Isometric cycles, cutsets, and crowning of bridged graphs

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Abstract

A graph G is *bridged* if every cycle C of length at least 4 has vertices x, y such that $d_G(x, y) < d_C(x, y)$. A cycle C is *isometric* if $d_G(x, y) = d_C(x, y)$ for all $x, y \in V(C)$.

We show that every graph contractible to a graph with girth g has an isometric cycle of length at least g. We use this to show that every minimal cutset S in a bridged graph G induces a connected subgraph.

We introduce a "crowning" construction to enlarge bridged graphs. We use this to construct examples showing that for every connected simple graph H with girth at least 6 (including trees), there exists a bridged graph G such that G has a unique minimum cutset S and that $G[S] = H$. This provides counterexamples to Hahn's conjecture that $d_G(u, v) \le 2$ when u and v lie in a minimum cutset in a bridged graph G.

We also study the convexity of cutsets in bridged graphs.

1 Introduction

A graph G is *chordal* if every cycle of length at least 4 has a chord, where a *chord* of a cycle is an edge not in the cycle with endpoints on the cycle. A subgraph H of a graph G is *isometric* if $d_H(x, y) = d_G(x, y)$ for all $x, y \in V(H)$, where $d_H(x, y)$ denotes the distance between x and y in H. A graph G is *bridged* if it has no isometric cycles of length at least four. Given a cycle C in G and two vertices x, y on C, an *x, y-bridge* of C is an x, y-path in G of length less than $d_C(x, y)$ that is internally disjoint from C. (In [6], this is called a

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proper bridge, and it is observed that the existence of proper bridges is equivalent to the definition of bridged graph given there.)

All our graphs are simple graphs. A *cutset* or *separating set* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The *girth* of a graph with a cycle is the length of a shortest cycle. A graph with no cycle has infinite girth.

Since a chord is a bridge of length 1, the family of bridged graphs contains the family of chordal graphs. It is well known that every minimal cutset of a chordal graph induces a complete subgraph. In seeking to generalize properties of chordal graphs, we study minimal cutsets of bridged graphs. In Section 3 we show that every minimal cutset in a bridged graph induces a connected subgraph.

In Section 4 we study which graphs can occur as subgraphs induced by minimum cutsets, motivated by Jamison's suggestion of exploring the relationship between cutsets and convexity in bridged graphs. For every connected simple graph H with girth at least 6 (including trees), we construct a bridged graph G such that G has a unique minimum cutset S and $G[S] = H$. It remains open whether the construction can be extended to all H that satisfy the necessary condition of having no induced cycle of length 4 or 5 (a bridge of a cycle of length 4 or 5 must be a chord).

Motivated by his negative answer to Aharoni's question of whether minimum cutsets in bridged graphs must be convex, Hahn [7] conjectured that $d_G(u, v) \leq 2$ when u and v lie in a minimum cutset in a bridged graph G. In Section 4 we provide counterexamples, constructing, for each k, a bridged graph such that the minimum cutset is unique and is convex and induces a path of length k. (A set $S \subseteq V(G)$ is *convex* if all shortest paths joining vertices of S are contained in $G[S]$.)

Furthermore, each cycle with length at least 4 in this graph has a bridge of length at most 2. Thus imposing the additional restriction that bridges have length at most two still will not guarantee that the diameter of a minimum cutset is bounded.

Norbert Polat independently also found counterexamples to Hahn's conjecture.

2 Collapsible walks

A *walk of length k* is an alternating list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such the endpoints of each edge e_i are v_{i-1} and v_i. In a simple graph, a walk can be specified by its ordered list of vertices alone. We name walks in this way.

A *trail* is a walk with no repeated edge. A *u,v-walk* is a walk with first vertex u and last vertex v. It is *closed* if $u = v$. Listing the vertices and edges of a cycle in order yields a closed trail of length at least one in which "first = last" is the only vertex repetition. We use the term *ordered cycle* to describe a walk that follows a cycle, starting and ending at the same vertex. A walk W *contains* a walk W' (such as an ordered cycle) if the vertices and edges of W' occur in order in W (with the edges occurring in the proper direction).

Definition 1 Given a walk W, an *(elementary) reduction* of W is the replacement of consecutive vertices x, y, x (and the intervening edge repetition) with the single vertex x. A walk is *collapsible* if it can be reduced to a single vertex by elementary reductions. A walk is *irreducible* if no reduction can be performed on it. A *subwalk* of a walk W is a walk obtained from W by applying (0 or more) reductions.

