A NOTE ON GENERALIZED CHROMATIC NUMBER AND GENERALIZED GIRTH

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Abstract. Erdős proved that there are graphs with arbitrarily large girth and chromatic number. We study the extension of this for generalized chromatic numbers.

Generalized graph coloring describes the partitioning of the vertices into classes whose induced subgraphs satisfy particular constraints. When $\mathbf{P}$ is a family of graphs, the $\mathbf{P}$ *chromatic number* of a graph G, written $\chi_{\mathbf{P}}$, is the minimum size of a partition of $V(G)$ into classes that induce subgraphs of G belonging to $\mathbf{P}$. When $\mathbf{P}$ is the family of independent sets, $\chi_{\mathbf{P}}$ is the ordinary chromatic number. General aspects are studied in [1-3,7-9,11-14,17-18]. Many additional results are known about particular generalized chromatic numbers.

One aim in the study of generalized chromatic numbers is the extension of classical coloring results. Erdős [4] proved that there exist graphs of large chromatic number and large girth. We study the extension of this for a class of generalized coloring parameters. We consider the family $\mathbf{P}$ consisting of all graphs not containing H as a subgraph; we call the corresponding parameter the H-chromatic number and write it as χ_H.

The natural extension requires an appropriate definition for generalized girth. For $j \geq 2$, an (H,j)-*cycle* in a graph G is a list of distinct subgraphs $H_1, \ldots, H_j$, each isomorphic to H, such that $\bigcup_{i=1}^{j} H_i$ contains a cycle that decomposes into j nontrivial paths with the ith path in H_i (any two successive paths in the decomposition share one vertex). The H-*girth* of G, written $g_H(G)$, is the minimum j such that G contains an (H,j)-cycle, if this exists; otherwise $g_H(G) = \infty$.

One might prefer a weaker notion of cycle. For $j \geq 2$, a *weak* (H,j)-*cycle* in G is a list of distinct subgraphs $H_1, \ldots, H_j$, each isomorphic to H, and a selection of distinct vertices $x_1, \ldots, x_j \in V(G)$ such that $x_i \in V(H_{i-1}) \cap V(H_i)$, with subscripts taken modulo j. The

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weak H-girth $g^*_H(G)$ is the minimum j such that G contains a weak (H,j)-cycle, if this exists; otherwise $g^*_H(G)=\infty$.

Every (H,j)-cycle is a weak (H,j)-cycle, so $g^*_H(G)\le g_H(G)$. The weak H-girth may be considerably smaller than the H-girth. Trivial examples arise when H is disconnected. Also, when H is a 3-vertex path and $G=K_{1,3}$, we have $g_H(G)=\infty$ and $g^*_H=2$.

One naturally seeks the existence of graphs with arbitrarily large H-chromatic number and arbitrarily large H-girth. Such a result does not hold for weak H-girth, even when H is r-edge-connected.

Example *If H is the union of two copies of K_{r+1} sharing a vertex, then $\chi_H(G)\ge 4r+2$ implies $g^*_H(G)\le 2$.* We prove the contrapositive. The union of three $r+1$-cliques sharing a single vertex has weak H-girth 2, so $g^*_H(G)>2$ forbids this as a subgraph. Thus in G there is no vertex x whose neighborhood induces a subgraph containing three disjoint r-cliques. Thus in $G[N(x)]$ there is a set S_x of at most $2r$ vertices (the vertices of a maximal set of disjoint r-cliques) that together contain some vertex of each r-clique in $N(x)$.

Let G' be the spanning subgraph of G whose edges include the edges from each x to S_x for each $v\in V(G)$. Since each S_x has size at most $2r$, each m-vertex subgraph of G' has at most $2rm$ edges and thus minimum degree at most $4r$. By the Szekeres-Wilf Theorem, G' has ordinary chromatic number at most $4r+1$. Every proper ordinary coloring of G' uses at least two colors on every $r+1$-clique in G, since $G'[S]$ has no isolated vertex when $G[S]$ is an $r+1$-clique. Thus $\chi_{K_{r+1}}(G)\le 4r+1$. Since H contains K_{r+1}, we have $\chi_H(G)\le\chi_{K_{r+1}}(G)\le 4r+1$. ■

Similar examples occur whenever H is not 2-connected, but weak H-girth equals H-girth when H is 2-connected. We prove the desired result using the stronger concept of H-girth. This also follows from the result of Erdős and Hajnal [5] establishing the existence of r-uniform hypergraphs with large girth and chromatic number (constructive proofs appear in [10,16]). If H has order r, then a graph G with H-girth g and H-chromatic number k (where g is sufficiently large in terms of H) can be obtained from an r-uniform hypergraph $\mathbf{H}$ with girth g and chromatic number k by taking the union of copies of H defined on the vertex sets of the edges of $\mathbf{H}$.

