

# THE BRICKLAYER PROBLEM AND THE STRONG CYCLE LEMMA

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## 1. THE PROBLEM

One of the most familiar brand names in the world of toys is *Lego*<sup>TM</sup>; the name immediately brings interlocking building blocks to mind. Interlocking blocks fit together in restricted ways. Typically, a block of order  $q$  (length  $q + 1$ ) has  $q + 1$  protrusions on its top and  $q + 1$  indentations on its bottom, so that the indentations on the bottom of one block can lock on to the protrusions on the top of another. Here  $q$  is a positive integer, and the width and height of a block are unimportant. For  $q = 1$  and  $q = 2$ , Figure 1 shows the  $q$  ways in which one block can sit on top of two others.

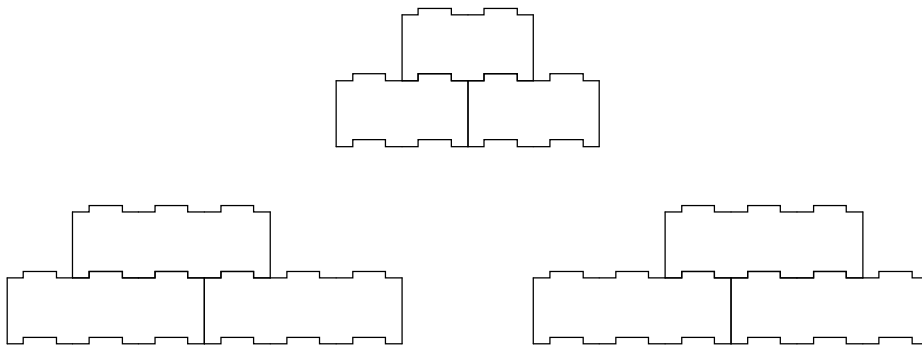


Figure 1. Bricks of length  $q + 1$ , for  $q = 1$  and  $q = 2$ .

In this paper, we solve a counting problem about building stacks of such blocks, which we call *bricks*. We have a linear base of length  $m$  on which we can place bricks. The bricks sitting directly on the base can start at any integer position, as long as they fit on the base and don't overlap. When two bricks are contiguous, sharing a common end, we can place another brick on top of them. Since we want the blocks to interlock, it must cover part of each brick below it. The protrusions and indentations then restrict it to  $q$  possible

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positions, covering a positive integer amount of each brick it rests on. Again bricks cannot overlap. We call a configuration built by these rules a *q-stack*.

**The bricklayer problem.** How many different *q*-stacks can be built on a base of length *m*?

For example, when  $m = 4$  and  $q = 1$ , there are five *q*-stacks, including three with one brick, one with two bricks, and one with three bricks that appears in Figure 1. When  $m = 5$  and  $q = 1$ , there are nine *q*-stacks. When  $m = 6$  and  $q = 2$ , there are seven *q*-stacks, two of which appear in Figure 1.

The bricklayer problem is related to a host of other problems in combinatorial enumeration. We solve it by establishing a bijection from the set of *q*-stacks on a base of length *m* to a special set of sequences of 0's and 1's. This leads us to related counting problems because these special sequences generalize the solutions to the famous *Ballot Problem*.

Consider an election that ends in a tie, with *k* votes for each of two candidates. If the votes are counted in a random order, with all sequences equally likely, we may wonder what the probability is that the first candidate never trails. Using 0's to designate votes for the first candidate and 1's to designate votes for the second, we are asking for the fraction of sequences with *k* 1's and *k* 0's such that every prefix (initial segment) has at least as many 0's as 1's. These sequences are the *ballot sequences* of length  $2k$ . To compute the desired probability, we count the ballot sequences and divide by  $\binom{2k}{k}$ . We will see shortly that the number of ballot sequences of length  $2k$  is the *k*th Catalan number  $C_k$ , defined by

$$C_k = \frac{1}{k+1} \binom{2k}{k}.$$

More generally, a sequence is *q-satisfying* if in each prefix the number of 0's is at least *q* times the number of 1's. We will establish a bijection between the *q*-stacks on a base of length *m* and the *q*-satisfying sequences of length *m*. Thus we reduce the bricklayer problem to the counting of *q*-satisfying sequences.

The *Cycle Lemma* of Dvoretzky and Motzkin [5] provides one of the many proofs that the Catalan numbers count the ballot sequences. With equal ease it enables us to count the *q*-satisfying sequences of length *m*. After doing so, we develop the bijection to solve the Bricklayer Problem. Subsequently, we explore generalizations of the Cycle Lemma and applications of these generalizations.

## 2. THE CYCLE LEMMA AND GENERALIZED CATALAN NUMBERS

To facilitate our discussion, we introduce terminology to describe various arrangements of 0's and 1's. A *q-dominating* sequence (as defined in [3]) is a sequence of 1's and 0's such that in each prefix the number of 0's is more than *q* times the number of 1's. A  $(k, l)$ -*sequence* is a sequence of *k* 1's and *l* 0's. A  $(k, l)$ -*arrangement* is a cyclic arrangement of *k* 1's and *l* 0's. By this we mean that rotating the arrangement does not change it, but we maintain a fixed direction; a flip or reversal produces a different arrangement.