Every collapsible walk is a closed walk. The converse is false, since every ordered cycle is irreducible. Our applications involve distance in graphs, but collapsible walks have also been applied by Imrich [8] to obtain elementary proofs of several theorems in combinatorial group theory.

Lemma 2 *Every walk has a unique irreducible subwalk.*

Proof. We use induction on the length of the walk W. If W has length at most 1 or is irreducible, then the claim holds. If there is a unique elementary reduction available, then the claim follows by the induction hypothesis. Otherwise, there are at least two reductions available. Any two reductions may be intertwined (x, y, x, y), or independent $(x, y, x$ and z, w, z in parts of W sharing no steps).

In the first case, both elementary reductions yield the same subwalk. In the second case, the two reductions yield different subwalks W_1 and W_2. By the induction hypothesis, each has a unique irreducible subwalk. To obtain this subwalk, we can begin by performing the other of the two specified reductions. This yields the same subwalk with length four less than W. Hence the unique irreducible subwalks of W_1 and W_2 are the same.

We have shown that the irreducible subwalk obtained by the reduction process does not depend on the choice of the first reduction, and hence by the induction hypothesis it is unique. □

Corollary 3 *If W' is obtained from W by an elementary reduction, then W' is collapsible if and only if W is collapsible.*

Proof. By Lemma 2, the unique irreducible subwalk of W is obtained by reducing W'. □

Lemma 4 *If a closed walk contains no ordered cycle, then it is collapsible.*

Proof. We use induction on the length of the walk. If the length is 0, then the walk is already collapsed. For the induction step, let W be a closed walk of positive length. Consider the first position in the list where a vertex occurs for the second time. Since W contains no ordered cycle, this repetition must be an immediate return along the edge just used. We perform the elementary reduction to eliminate this edge repetition and obtain a subwalk W'. Since W' is a subwalk of W, it contains no ordered cycle. By the induction hypothesis, it is collapsible, and hence W is collapsible. □

It is natural to think that a closed walk is collapsible if and only if it contains no ordered cycle. However, the condition is not necessary, as shown by the vertex list v, w, v, x, w, x, v, which contains the ordered cycle w, v, x, w.

A walk that starts at a vertex v can be concatenated with a walk that ends at v by appending it without its first vertex.

Lemma 5 *Suppose that P is an x, y-walk and Q is a y, x-walk. If P and Q are irreducible and the concatenation (P followed by Q) is collapsible, then Q is the reverse of P.*

Proof. Since the concatenation PQ is collapsible, we can apply an elementary reduction to it. Since no such reduction is possible within P or Q, the reduction must be at w, y, w, where y is the end of P and start of Q. Hence Q begins with the edge on which P ends.

Since every consecutive portion of an irreducible walk is irreducible, iterating this argument shows that Q is the reverse of P. □

Corollary 6 *Let A, B, and P be x, y-walks. Let P' and B' denote the reverse of P and B, respectively. If the concatenations AP' and PB' are both collapsible, then AB' is also collapsible.*

Proof. By Corollary 3, we may assume that A, B, and P are irreducible. Lemma 5 now implies that $A = P$ and $P = B$. Hence $A = B$, and AB' is collapsible. □

3 Isometric cycles

When we contract an edge in a simple graph, we delete any resulting extra copies of edges.

Definition 7 A simple graph G is *contractible* to a simple graph H if H can be obtained from G by contracting edges of G. Given a partition of $V(G)$ into subsets $V_1, \ldots, V_m$ inducing connected subgraphs, we use $G_{V_1,\ldots,V_m}$ to denote the simple graph obtained from G by contracting all edges within each $G[V_i]$.

If G is contractible to H, then $H = G_{V_1,\ldots,V_p}$ for some partition $V_1, \cdots, V_p$ of $V(G)$.

Theorem 8 *If G is a graph that is contractible to a graph H with finite girth g, then G contains an isometric cycle with length at least g.*

Proof. Since G is contractible to H, we have $H = G_{V_1,\ldots,V_m}$ for some partition $V_1, \ldots, V_m$ of $V(G)$. Let $v_1, \ldots, v_m$ be the corresponding vertices in H. For a cycle C in G, let $W(C)$ denote the image of C in H under the contraction; $W(C)$ is a closed walk in H. Let $\mathcal{T}$ be the set of cycles in G whose image in H is irreducible.

