vertices of each class in order.

Note: In fact, \( \overline{G} \) contains \( d \)th power of a spanning cycle.
Special Classes

Def: Kneser graph $K(n,k)$:
Vertex set: $k$-subsets of $\{1, \ldots, n\}$; adjacent if disjoint.

Lovász 1978: $\chi(K(n,k)) = n-2k+2$

Conj (Johnson-Holroyd-Stahl 1997) $\chi_c(K(n,k)) = \chi(K(n,k))$
True for $n \geq 2k^2(k-1)$ (Hajiabolhassan-Zhu 2003)

Def: Mycielski's construction $M(G)$:
$\omega(M(G)) = \omega(G)$
$\chi(M(G)) = \chi(G) + 1$

Chang-Huang-Zhu 1999 $\chi(G) = 3 \Rightarrow \chi_c(M(G)) = \chi(M(G)) = 4$
Huang-Chang 1999 close to $x-1$ $\Rightarrow \chi_c(M(G_{k,d})) = \chi(M(G_{k,d})) = \lceil \frac{k}{d} \rceil + 1$
Hajiabolhassan-Zhu 2003 $\chi_c(M^t(K_n)) = \chi(M^t(K_n))$ when $n$ is sufficiently large
$\text{Conj } n \geq t+2$
$\text{False } n = t+1$

Fan 2004 $\chi(G) \geq \frac{n(G) + 3}{2} \Rightarrow \chi_c(M(G)) = \chi(M(G))$
Orientations - Another model

Minty 1962: $\chi(G) = 1 + \min_D \max_C \left[ \frac{|C+1|}{|C-1|} \right] = \min_D \max_C \left[ \frac{|C|}{|C-1|} \right]

Goddyn-Tarsi-Zhang 1998 $\chi_c(G) = \min_D \max_C \frac{|C|}{|C-1|}

Tuza 1992: $G$ is $k$-colorable if $\exists D$ s.t. $\left[ \frac{|C|}{|C-1|} \right] \leq k$
whenever $|C| = 1 \mod k$

Zhu 2002: $G$ is $(k,d)$-colorable if $\exists D$ s.t. $\frac{|C|}{|C-1|} \leq \frac{k}{d}$
whenever $d|C| \in \{1, 2, \ldots, d-1\} \mod k$.

(a spanning tree minimizing an appropriate weight function yields a $(k,d)$-coloring)

---

Duality circular flow number

$\phi_c(G) = 1 + \min_D \max_S \frac{|[S, S]|}{|S, S|}$

see DeVos-Goddyn-Mohar-Vertigan-Zhu 2004
Other Relations to $\chi$

**Cartesian Product:**

\[ \chi(G \square H) = \max \{ \chi(G), \chi(H) \} \]

\[ \chi_c(G \square H) = \max \{ \chi_c(G), \chi_c(H) \} \]

- Subidussi 1957
- Vizing 1963
- Ather 1964
- Zhu 1992

**Vertex deletion:** $\chi(G-v) \geq \chi(G) - 1$, but →

Does $\exists v$ s.t. $\chi_c(G-v) \geq \chi(G) - 1$ ?

No! An infinite family has

$\chi_c(G) = 4$ & $\chi_c(G-v) = \frac{8}{3}$ \(\forall v\) Zhu 2004

**Hajós Theorem:** Generating the non-$k$-colorable graphs from operations on cliques. (Zhu 2003, Mohar 2005)

**Pre-Coloring Extension:**

$d(W) = \min$ pairwise distance

Albertson 1998: $\chi(G) \leq k$, $d(W) \geq 4$ → every $(k+1)$-coloring of $W$ extends to proper $(k+1)$-coloring of $G$

Albertson-West 2006: $\chi_c(G) \leq \frac{k}{d}$, $\frac{k'}{d'} > \frac{k}{d}$, $d(W) = 2 \left[ \frac{kk'}{2(kd-kd')} \right]$

→ every $Z_k$-coloring of $W$ extends to $(k',d')$-coloring of $G$. Nearly sharp when $kd-kd' = 1$
Forbidden Minors, Large Girth

Thm (Galluccio-Goddyn-Hell 2001) For any graph $H$, $\exists f(k)$ s.t. $\chi_c(G) \leq 2 + \frac{1}{k}$ when $G$ is $H$-minor-free and has girth $\geq f(k)$.

Ex: $H = K_4$ ("series-parallel graphs")

Hell-Zhu 2000: $\chi_c(G) = 3$ or $\chi_c(G) \leq \frac{8}{3}$

Pan-Zhu 2003: All values in $[2, \frac{8}{3}]$ are attainable

Pan-Zhu 2002: odd girth $\geq 6k-1 \implies \chi_c(G) \leq \frac{8k}{4k-1}$

$\geq 6k+1 \implies \chi_c(G) \leq \frac{4k+1}{2k}$

$\geq 6k+3 \implies \chi_c(G) \leq \frac{4k+3}{2k+1}$

Thm (Grötzsch 1959) $G$ planar & triangle-free $\implies \chi(G) \leq 3$.