Erdős and Hajnal did not directly define cycles in hypergraphs; instead they said that an r-uniform hypergraph is *s-circuitless* if for $t\le s$ every set of t edges contains at least $1+(r-1)t$ vertices in its union. When H has r vertices, an (H,j)-cycle has at most $(r-1)j$ vertices, so their definition of s-circuitless is equivalent to $g_H(G)>s$ when the edges of the hypergraph correspond to copies of H in G.

Using the strong version of H-girth, we give a short direct existence argument for graphs with large H-chromatic number and large H-girth. When H is a clique, this becomes a proof of the Erdős-Hajnal result. The main idea of the construction is similar to theirs, but the computations are somewhat different, and our presentation is perhaps more self-contained. When H has order r, the order of our graphs with $g_H>s$ and $\chi_H>k$ is about k^{rs}. The minimum order of such graphs is addressed in [6], so the point of our note is its alternative computations.

We summarize the approach. When seeking $g_H(G)>s$, we say that an (H,j)-cycle is a *short H-cycle* if $j\le s$. We will use the "deletion method", generating an n-vertex graph having many copies of H but few short H-cycles. To do this, we let r-sets receive copies of H with probability p. For appropriate p, we expect so many copies of H that in some graph every set of size $\lceil n/k \rceil$ contains more copies of H than the number of short H-cycles in the graph. After deleting edges in copies of H to break all the short H-cycles, every set of size at least n/k still contains a copy of H. By the pigeonhole principle, the H-chromatic number of the resulting graph exceeds k.

We need a numerical lemma about tail probabilities in the binomial distribution.

LEMMA If X has the binomial distribution with N trials and success probability p, then $\mathrm{Prob}(X \le pN/2) < 2(2/e)^{pN/2}$.

Proof: For $1 \le k \le pN/2$, we have

$$\frac{\binom{N}{k} p^k (1-p)^{N-k}}{\binom{N}{k-1} p^{k-1}(1-p)^{N-k+1}} = \frac{N-k+1}{k}\frac{p}{1-p} > 2.$$

In particular, $\mathrm{Prob}(X = pN/2 - k) < 2^{-k}\mathrm{Prob}(X = pN/2)$. Summing, we obtain

$$\mathrm{Prob}(X \le pN/2) < 2\binom{N}{pN/2} p^{pN/2}(1-p)^{(1-p/2)N}.$$

Because $\binom{N}{\alpha N} < \frac{(1/\alpha)^{\alpha N}}{(1-\alpha)^{(1-\alpha)N}}$ and $1-\beta < e^{-\beta}$, we conclude

$$\mathrm{Prob}(X \le pN/2) < 2 \cdot 2^{pN/2}(\frac{1-p}{1-p/2})^{(1-p/2)N}$$

$$= 2 \cdot 2^{pN/2}(1 - \frac{p/2}{1-p/2})^{(1-p/2)N} < 2 \cdot 2^{pN/2} e^{-pN/2}.$$

■

THEOREM Let H be a graph of order r. If s, k are positive integers with $s > r$, then there is a graph G with $g_H(G) > s$ and $\chi_H(G) > k$. Furthermore, if n is sufficiently large, then there is a graph G of order n with $g_H(G) > s$ and $\chi_H(G) \ge n^{1/(r^\epsilon s)}$, where $\epsilon = 1$ if H is 2-connected and $\epsilon = 2$ if H is not 2-connected.

Proof: From vertex set $[n]$ we select r-subsets, independently, each with probability p. This yields a random r-uniform hypergraph R.

