

# Chvátal's Condition Cannot Hold for Both a Graph and Its Complement

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## Abstract

Chvátal's Condition is a sufficient condition for a spanning cycle in an  $n$ -vertex graph. The condition is that when the vertex degrees are  $d_1, \dots, d_n$  in nondecreasing order,  $i < n/2$  implies that  $d_i > i$  or  $d_{n-i} \geq n - i$ . We prove that this condition cannot hold in both a graph and its complement, and we raise the problem of finding its asymptotic probability in the random graph with edge probability  $1/2$ .

This note is motivated by a discussion in the book of Palmer [7, page 81–85]. A theorem is strong if the conclusion is satisfied only when the hypothesis is satisfied, because then the hypotheses cannot be weakened. Palmer defines the *strength* of a theorem to be the probability that its hypotheses hold divided by the probability that its conclusion holds.

We use the standard random graph model for generating  $n$ -vertex simple graphs: the vertex set is  $\{1, \dots, n\}$ , and edge  $ij$  occurs with probability  $p$ , independently of other edges. Let  $\mathbb{G}_{n,p}$  denote the random variable for the resulting graph. In general, “ $\mathbb{G}_{n,p}$

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almost always satisfies  $Q$ ” means that the probability of  $\mathbb{G}_{n,p}$  satisfying  $Q$  tends to 1 as  $n \rightarrow \infty$ . We restrict our attention to constant  $p$ .

A graph is *Hamiltonian* if it has a spanning cycle. When  $p$  is constant,  $\mathbb{G}_{n,p}$  is almost always Hamiltonian. Dirac [3] proved that an  $n$ -vertex graph is Hamiltonian when every vertex degree is at least  $n/2$ . When  $p > 1/2$ , this condition holds almost always; when  $p \leq 1/2$ , it fails almost always. Hence the asymptotic strength of Dirac’s Theorem is 0 when  $p \leq 1/2$ . Ore [6] proved the stronger theorem that an  $n$ -vertex graph is Hamiltonian when the degrees of any two nonadjacent vertices sum to at least  $n$ . The asymptotic strength of Ore’s Theorem also is 0 when  $p \leq 1/2$ .

The strongest possible result using the degree list alone is that of Chvátal [2]. Chvátal proved that if the vertex degrees of an  $n$ -vertex graph are  $d_1, \dots, d_n$  in nondecreasing order, and for  $i < n/2$  it holds that  $d_i > i$  or  $d_{n-i} \geq n - i$ , then the graph is Hamiltonian. (If the condition just barely fails anywhere, then the graph can fail to be Hamiltonian.)

Palmer [7, page 85] states that Chvátal’s Condition almost always fails when  $p < 1/2$ . It is implied by Dirac’s Condition and hence almost always holds when  $p > 1/2$ .

**Question:** What is the asymptotic probability of Chvátal’s Condition for the random graph with edge probability  $1/2$ ?

There are other sufficient conditions for Hamiltonian cycles that almost always hold when  $p = 1/2$  and hence yield stronger theorems. Given a graph  $G$  with  $n$  vertices, let  $C(G)$  be the graph obtained by adding to  $G$  all edges joining two nonadjacent vertices in  $G$  whose degrees sum to at least  $n$ . Ore’s Condition is that  $C(G) = K_n$ . Let  $C^*(G)$  be the result of repeatedly applying the operator  $C$  until no further change occurs; Chvátal’s Condition implies that  $C(G) = K_n$ . Bondy and Chvátal [1] observed that  $G$  is Hamiltonian if and only if  $C^*(G)$  is Hamiltonian.

The value  $p = 1/2$  is critical here, as shown by Gimbel, Kurtz, Lesniak, Scheinerman, and Wierman [4]. If  $p < 1/2$ , then almost always  $C(\mathbb{G}_{n,p}) = \mathbb{G}_{n,p}$ . If  $p > 1/2$ , then almost always  $C(\mathbb{G}_{n,p}) = K_n$ . If  $p = 1/2$ , then almost always  $C(C(C(\mathbb{G}_{n,p})))$  is complete but  $C(C(\mathbb{G}_{n,p}))$  is not. Hence the theorem that  $C^*(G) = K_n$  suffices for Hamiltonicity of  $n$ -vertex graphs has asymptotic strength 1 when  $p \geq 1/2$ .

By proving that Chvátal’s Condition cannot hold for both a graph and its complement, we prove that the condition always has probability less than  $1/2$  (strict inequality, because

it fails for both  $G$  and  $\overline{G}$  when each has a vertex of degree at most 1). Since the degree list is almost always constant for a while near the middle, the asymptotic probability may rest on the probability that Chvátal's Condition fails at the middle values. When  $n$  is even, by complementation  $d_{n/2-1} \geq n/2$  with probability nearly  $1/2$ . When  $n$  is odd, however, Brendan McKay observes that the asymptotic probability of  $d_{(n-1)/2} = d_{(n+1)/2} = d_{(n+3)/2} = (n-1)/2$  (for both  $G$  and  $\overline{G}$ ) is a nontrivial constant (bounded away from 1) that can be determined by the method in [5]; this bounds the asymptotic probability of Chvátal's Condition below  $1/2$  when  $n$  is odd.

**Theorem 1** *If a graph  $G$  with at least three vertices satisfies Chvátal's Condition, then  $\overline{G}$  does not.*

**Proof.** Let  $n$  be the order of  $G$ . Let  $d_1, \dots, d_n$  be the vertex degrees, indexed so that  $d_1 \leq \dots \leq d_n$ . Let  $d'_1, \dots, d'_n$  be the vertex degrees of  $\overline{G}$ , also indexed in nondecreasing order, so that  $d'_{n+1-i} = n-1-d_i$ . Chvátal's Condition for  $G$  states that if  $i < n/2$ , then  $d_i > i$  or  $d_{n-i} \geq n-i$ .

The claim is easy to show for odd  $n$ , so we show this first in order to simplify notation for the other case. Consider  $i = (n-1)/2$ , so  $n-i = (n+1)/2$ . If  $d_{(n-1)/2} > (n-1)/2$ , then  $d_{(n+1)/2} \geq (n+1)/2$ , so we may assume the latter condition. This in turn implies  $d'_{(n+1)/2} \leq (n-3)/2$ , which also implies  $d'_{(n-1)/2} \leq (n-3)/2$ . Hence  $\overline{G}$  fails Chvátal's Condition at  $i = (n-1)/2$ .

Now consider  $n$  even. Let  $j = \max\{i: d_i > i \text{ and } i < n/2\}$ . Such an index exists, since  $d_1 \leq 1$  implies  $d_n \geq d_{n-1} \geq n-1$  by Chvátal's Condition for  $G$ , but this contradicts  $d_1 \leq 1$ .

If  $j = n/2 - 1$ , then  $d_{n/2-1} \geq n/2$ , so complementation yields  $d'_{n/2+2} \leq n/2 - 1$ . Hence also  $d'_{n/2+1} \leq n/2 - 1$  and  $d'_{n/2-1} \leq n/2 - 1$ , so Chvátal's Condition fails.

If  $j < n/2 - 1$ , then  $d_{j+1} \leq j+1$ . By Chvátal's Condition,  $d_{n-1-j} \geq n-1-j$ . Now complementation yields  $d'_{j+2} \leq j$ , and hence also  $d'_j \leq j$ . If Chvátal's Condition holds for  $\overline{G}$ , then  $d'_{n-j} \geq n-j$ . Now complementation yields  $d_{j+1} \leq j-1$ . This implies  $d_j < j$ , which contradicts the choice of  $j$ . ■

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