We first show that $\mathcal{T}$ is nonempty. Let L be an ordered cycle in H. Without loss of generality, let L have vertices $v_1, \cdots, v_q$ in order. For $j \in \{1, \cdots, q\}$, there is an edge $x'_j x''_{j+1}$ in G with $x'_j \in V_j$ and $x''_{j+1} \in V_{j+1}$ (subscripts taken modulo q). For $j \in \{1, \cdots, q\}$, let P_j be a path in $G[V_j]$ connecting x'_j and x''_j; these paths exists since each $G[V_j]$ is connected. Now $C' = \bigcup_{j=1}^{q}(P_i \cup x'_j x''_{j+1})$ is a cycle in G with $W(C') = L$. Since L is irreducible, $C' \in \mathcal{T}$. Thus, $\mathcal{T}$ is nonempty.

Next we show that every cycle in $\mathcal{T}$ has length at least g. If C is a cycle of length less than g in G, then $W(C)$ is a closed walk of length less than g in H. Since H has girth g, $W(C)$ is a closed walk in H that contains no cycle. By Lemma 4, $W(C)$ is reducible, and so $C \notin \mathcal{T}$.

Now let C^* be a cycle in $\mathcal{T}$ of minimum length. It suffices to show that C^* is isometric. If not, then C^* has a bridge P from some vertex x to some vertex y. Let A and B denote the x, y-paths on C^*. Since A, B, and P are paths, they are irreducible as walks. Since AP' and PB' are cycles shorter than C, they are not in $\mathcal{T}$, and hence their images are collapsible. Corollary 6 now implies that $W(C^*)$ is reducible, which contradicts $C^* \in \mathcal{T}$. $\square$

A graph G is *triangle-free* if it does not contain K_3. Since a bridged graph has no isometric cycle of length at least 4, we have the following corollary from the theorem.

Corollary 9 *If G is a bridged graph, then G is not contractible to any triangle-free graph that contains a cycle.*

Corollary 10 *If G is a bridged graph and S is a minimal cutset of G, then $G[S]$ is connected.*

Proof. Let $A_1, \ldots A_p$ denote the components of $G[S]$, and let $B_1 \ldots, B_q$ denote the components of $G - S$. Since S is a cutset, $q \geq 2$. Since S is a minimal cutset, $G_{A_1,\ldots,A_p,B_1,\ldots,B_q}$ is the complete bipartite graph $K_{p,q}$. By Corollary 9, $p = 1$. □

4 Minimal cutsets in bridged graphs

It is well known that a graph G is a chordal graph if and only if every minimal cutset of G induces a complete subgraph. In light of this, it is natural to ask whether a minimal cutset of a bridged graph also must be restrictive in structure. It is perhaps surprising how little restriction there is.

In this section, we show that for every connected simple graph H with girth at least 6 (including trees), there exists a bridged graph G such that G has a unique minimum cutset S and $G[S] = H$. Note that the exclusion of induced cycles of lengths 4 and 5 is necessary, since cycles of these lengths in bridged graphs must have chords. It remains open whether our result extends to all graphs having no induced 4-cycles or 5-cycles.

Our result was motivated both by Hahn's conjecture and by the issue of convexity in bridged graphs, studied by Farber and Jamison [6]. An induced subgraph H in a graph G is *convex* if for any pair $u, v \in V(H)$, every shortest u,v-path in G lies completely in H. If H is a convex subgraph of G, then $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. If G is a bridged graph and H is an induced cycle in G of length at least six, then H is not a convex subgraph of G. Hence our construction shows that minimum cutsets cannot be guaranteed to be convex. On the other hand, when H is a tree we do obtain a bridged graph having a unique minimum cutset that is convex and induces a subgraph isomorphic to H.

Definition 11 Given a graph G and a vertex v in G, *duplicating* v means adding a new vertex v' and making it adjacent to v and to all the neighbors of v in G.

Lemma 12 *If G' is obtained from a bridged graph G by duplicating a vertex v, then G' is also bridged. Furthermore, if $x, y \in V(G) - v$, then $d_{G'}(x, y) = d_G(x, y)$.*

Proof. Let v' be the vertex added to duplicate v. Let C be a cycle of length at least four in G'. If $V(C)$ contains at most one of v and v', then C is a copy of a cycle in G and has in G' a copy of each bridge of that cycle in G.

If v and v' are both in $V(C)$ but not consecutive on C, then vv' is a chord of C and hence is a bridge of C. If v and v' are consecutive on C, then the neighbors of these vertices on C are adjacent to both v and v' in G'; this again yields chords.