If H is 2-connected, then set $\ell = s$; otherwise, set $\ell = rs$. A *set-cycle* of length j in G' is a cyclic arrangement of j selected r-sets such that the intersections of successive pairs yield a system of j distinct vertices as representatives. A set-cycle is *short* if it has length at most ℓ. Let X be the number of short H-cycles in R. When n is sufficiently large and N and p are appropriately chosen in terms of n, we claim that $E(X) < pN/4$, and that with probability at least $1/2$ every set of $\lceil n/k \rceil$ vertices contains at least $pN/2$ selected r-sets. Thus in some R every $\lceil n/k \rceil$-set contains at least $pN/2$ edges (r-sets) of R. Let R' be the hypergraph obtained from R by deleting some edge (r-set) from every short set-cycle in

R. On each edge (r-set) in R', place a copy of H, and let G be the graph formed by the union of these copies of H.

If H is 2-connected, then the only copies of H in G are the ones we have placed into the r-sets of R', so $g_H(G) > \ell = s$. If H is not 2-connected, then every cycle in G is either entirely in an r-set in R', or else it goes through all the sets of a set-cycle in R'. In the latter case this cycle has length at least $\ell + 1$, so it is not contained in the union of s subgraphs isomorphic to H. Hence $g_H(G) > s$ in this case as well.

Finally, the H-chromatic number of G is large; since every set of $\lceil n/k \rceil$ vertices of R' contains an edge of R', we have $\chi_H(G) > k$.

All that remains is to justify the inequalities claimed earlier. It suffices to show that our assertions hold when n is sufficiently large and $k = \lfloor n^{1/rs} \rfloor$. Set $N = \binom{n/k}{r}$ and $p = 8(1 + \log k)n/(kN)$. The expected number of $(H, 2)$-cycles is less than $\frac{n^2}{2}\binom{n}{r-2}^2 p^2$, and the expected number of (H, j)-cycles is less than $\frac{n^j}{2j}\binom{n}{r-2}^j p^j$. Letting $\beta = \frac{2}{s}\frac{n^{rs-s}p^s}{(r-2)!^s}$, we have

$$\begin{aligned} E(X) &< \frac{n^2}{4}\binom{n}{r-2}p^2 + \sum_{j=2}^{s} \frac{n^j}{2j}\binom{n}{r-2}^j p^j \\ &\le \frac{n^s}{s}\binom{n}{r-2}^s p^s \le \beta/2 \end{aligned}.$$

By Markov's Inequality, the probability now exceeds $1/2$ that $X \le \beta$.

Consider the $\lceil n/k \rceil$-sets. By the Lemma, the probability that *some* $\lceil n/k \rceil$-set of vertices contains at most $pN/2$ selected k-sets is less than α, where $\alpha = 2(2/e)^{pN/2}\binom{n}{\lceil n/k \rceil}$. When we have $k \le n^{1/6}$, we can use Stirling's approximation to conclude that $\alpha < \frac{1}{2}(2/e)^{pN/2}e^{(1+\log k)n/k}$. With $pN = 8(1 + \log k)n/k$, this guarantees $\alpha \le 1/2$.

Hence there exists an n-vertex graph G having at most $pN/2$ selected r-sets and having $X \le \beta$. It remains only to prove that $\beta < pN/2 = 4(1 + \log k)n/k$. We need $p^s \le \frac{2s(1+\log k)}{k}n^{-s(r-1)+1}(r-2)!^s$. Using the definition of p, it suffices to have

$$16(\log k)k^{r-1}r!n^{-r+1} \le (\frac{2s(1+\log k)}{k})^{1/s}(r-2)!n^{-r+1+1/s},$$

or

$$\Big(16(\log k)k^{r-1}r(r-1)\Big)^s \frac{k}{2s(1+\log k)} \le n.$$

This is satisfied by the condition on k in terms of n. ∎

Let H be a connected graph. We say that G is an *H-forest* if G is a subgraph of a union of copies of H such that any two of the specified copies of H have at most one common vertex and every cycle in G is contained in one of the specified copies of H.

COROLLARY Let H be a graph, and let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a family such that no F_i is an H-forest. Let $\mathcal{G}$ be the family of graphs containing no graph in $\mathcal{F}$. For every $k \ge 1$, there is a graph $G \in \mathcal{G}$ such that $\chi_H(G) = k$.

Proof: If s is the maximum order of the graphs in $\mathcal{F}$, then every graph constructed for s in the proof of the theorem belongs to $\mathcal{G}$. ∎

In particular, if m is less than the girth of H, then C_m is not an H-forest, and hence there exist C_m-free graphs with arbitrarily large H-chromatic number. Also, $\chi_{\mathbf{P}}$ is unbounded for C_m-free graphs when $\mathbf{P}$ is the family of graphs not containing H as an *induced* subgraph.

The Corollary also follows immediately from the result of Nešetřil and Rödl [15] that the class of graphs avoiding a fixed finite set of 2-connected graphs has the vertex Ramsey property. They proved their theorem using the result of Erdős and Hajnal [5].

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