A special case of the Cycle Lemma states that every  $(k, k + 1)$ -arrangement can be cut in exactly one position to obtain a 1-dominating  $(k, k + 1)$ -sequence. The first element of this sequence (the element after the cut) must be a 0. Deleting this 0 yields a ballot sequence, and the process is reversible. Thus the ballot sequences and the  $(k, k + 1)$ -arrangements are equinumerous. Since  $k$  and  $k + 1$  are relatively prime, the number of  $(k, k + 1)$ -arrangements (and ballot sequences of length  $2k$ ) is exactly the Catalan number

$$\frac{1}{2k + 1} \binom{2k + 1}{k + 1} = \frac{1}{k + 1} \binom{2k}{k} = C_k.$$

The Cycle Lemma thus “explains” one 0 in a  $(k, k + 1)$ -arrangement. Kierstead and Trotter [9] generalized this to give combinatorial meaning to each 0. We present their result in Section 4 and extend it slightly. Our task at present is to prove the Cycle Lemma and use it to count the  $q$ -satisfying sequences of length  $m$ . Figure 2 illustrates the Cycle Lemma (and its proof) when  $(k, q, p) = (2, 2, 3)$ ; the underscored 0’s are those that begin 2-dominating sequences when the arrangement is read clockwise.

**THEOREM 1** (The Cycle Lemma - Dvoretzky and Motzkin [5]) For  $k, q, p \geq 0$ , every  $(k, qk + p)$ -arrangement breaks to form a  $q$ -dominating sequence in exactly  $p$  places.

**Proof:** The statement is trivial for  $k = 0$ ; we proceed by induction. For  $k > 0$ , let  $a$  be a  $(k, qk + p)$ -arrangement. By the pigeonhole principle, between some pair of the 1’s in  $a$  there are more than  $q$  0’s. Let  $S$  be a set of  $q + 1$  consecutive positions consisting of  $q$  0’s followed immediately by a 1 (illustrated by the outside arc in Figure 2).

None of the  $q + 1$  positions of  $S$  can start a  $q$ -dominating sequence. A position outside  $S$  starts a  $q$ -dominating sequence if and only if it starts one in the  $(k - 1, q(k - 1) + p)$ -arrangement  $a'$  obtained from  $a$  by deleting  $S$ . The number of  $q$ -dominating starting places in  $a$  thus equals the number of  $q$ -dominating starting places in  $a'$ , which by the induction hypothesis is  $p$ . ■

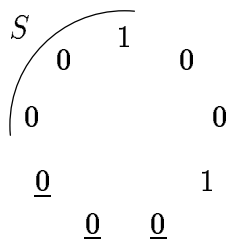


Figure 2. 2-dominating sequences from a  $(2, 7)$ -arrangement.

**COROLLARY 2** The number of  $q$ -satisfying  $(k, qk + p - 1)$ -sequences and the number of  $q$ -satisfying sequences of length  $m$ , respectively, are

$$\frac{p}{qk + p} \binom{(q+1)k + p - 1}{k} \quad \text{and} \quad \sum_{k=0}^{\lfloor m/(q+1) \rfloor} \frac{m - (q+1)k + 1}{m - k + 1} \binom{m}{k}.$$

**Proof:** A sequence is  $q$ -satisfying if and only if the sequence obtained by adding a 0 at the front is  $q$ -dominating. Thus the  $q$ -satisfying  $(k, qk + p - 1)$ -sequences and the  $q$ -dominating  $(k, qk + p)$ -sequences are equinumerous. By the Cycle Lemma, each  $(k, qk + p)$ -arrangement has  $p$  positions that yield  $q$ -dominating sequences. Thus in each class of  $(k, qk + p)$ -sequences equivalent under cyclic rotation, the fraction that are  $q$ -dominating is  $p/(k + qk + p)$ . Since the fraction is the same over all classes, we need not worry about periodicity. We conclude that the number of  $q$ -dominating  $(k, qk + p)$ -sequences is  $\binom{(q+1)k + p}{k} p / (k + qk + p)$ , which equals the formula claimed.

Each  $q$ -satisfying sequence of length  $m$  has  $k$  1's and  $m - k$  0's, for some  $k$ . By setting  $m - k = qk + p - 1$ , we obtain  $p = m - (q + 1)k + 1$  and can use the preceding formula to count these sequences. Thus the term for  $k$  in the summation is precisely the number of  $q$ -satisfying sequences of length  $m$  that have  $k$  1's. ■

Dershowitz and Zaks [3] gave two applications of the Cycle Lemma and provided references to several proofs of it. Many applications (and many proofs) have been given for the special case  $p = 1$ . These lead to the *generalized Catalan numbers*, defined by

$$C_n^q = \frac{1}{qn + 1} \binom{(q+1)n}{n}.$$

By Corollary 2,  $C_n^q$  is the number of  $q$ -satisfying  $(n, qn)$ -sequences. There are other ways to generalize the Catalan numbers by introducing additional parameters, but this is the generalization appropriate for our discussion.

For example, the generalized Catalan numbers arise in counting the rooted plane trees with  $qn + 1$  leaves in which every non-leaf vertex has exactly  $q + 1$  children. By “plane tree”, we mean that the left-to-right order of children matters. Figure 3 shows the three such trees when  $q = 2$  and  $n = 2$ .

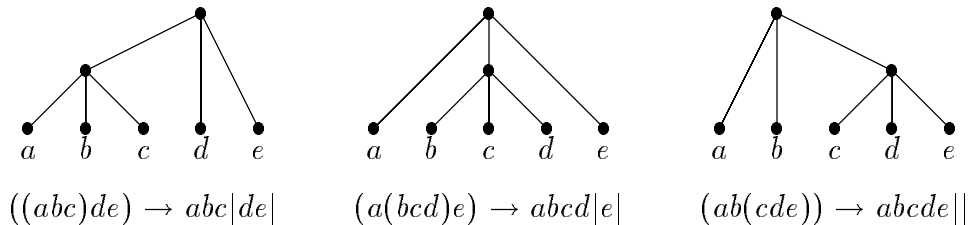


Figure 3. The rooted plane ternary trees with five leaves.