For the second statment, a shortest x, y-path in G' cannot contain both v and v'; this would yield a chord. If it contains v', then replacing v' with v yields an x, y-path in G of the same length. □

Construction 13 Let G be a graph and H be a subgraph of G. The operation of *crowning H in G* produces a graph $G \star H$ from G in two steps as follows:

Step 1. For each $xy \in E(H)$, add a new vertex a_{xy} adjacent to x and y.

Step 2. For each $u \in V(H)$, add a new vertex u' adjacent to u and to each a_{uv} such that $uv \in E(H)$. (In this construction, we call u the *parent* of u'.)

In $G \star H$, we call vertices from G *Type 0* vertices, vertices added in Step 1 *Type 1* vertices, and vertices added in Step 2 *Type 2* vertices.

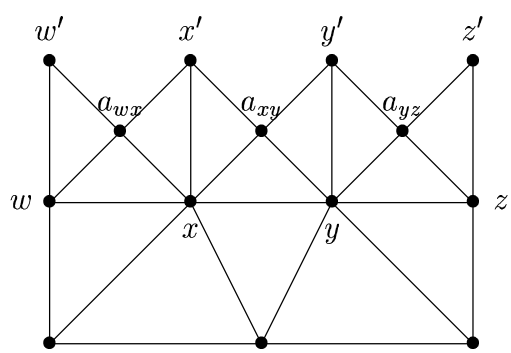

Fig. 1. Crowning a path H in G.

Remark 14 *If H is a connected subgraph of G, then the subgraph of $G \star H$ induced by the vertices of Types 1 and 2 introduced by crowning H is also connected.*

We use $l(P)$ to denote the length of a path P.

Lemma 15 *If G is a bridged graph, and H is a triangle-free subgraph of G, then $G \star H$ is also bridged. Furthermore, if $u, v \in V(G)$, then $d_{G \star H}(u, v) = d_G(u, v)$. Also, if H is convex in G, then H is convex in $G \star H$.*

Proof. Let $G' = G \star H$, and let C' be a cycle of length at least four in G'. We show that C' has a bridge in G'. Since the addition of a vertex whose neighborhood is a clique introduces no chordless cycle, the graph obtained after Step 1 is bridged. Hence we may assume that C' has at least one Type 2 vertex. Also, every cycle of length at least 4 containing both a Type 2 vertex and its parent has a chord, so we may exclude this possibility.

We create from C' a cycle C in G. First replace each Type 2 vertex with its parent (a Type 0 vertex). Since the neighborhood of a Type 2 vertex is contained in the neighborhood of its parent, and since C' does not contain both a Type 2 vertex and its parent, the result is a cycle C_1. Now form C by dropping each Type 1 vertex from C_1. Since the neighborhood of each Type 1 vertex among Type 1 and Type 0 vertices consists of the two endpoints of the edge that generated it, each such change merely shortens the cycle by one step, and C is a cycle in G. The difference between the lengths of C' and C is the number of Type 1 vertices that have been deleted.

Since C' uses at least one Type 2 vertex and H is triangle-free, C has length at least four. Thus C has a bridge P in G. Let u and v be the endpoints of P, and let A and B be the u,v-paths on C. Note that P is shorter than both A and B.

Note that $V(C')$ contains one of $\{u, u'\}$ and one of $\{v, v'\}$. Let A' and B' be the portions of C' from $\{u, u'\}$ to $\{v, v'\}$, corresponding to A and B, respectively. Let $r = |\{u', v'\} \cap V(C')|$. Since each Type 2 vertex on C' is immediately preceded and followed by Type 1 vertices, we have $l(A') \geq l(A) + r$ and $l(B') \geq l(B) + r$. This allows us to obtain a bridge of C' by extending P to $\{u', v'\}$ at each end where C' contains the Type 2 vertex rather than its parent. The resulting path P' has length $l(P) + r$, and hence it is shorter than A' and B'.

To prove the second statement, consider $u, v \in V(G)$, and let Q' be a shortest u,v-path in G'. We apply essentially the same construction. We may assume that if Q' contains a Type 2 vertex w', then Q' does not contain its parent w; otherwise Q' has a chord. We now replace Type 2 vertices in Q' by their parents and skip the Type 1 vertices. As in the comparison of C' and C above, this produces a u,v-path Q that is no longer than Q'. Furthermore, if Q' contains a vertex of Type 1 or 2, then Q is strictly shorter than Q', which yields the final statement. □

Lemmas 12 and 15 yield the main results in this section.