 Conj (Jaeger 1984) $G$ planar & girth $\geq 4r \implies \chi_c(G) \leq 2 + \frac{1}{r}$

Sharpness (DeVos) 4r-1 spokes of length 2r-1

Need $(2r+1,r)$-coloring $C_{2r+1}$-coloring

Outer vertices can't use center color

Odd cycle doesn't map into bipartite graph.
Results on Jaeger's Conjecture

Nešetřil-Zhu 1996
Galuccio-Goddyn-Hell 2001
Klostermeyer-Zhang 2000
Zhu 2001
Borodin-Kim-Kostochka-West 2004

\[ \text{girth} \geq 13 \implies \chi_c(G) \leq \frac{5}{2} \]

\[ \text{girth} \ \text{10r-4 suffices} \]
\[ \text{odd girth} \ \text{10r-4} \]
\[ \text{odd girth} \ \text{8r-3} \]
\[ \frac{20r-2}{3} \]

also on higher surfaces
Folding Lemma
discharging
same for proj. plane
extends to higher surf.
models beyond homom.
other applications
Brief Outline of Proof

Fix $t = 2r$; seek homomorphism to $C_{t+1}$

Thread = path w. internal verts of degree 2

Given $\phi(u)$

\[\begin{array}{c|cccc}
\text{colors} & 1 & 2 & 3 & t \\
\hline
\text{allowed} & \_ & \_ & \_ & t \\
\text{forbidden} & t-1 & t-2 & t-3 & t-4 \\
\end{array}\]

---

Lem 1 Minimal counterexample has no thread of length $\geq t$

Lem 2 Minimal counterex has no vertex $u$ w. threads s.t. $\Sigma (t-l_i) \leq t$

Pf: Homomorphism on $G$-threads has common extension to $u$

Lem 3 a second neighborhood constraint

---

These "forbidden (reducible) configurations" restrict usage of 2-valent vertices

Discharging $\Rightarrow$ avg. degree $\geq 2 + \frac{6}{5t-4}$ when these configurations avoided

Euler $\Rightarrow$ girth $\leq \frac{10t-2}{3}$
Circular Edge-Coloring

same motivation: schedule time-cycle of pairwise interactions of unit duration, no conflicts

\( (k,d) \)-edge-coloring \( f : E(G) \to \mathbb{Z}_k \) s.t. \( |f(e) - f(e')| \geq d \) when \( e, e' \) are incident

\( \chi_e'(G) = \inf \left\{ \frac{k}{d} : f \right\} \)  
Observe: \( \chi_e'(G) = \chi_e(L(G)) \)

Hackmann-Kemnitz 2004

\( \chi_e'(G) = \chi'(G) = \Delta(G) \) for Class 1 graphs

\( \Delta(G) < \chi_e'(G) \leq \Delta(G) + 1 \) for Class 2 graphs

\( \chi_e'(K_n) = \chi'(K_n) \)  
\( \chi'(P_{13}) = \frac{11}{3} \)

Nadolski  
Bounds for special families \( w. \Delta(G) \leq 3 \).

Afshani-Hatami-Tusserkhani  
Ghandehari-Ghandehari-Hatami-Zhu

If \( \Delta(G) = 3 \), then \( \chi_e'(G) \leq \frac{11}{3} \) unless \( \nabla \) or \( \nabla \) is subgr.

Kaiser-Král-Škrekovský  
\( \forall \varepsilon > 0, \exists g \) s.t. \( \chi_e'(G) \leq 3 + \varepsilon \) when \( G \) is 3-regular, 2-edge-connected, girth \( \geq g \)

Does it generalize to \( k \)-regular?
Cartesian Product

$G \square H$ is Class 1 if G or H is Class 1 or both have 1-factor.

Consider $G = C_{2m+1}$, $H$ = regular graph of odd order $(4d+2)$.

Thm (West-Zhu) Lower bound for $\chi'_c (C_{2m+1} \square H)$.

Cor: $\chi'_c (C_{2m+1} \square C_{2k+1}) = 4 + \frac{1}{\lceil (6k+3)/4 \rceil}$ if $m \geq 3k+1$.

For a graph $F$, let $\vartheta (F) = \chi'_c (F) - \Delta (F)$.

 Conj (W-Z) $\vartheta (G \square H) \leq \max \{ d(G), d(H) \}$.

True when G or H is Class 1

Lemma: $\chi'_c (C_{2m+3} \square H) \leq \chi'_c (C_{2m+1} \square H)$