By working up from the leaves, combining  $q+1$  subtrees at each non-leaf vertex, we see that each tree corresponds to a product of  $qn+1$  elements in order using a non-associative  $q+1$ -ary operator. Conversely, from such a product we obtain the corresponding tree. Sands [14] was interested in counting the ways to form such a product.

In order to count the products, we convert the tree to a sequence of objects and markers. We do this by traversing the tree; beginning at the root, walk around the tree, keeping our left hand on it, until we have traversed every edge twice. We record an object for each leaf when it is visited and a marker for each non-leaf vertex when the subtree rooted at that node is completed. Figure 3 shows the resulting sequences when  $q = n = 2$ . The sequence corresponding to each tree (that is, each bracketing) is “post-fix” notation for the formation of the product; the marker specifies application of the operation to the  $q$  most recent objects on the stack, replacing them by a single object.

By converting markers to 1’s and objects to 0’s, we obtain an  $(n, qn+1)$ -sequence. For each such sequence produced in this way, the number of objects that precede the  $i$ th marker must exceed  $qi$ , since applying a  $q+1$ -ary operator  $i$  times converts  $qi+1$  objects into one object. Sands observed by an easy induction on  $n$  that the  $q$ -dominating condition is also sufficient. Thus these bracketings (or these trees) are equinumerous with the  $q$ -satisfying  $(n, qn)$ -sequences, and there are  $C_n^q$  of them.

When  $q = 1$ , we obtain the ballot sequences and the ordinary Catalan numbers. Other counting problems solved by the Catalan numbers generalize in analogous ways. Hilton and Pedersen [8] observed that  $C_n^q$  counts the subdivisions of a convex polygon into  $n$  disjoint  $(q+1)$ -gons by noncrossing diagonals. This generalizes a bijection between binary trees with  $n+1$  leaves and dissections of an  $n+2$ -gon into triangles, where the root becomes one edge of the polygon and the leaves become the other edges. In light of these generalizations, we use the name *q-ballot sequences* for the  $q$ -satisfying sequences with exactly  $n$  1’s and  $qn$  0’s.

### 3. $q$ -SATISFYING SEQUENCES AND THE BRICKLAYER PROBLEM

Before attacking the full generality of the bricklayer problem, we consider how the bijection works when  $q = 1$ . Propp (see [12]) observed that the ordinary Catalan numbers solve a simple coin-stacking problem. We begin with a base row of  $n$  coins. Each coin not in the base row rests on two coins in the row immediately below it. Figure 4 illustrates such a stack with a base row of 6 coins.



Figure 4. A stack of coins and its conversion.

Let  $a_n$  be the number of distinct stacks that can be built on a base of length  $n$ , with  $a_0 = 1$ . If the number of contiguous coins at the beginning of the first row above the base is  $k-1$ , then the stack is completed by building one stack based on these  $k-1$  coins and

another based on the last  $n - k$  coins of the original base. Summing over the possible values of  $k$  yields  $a_n = \sum_{k=1}^n a_{k-1} a_{n-k}$  for  $n \geq 1$ . This is the well-known recurrence satisfied by the Catalan numbers. It arises for the ballot sequence model by letting  $2k$  be the minimum length of a nonempty prefix with the same number of 0's and 1's.

The recurrence suggests a natural bijection. Replace each coin by a wedge, as shown on the right in Figure 4. View each wedge as having width 2. The wedge for a coin above the base rests on the apexes of the wedges for its supporting coins. Now follow the outline of the mound of wedges. Each step up is a 0, each step down is a 1. Each step moves one unit right, so there are  $2n$  steps. We end at the base, so there are  $n$  steps of each type. Since the mound never dips below the base, the result is a ballot sequence. That this is a bijection is a special case of our main theorem.

We generalize to the bricklayer problem by viewing the coins as bricks of length  $2 = q + 1$ . We drop the requirement that the base be full to allow bases of all lengths rather than merely multiples of  $q + 1$ . For this more general problem, the proof is simpler. In the special case where  $m = (q + 1)n$  and the base is filled by  $n$  bricks, there will be  $C_n^q$  stacks, corresponding to the  $q$ -ballot sequences.

Recall our conditions for  $q$ -stacks in the bricklayer problem:

- 1) The base has length  $m$ .
- 2) The bricks have length  $q + 1$ .
- 3) Each brick not directly resting on the base rests on two contiguous bricks immediately below it and covers a positive integer amount of the tops of each.

Thus a brick has  $q$  possible positions in relation to the two bricks below it.

**THEOREM 3** In the bricklayer problem with a base of length  $m$ , the number of  $q$ -stacks with  $n$  bricks resting directly on the base, and the total number of  $q$ -stacks, respectively, are

$$\frac{m - (q + 1)n + 1}{m - n + 1} \binom{m}{n} \quad \text{and} \quad \sum_{n=0}^{\lfloor \frac{m}{q+1} \rfloor} \frac{m - (q + 1)n + 1}{m - n + 1} \binom{m}{n}.$$

**Proof:** Fix  $q$ . Let the *length* of a  $q$ -stack be the length of the base. Let  $S_m$  be the set of  $q$ -stacks of length  $m$ . Let  $Q_m$  be the set of  $q$ -satisfying sequences of length  $m$ . We establish a bijection  $f: S_m \rightarrow Q_m$ . Furthermore,  $f$  restricts to a bijection from  $S_m^n$  to  $Q_m^n$ , where  $S_m^n$  is the subset of  $S_m$  consisting of stacks having  $n$  bricks resting on the base and  $Q_m^n$  is the subset of  $Q_m$  consisting of the sequences with  $n$  1's. By Corollary 2, establishing this bijection completes the proof.