Theorem 16 *If H is a connected graph with girth at least 6 (including trees), then there exists a bridged graph G with a minimal cutset S such that $G[S] = H$. Furthermore, if H is a tree, then also $G[S]$ is convex in G.*

Proof. *Case 1: H is not a tree.* Form G_1 from H by adding a single vertex w adjacent to all of $V(H)$. All cycles in G_1 have bridges involving w, so G_1 is bridged. Now, let $G = G_1 \star H$. By Lemma 15, G is bridged. By Remark 14, the subgraph F induced by the vertices of Types 1 and 2 is connected. Let $S = V(H)$. The set S is a cutset of G, with $G - S$ consisting of two components, $\{w\}$ and F. Furthermore, each vertex of S has neighbors in both $\{w\}$ and F, so S is a minimal cutset. Finally, G is constructed so that $G[S] = H$.

Case 2: H is a tree. Every tree is a bridged graph. Let $G_1 = H \star H$, and let F_1 denote the subgraph of G_1 induced by the vertices of Types 1 and 2. As in Case 1, G_1 is bridged and F_1 is connected. Now, let $G = G_1 \star H$, and let F_2 denote the subgraph of G induced by the vertices of Types 1 and 2 introduced by crowning H. As before, G is bridged and F_2 is connected. Letting $S = V(H)$, again S is a minimal cutset of G, with F_1 and F_2 being the two components of $G - S$. By the last part of Lemma 15, each crowning operation preserves convexity. □

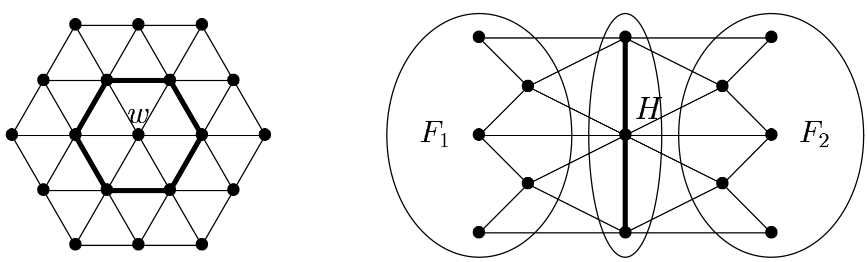

Fig. 2. Constructions for $H = C_6$ and $H = P_3$.

Corollary 17 *If H is a connected graph with girth at least 6 (including trees), then there exists a bridged graph G such that G has a unique minimum cutset S and that $G[S] = H$. Furthermore, if H is a tree, then also $G[S]$ is convex in G.*

Proof. Let $S = V(H)$. Let G' be the graph constructed in Theorem 16. Duplicate each vertex created in forming G' from H at least m times, where $m > |S|$. By Lemma 12, the resulting graph G is bridged. Since duplicating a vertex increases the minimum size of a separating set containing it, S is the unique minimum cutset in G. The other claims follow from the statement of Theorem 16. □

Corollary 17 immediately yields counterexamples to the conjecture that motivated our investigation.

Conjecture 18 *(G. Hahn [7]) If u and v are vertices of a minimum cutset in a bridged graph G, then $d_G(u, v) \le 2$.*

To obtain counterexamples, apply Corollary 17 with $H = P_k$. The unique minimum cutset S_k of the resulting graph G_k induces the path P_k of length $k-1$. Furthermore, since S_k is convex in G_k, the distance between two vertices in S_k is the same in the k-connected graph G_k as in $G_k[S_k]$, and it can be as large as $k-1$.

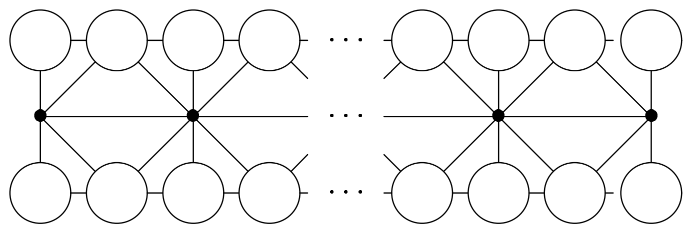

Fig. 3. Minimum cutsets with vertices at large distance.

Note also that in G_k every cycle of length at least four has a bridge of length at most two. One may consider generalizing the definition of bridged graphs as follows: For a positive integer d, a graph G is *d-bridged* if every cycle of length at least four in G has a bridge of length at most d in G. Thus the 1-bridged graphs are the chordal graphs. The graphs G_k constructed above are 2-bridged graphs. Thus Conjecture 18 fails even in the class of 2-bridged graphs.

Finally, we remark that the crowning operation can be generalized to produce larger classes of bridged graphs. For example, omitting any subset of the Type 2 vertices will still produce a bridged graph.

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