To define  $f$  on a stack  $A \in S_m$ , we begin by shaving the top corners of each brick along a line from the bottom corner to a point at distance one from the top corner along the top edge. Each brick becomes a symmetric trapezoid with upper edge of length  $q - 1$  and lower edge of length  $q + 1$  (a wedge when  $q = 1$ ). Shaving the bricks is an invertible process (mathematically, not physically), so we henceforth treat  $S_m$  in this form.

For  $A \in S_m$  (shaved), we define  $f(A)$  by reading the top outline of the stack, recording a 0 for each up-slant or horizontal step, and recording a 1 for each down-slant. Since we record one bit for each increase in the horizontal coordinate, the length of  $f(A)$  is  $m$ . Figure 5 illustrates the correspondence for one stack with  $(q, m, n) = (2, 12, 3)$  and for all stacks with  $(q, m, n) = (2, 9, 3)$ .

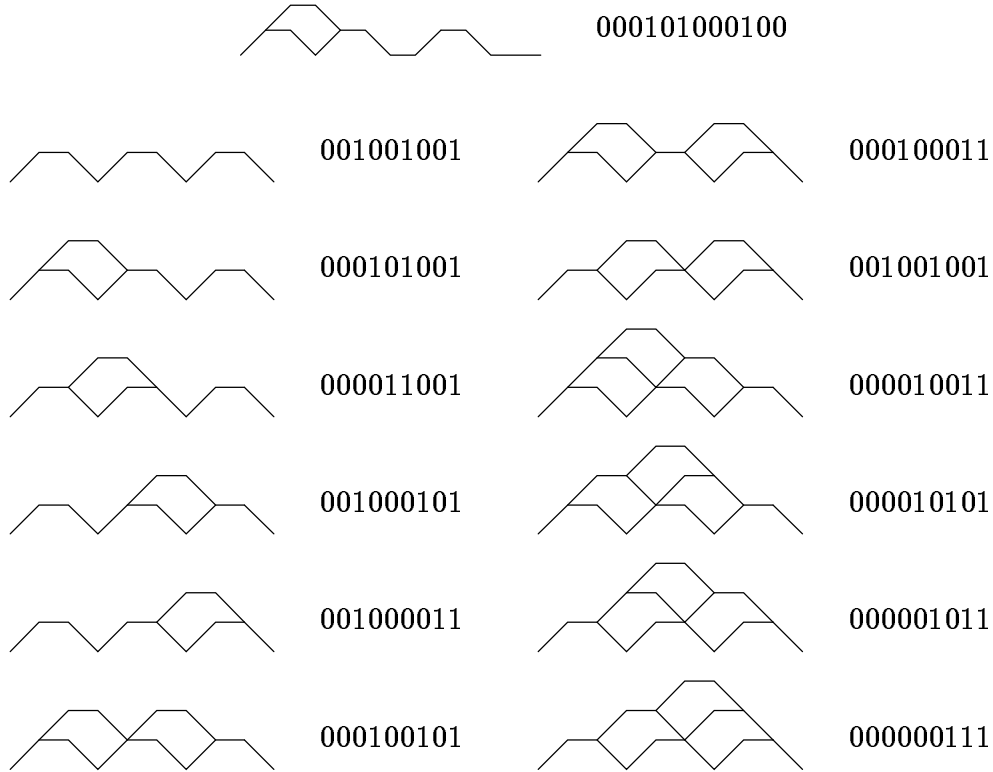


Figure 5. Shaved  $q$ -stacks and the corresponding  $q$ -satisfying sequences.

Among the  $q$ -satisfying sequences, the  $q$ -dominating sequences are those that have no  $q$ -ballot prefix. We prove the further property that, within each pair  $(S_m^n, Q_m^n)$ ,  $f$  matches the  $q$ -dominating sequences with the stacks not covering the first space on the base. We use induction on  $m$ . When  $m = 1$ , there are no bricks, and the empty stack in  $S_1^0$  maps to the sequence 0 in  $Q_1^0$ . For  $m > 1$ , we consider three types of stacks and the corresponding sequences, showing first that  $f$  restricts as desired.

If  $A \in S_m^n$  does not cover the first space of the base, then  $f(A)$  consists of 0 followed by  $f(A')$ , where  $A'$  is obtained from  $A$  by deleting the first space of the base. By the induction hypothesis,  $f(A') \in Q_{m-1}^n$ , so  $f(A) \in Q_m^n$  and  $f(A)$  is  $q$ -dominating.

If  $A \in S_m^n$  covers the first space, let  $m' = k(q + 1)$  be the step on which the outline of  $A$  first returns to the base (in the top example of Figure 5,  $k = 2$  and  $m' = 6$ ). If  $m' < m$ , then  $A$  is the concatenation of a stack  $A' \in S_{m'}^k$ , and a stack  $A'' \in S_{m-m'}^{n-k}$ , and  $f(A)$  is the concatenation of  $f(A')$  and  $f(A'')$ . By the induction hypothesis,  $f(A')$  is a  $q$ -ballot sequence and  $f(A'')$  is  $q$ -satisfying, so  $f(A) \in Q_m^n$  and  $f(A)$  is not  $q$ -dominating.

If  $m' = m$ , then  $m = (q + 1)n$  and every notch in the lowest row of bricks is covered by a higher brick. Let  $A'$  be the stack obtained from  $A$  by deleting the bottom row of bricks and the first and last space of the base. Since every brick above the first row covers one notch,  $A' \in S_{m-2}^{n-1}$ . Also,  $f(A)$  is obtained from  $f(A')$  by adding 0 at the beginning and 1 at the end. By the induction hypothesis,  $f(A') \in Q_{m-2}^{n-1}$ . Adding 0 at the beginning and 1 at the end of a  $q$ -satisfying sequence with these parameters yields a  $q$ -satisfying sequence, so  $f(A) \in Q_{(q+1)n}^n$ . The prefix without the final 1 is  $q$ -dominating and has no proper  $q$ -ballot prefix, but the full sequence  $f(A)$  is a  $q$ -ballot sequence and hence is not  $q$ -dominating.

To show that  $f$  is invertible, consider  $a \in Q_m^n$ . If  $a$  is  $q$ -dominating, then no stack covering the first space has image  $a$ . Let  $a'$  be the  $q$ -satisfying sequence obtained by deleting the initial 0 of  $a$ . By the induction hypothesis, there exists one stack  $A' \in S_{m-1}^n$  such that  $f(A') = a'$ . Among the stacks in  $S_m^n$  not covering the first space, there is thus one stack  $A$  such that  $f(A) = a$ .

If  $a$  is not  $q$ -dominating, then  $a$  has a  $q$ -ballot prefix. Let  $m'$  be the length of the shortest  $q$ -ballot prefix  $a'$  of  $a$ , having  $k$  1's. Since  $a'$  has  $qk$  0's, the remainder  $a''$  of  $a$  is in  $Q_{m-m'}^{n-k}$ . We have observed that a proper  $q$ -ballot prefix of length  $m' < m$  arises in  $f(A)$  if and only the outline of  $A$  first returns to the base after  $m'$  steps. By the induction hypothesis, there is one  $A \in S_{m'}^k$  such that  $f(A) = a'$ , and there is one  $A'' \in A_{m-m'}^{n-k}$  such that  $f(A'') = a''$ , and the concatenation yields the unique  $A$  such that  $f(A) = a$ .

If  $a$  is not  $q$ -dominating and  $m' = m$ , then  $m = (q + 1)n$  and  $a$  is a  $q$ -ballot sequence with no proper  $q$ -ballot prefix. Before the final 1,  $a$  is  $q$ -dominating, and deleting the initial 0 and final 1 yields a  $q$ -satisfying sequence  $a'$  of length  $m - 2$ . We have observed that if  $f(A)$  is a  $q$ -ballot sequence with no proper  $q$ -ballot prefix, then  $A$  has  $n - 1$  bricks in the second row, and  $f(A)$  is obtained by adding 0 at the beginning and 1 at the end of  $f(A')$  for the stack  $A' \in S_{m-2}^{n-1}$  obtained by deleting the bottom row of  $A$  and shortening the base at both ends. By the induction hypothesis, there is one  $A' \in S_{m-2}^{n-1}$  such that  $f(A') = a'$ , and thus there is one  $A$  such that  $f(A) = a$ . ■

The proof yields a recurrence for the number  $c_{m,n}$  of  $q$ -satisfying sequences of length  $m$  with  $n$  1's. We have  $c_{1,0} = 1$ , and  $c_{1,n} = 0$  for  $n \neq 1$ . For  $m > 1$ ,

$$c_{m,n} = c_{m-1,n} + \epsilon_{m,n} c_{m-2,n-1} + \sum_{0 \leq k < m/(q+1)} c_{(q+1)k,k} c_{m-(q+1)k,n-k}, \quad (*)$$

where  $\epsilon_{m,n}$  is 1 if  $m = (q + 1)n$  and is 0 otherwise. Bailey [1] obtained another recurrence for the case  $q = 1$  and used it to obtain the first statement of Corollary 2 in that case.

The Catalan recurrence  $a_n = \sum_{k=1}^n a_{k-1} a_{n-k}$  is simpler than (\*) because removing the initial 0 and trailing 1 from a ballot sequence that has no balanced prefix yields a shorter ballot sequence. This statement does not generalize; when a  $q$ -ballot sequence of length  $(q + 1)n$  has no proper  $q$ -ballot prefix (such as 000100011 for  $q = 2$  and  $n = 3$ ), removing the bits after the penultimate 1 and removing enough leading 0's to reduce to length  $(q + 1)(n - 1)$  need not produce a  $q$ -ballot sequence.



We can obtain a natural recurrence for generalized Catalan numbers by modeling the formation of bracketings. We know that  $C_n^q$  counts both the  $q$ -ballot sequences with  $n$  1's and the bracketings of a product involving  $n$  applications of a  $q + 1$ -ary operator. Hilton and Pedersen [8] observed that the last application of the operator combines  $q + 1$  segments, each of which is a shorter  $q$ -dominating  $(n_i, qn_i + 1)$ -sequence. Thus

$$C_n^q = \sum \prod_{i=1}^{q+1} C_{n_i}^q,$$

where the summation runs over all choices of  $q + 1$  positive integers  $n_1, \dots, n_{q+1}$  that sum to  $n - 1$ . The initial condition is  $C_0^q = 1$ .

#### 4. $(k, qk+1)$ -ARRANGEMENTS

For  $p = 1$ , Kierstead and Trotter strengthened the Cycle Lemma on  $(k, qk + p)$ -arrangements, showing that each 0 plays a special role. Within a cyclic arrangement  $a = a_0, \dots, a_{n-1}$  of 0's and 1's, we denote the list  $a_{i+1}, \dots, a_j$  (indices modulo  $n$ ) by  $(i, j]$ . The *linearization* of  $a$  ending at position  $i$  is  $(i, i]$ . For a fixed linearization, we use *0-interval* to mean a prefix ending at a 0. If  $a_i = 0$ , then  $(i, i]$  is a *0-linearization*, and the full list  $(i, i]$  is the *trivial* 0-interval. We use  $w_0(I)$  and  $w_1(I)$  to denote the number of 0's and 1's in  $I$ , respectively. A 0-interval  $I$  is *q-good* if  $w_0(I) > qw_1(I)$ . In every 0-linearization of a  $(k, qk + p)$ -arrangement with  $p > 0$ , the trivial 0-interval is  $q$ -good.

For  $q = p = 1$  and  $1 \leq i \leq k + 1$ , Kierstead and Trotter [9] proved that every  $(k, k + 1)$ -arrangement  $a$  has a unique 0-linearization such that exactly  $i$  of the 0-intervals are 1-good. They noted that this result is implicit in the work of Feller [6] and Narayana [11], and they used it to construct new explicit perfect matchings in the bipartite graph of the inclusion relation on the  $k$ -sets and  $k + 1$ -sets of a  $2k + 1$ -element set. Their elegant proof of a technically stronger statement extends directly to  $(k, qk + 1)$ -arrangements. Figure 6 shows a (clockwise)  $(3, 7)$ -arrangement and its 0-linearizations, indicating the number of 2-good 0-intervals in each and underscoring the positions that end 2-good 0-intervals.

			1	1 0 1 0 0 1 0 0 0 <u>0</u>
0	1	0	3	1 0 0 1 0 0 <u>0</u> <u>0</u> 1 <u>0</u>
0		1	6	<u>0</u> 1 0 <u>0</u> <u>0</u> <u>0</u> 1 <u>0</u> 1 <u>0</u>
0		0	4	1 0 0 <u>0</u> <u>0</u> 1 <u>0</u> 1 <u>0</u> <u>0</u>
0	1	0	7	<u>0</u> <u>0</u> <u>0</u> 1 <u>0</u> 1 <u>0</u> <u>0</u> 1 <u>0</u>
			5	<u>0</u> <u>0</u> 1 <u>0</u> 1 0 <u>0</u> 1 <u>0</u> <u>0</u>
			2	<u>0</u> 1 0 1 0 0 1 0 0 <u>0</u>

Figure 6. 2-good 0-intervals in 0-linearizations of a  $(3, 7)$ -arrangement.

**LEMMA 4** (Strong Cycle Lemma) If  $a$  is a  $(k, qk + 1)$ -arrangement and  $1 \leq i \leq qk + 1$ , then there is a unique 0-linearization  $(j_i, j_i]$  of  $a$  in which exactly  $i$  0-intervals are  $q$ -good. Furthermore, for  $i \leq qk$ , the 0's that end  $q$ -good 0-intervals in  $(j_i, j_i]$  also end  $q$ -good 0-intervals in  $(j_{i+1}, j_{i+1}]$  (we call this the *nesting property*).

**Proof:** Given a  $(k, qk + 1)$ -arrangement  $a$ , let the *deficiency* of an interval  $I = (r, s]$  be  $\delta(r, s] = qw_1(I) - w_0(I)$ . Given a 0-linearization  $(r, r]$ , let  $D(r) = \{j: \delta(r, j] < 0 \text{ and } a_j = 0\}$ ; this is the set of indices ending  $q$ -good 0-intervals for  $(r, r]$ . Note that  $r \in D(r)$ . Since  $1 \leq |D(r)| \leq qk + 1$  for all  $j$ , it suffices to prove that the sets  $\{D(j)\}$  are distinct and are linearly ordered by inclusion.

Let  $r, s$  be the positions of two 0's. Since  $\delta(r, s] + \delta(s, r] = -1$ , exactly one of  $(r, s]$  and  $(s, r]$  has negative deficiency and is  $q$ -good. When  $(r, s]$  is  $q$ -good, we claim that  $D(s) \subset D(r)$ . Note that  $s \in D(r)$ , but  $r \notin D(s)$ . Now consider  $j \in D(s)$ ; we have two cases. If  $j \in (r, s]$ , then  $\delta(r, j] = \delta(s, j] - \delta(s, r] < 0$ . If  $j \in (s, r]$ , then  $\delta(r, j] = \delta(s, j] + \delta(r, s] < 0$ . In each case, we obtain  $j \in D(r)$ . ■

The Strong Cycle Lemma can also be proved by constructing  $j_{i+1}$  explicitly from  $j_i$ , but that takes longer. As an application, we obtain a result of Chung and Feller that generalizes the Ballot Problem discussed earlier. We obtain the simple Ballot Problem by setting  $l = 0$  and interchanging A and B. The Chung-Feller proof used analytic methods.

**COROLLARY 5** (Chung-Feller [2]) Let  $l$  be an integer in  $\{0, \dots, n\}$ . In a random sequence of  $n$  A's and  $n$  B's, the probability is  $1/(n + 1)$  that there are exactly  $l$  choices of  $i$  such that the  $i$ th A precedes the  $i$ th B.

**Proof:** Given a sequence  $b$  of  $n$  A's and  $n$  B's, convert A's to 0's, B's to 1's, and append a 0 at the end; call this sequence  $b'$ . The sequence  $b'$  is a 0-linearization of an  $(n, n + 1)$ -arrangement  $a$ . Because  $n + 1$  and  $2n + 1$  are relatively prime, exactly  $n + 1$  sequences of  $n$  A's and  $n$  B's yield the same  $(n, n + 1)$ -arrangement. By the Strong Cycle Lemma, the  $n + 1$  0-linearizations of  $a$  have different numbers of 1-good 0-intervals. The  $i$ th B in  $b$  precedes the  $i$ th A in  $b$  if and only if the  $i$ th 0 in  $b'$  is a 1-good 0-interval. By grouping the sequences into sets of size  $n + 1$  yielding the same cyclic arrangement, we see that the number of sequences with exactly  $l$  values where the  $i$ th A precedes the  $i$ th B is independent of  $l$ . ■

We next extend the Strong Cycle Lemma by specifying an arbitrary set of 0's in a  $(k, qk + 1)$ -arrangement.

**LEMMA 6** (Stronger Cycle Lemma) If  $a$  is a  $(k, qk + 1)$ -arrangement,  $S$  is a set of  $t$  positions containing 0's, and  $1 \leq i \leq t$ , then there is a unique 0-linearization  $(j_i, j_i]$  of  $a$  such that  $j_i \in S$  and exactly  $i$  of the 0-intervals ending at elements of  $S$  are  $q$ -good. Furthermore, for  $i < t$ , the 0's that end  $q$ -good 0-intervals in  $(j_i, j_i]$  also end  $q$ -good 0-intervals in  $(j_{i+1}, j_{i+1}]$ .

**Proof:** Define deficiency as in Lemma 4, but let  $D(r) = \{j: \delta(r, j) < 0 \text{ and } j \in S\}$ . Since  $1 \leq D(r) \leq t$ , it suffices to show that the sets  $D(r)$  for  $r \in S$  are distinct and ordered by inclusion. Given  $r, s \in S$ , the proof of this is as in Lemma 4. ■

Viewing the elements of a  $(k, k + 1)$ -arrangement as exponents on  $-1$  yields a cyclic arrangement of  $k + 1$  positive 1's and  $k$  negative 1's. The ordinary Cycle Lemma provides a unique starting position such that all the partial sums are positive. Raney [13] proved more generally that every cyclic arrangement of integers summing to  $+1$  has a unique starting position such that all partial sums are positive. The Stronger Cycle Lemma for  $q = 1$  provides a short proof of a further generalization. In Figure 7, we indicate the number of positive partial sums in each successive linearization of a clockwise arrangement and underscore the positions that end positive partial sums.

			5	<u>2</u>	<u>-1</u>	<u>2</u>	-5	<u>3</u>	-2	1	-2	<u>3</u>	
			2	-1	<u>2</u>	-5	3	-2	1	-2	3	<u>2</u>	
	3	2	-1	3	<u>2</u>	-5	3	-2	1	-2	3	<u>2</u>	<u>-1</u>
	-2		2	1	-5	3	-2	1	-2	3	2	-1	<u>2</u>
				8	<u>3</u>	<u>-2</u>	<u>1</u>	-2	<u>3</u>	<u>2</u>	<u>-1</u>	<u>2</u>	<u>-5</u>
	1		-5	4	-2	<u>1</u>	-2	3	<u>2</u>	<u>-1</u>	<u>2</u>	-5	<u>3</u>
		-2	3	7	<u>1</u>	-2	<u>3</u>	<u>2</u>	<u>-1</u>	<u>2</u>	-5	<u>3</u>	<u>-2</u>
				6	-2	<u>3</u>	<u>2</u>	<u>-1</u>	<u>2</u>	-5	<u>3</u>	-2	<u>1</u>
				9	<u>3</u>	<u>2</u>	<u>-1</u>	<u>2</u>	<u>-5</u>	<u>3</u>	<u>-2</u>	<u>1</u>	<u>-2</u>

Figure 7. Positive partial sums in an arrangement summing to  $+1$ .

**COROLLARY 7** (Montágh [10]) Given a cyclic arrangement of  $n$  integers summing to  $+1$  and an integer  $l \in \{1, \dots, n\}$ , there is a unique linearization of the arrangement such that exactly  $l$  of the partial sums are positive.

**Proof:** Let  $b$  denote the arrangement of integers. Form a cyclic arrangement  $a$  of 1's and 0's by replacing each nonnegative integer  $b_i$  by a single 1 followed by  $1 + b_i$  consecutive 0's, and replacing each negative  $b_i$  by  $1 - b_i$  consecutive 1's followed by one 0. The resulting  $a$  is a  $(k, k + 1)$ -arrangement, where  $k = n + \lfloor \sum |b_i| / 2 \rfloor$ . Let  $S$  be the  $n$ -set of 0's that end maximal consecutive segments of 0's. The 0-linearizations ending in  $S$  correspond naturally to linearizations of  $b$ ; the number of positive partial sums in a linearization of  $b$  equals the number of 1-good 0-intervals ending in  $S$  in the corresponding 0-linearization of  $a$ . By Lemma 6, these numbers are distinct for the  $n$  linearizations ending in  $S$ . ■

In this application, the “nesting property” says that the numbers ending positive partial sums in the  $l - 1$ th arrangement also end positive partial sums in the  $l$ th arrangement.

Graham, Knuth, and Patashnik [7, p. 346] presented a geometric proof of Raney's original result, which upon closer examination also yields Montágh's generalization. (Der-showitz and Zaks [3] observed that the geometric approach can also be used to prove the

Cycle Lemma itself.) Encode the integer arrangement  $a_1, \dots, a_n$  as a walk in the plane, starting from the origin and moving  $(+1, +a_i)$  from the current position when the  $i$ th number is encountered. The ending position is  $(n, 1)$ . Figure 8 shows two periods of the walk for the sequence  $2, -1, 2, -5, 3, -2, 1, -2, 3$ .

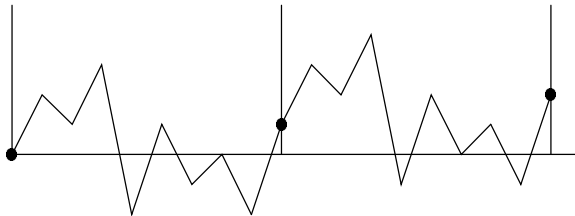


Figure 8. A geometric argument.

The unique starting position from which all the partial sums are positive follows the last occurrence of the minimum in the first period. All other positions have a non-positive partial sum ending at that position, but partial sums starting after it are positive. All but one partial sum is positive when we start after the previous occurrence of the minimum or, when the minimum is unique, after the last occurrence of the next smallest value. For each  $l < n$ , let  $b_l$  be the position in the first period from which  $l + 1$  partial sums are positive. Then  $b_{l-1}$  is obtained from  $b_l$  by moving to the previous occurrence of the same height as  $b_l$  or, if  $b_l$  is the first occurrence of that height, the last occurrence of the next larger height.

When  $n = (q + 1)k + 1$  and the sequence consists only of 1's and  $-q$ 's, summing to  $+1$  requires that exactly  $k$  terms equal  $-q$ . Following Graham, Knuth, and Patashnik (with a shift of index), we call such a sequence a  $q$ -Raney sequence if all the partial sums are positive (this requires the first term to be a 1). They prove that there are  $C_k^q$  such sequences. This follows immediately from Corollary 7 when  $l = n$ , since  $C_k^q$  equals the number of  $(k, qk + 1)$ -arrangements.

The Strong Cycle Lemma also yields a short direct proof that the number of  $q$ -ballot sequences with  $k$  ones is  $C_k^q$ . As before, prepending a 0 shows that these are equinumerous with the  $q$ -dominating  $(k, qk + 1)$ -sequences. By the Strong Cycle Lemma, the reverse  $a'$  of such a sequence  $a$  has a unique 0-linearization such that *no* 0-interval is  $q$ -good. The reverse of this 0-linearization is the unique cyclic permutation of  $a$  such that  $w_0(I) > qw_1(I)$  for every prefix  $I$  (whether ending at a 0 or a 1). Hence we conclude again that the  $q$ -ballot sequences of length  $(q + 1)k$  are equinumerous with the  $(k, qk + 1)$ -arrangements.

The result of Kierstead and Trotter, extended to the Strong Cycle Lemma, distinguishes the 0's of a  $(k, qk + 1)$ -arrangement in a combinatorial fashion. We close this section by presenting another combinatorial distinguishing of these 0's that extends to  $(k, l)$ -arrangements whenever  $k$  and  $l$  are relatively prime. In the case  $l = k + 1$ , it yields matchings different from the matchings of Kierstead and Trotter [9] between the middle levels of the lattice of subsets of a  $2k + 1$ -element set; further discussion appears in [4].

**THEOREM 8** (Snevily [15]) If  $k$  and  $l$  are relatively prime, then the position-sums of the 0-linearizations of a  $(k, l)$ -arrangement  $a$  belong to distinct congruence classes modulo  $l$ , where the *position-sum* of a 0,1-vector is the sum of the indices of its 1's.

**Proof:** We cycle through the 0-linearizations, decreasing the position-sum by  $k \bmod l$  for each successive 0-linearization. From one linearization of  $a$ , we move to the next by moving the bit in position 1 to position  $n = k + l$  and shifting each other bit down by one. If we have a 0-linearization with a 0 in position 1, then a single shift takes us to the next 0-linearization and decreases the position-sum by  $k$ . If the bit in position 1 is a 1, then we make additional shifts before moving the first 0 to the back. For each shift in which a 1 moves from the front to the back, we decrease the position by one for  $k - 1$  1's and increase it by  $k + l - 1$  for one 1. The net change in the position-sum is  $(k + l - 1) - (k - 1) = l$ . Thus this operation does not change the congruence class of the position sum. Only the last shift to reach the next 0-linearization changes the congruence class, again reducing it by  $k$  modulo  $l$ . ■

## 5. $(k, qk+p)$ -ARRANGEMENTS WITH $p > 1$

In light of our results about  $q$ -satisfying sequences of arbitrary lengths, it is natural to seek comparable extensions of the Strong Cycle Lemma to  $(k, qk + p)$ -arrangements. Unfortunately, when  $p > 1$  it is possible for complementary intervals to be  $q$ -good. The simple proof of the Strong Cycle Lemma used when  $p = 1$  thus fails in the general case, and generalizations for  $p > 1$  make weaker statements about  $q$ -good intervals. We mention two special cases of our final theorem: every 0-linearization of a  $(k, qk + p)$ -arrangement has at least  $p$  0-intervals that are  $q$ -good, and there are at least  $p$  0-linearizations in which every 0-interval is  $q$ -good.

**THEOREM 9** (Extended Strong Cycle Lemma) If  $a$  is a  $(k, qk + p)$ -arrangement and  $p \leq i \leq qk + p$ , then  $a$  has at least  $qk + 2p - i$  0-linearizations that have at least  $i$   $q$ -good 0-intervals.

**Proof:** The crux of the proof is the *augmentation property*: If  $b$  is a 0-linearization of a  $(k, l)$ -arrangement with  $l > qk$ , and  $b'$  is obtained from  $b$  by inserting a 0, then  $b'$  has more  $q$ -good 0-intervals than  $b$ . To prove this, we partition  $b$  as  $I_1 0 I_2 0 \cdots I_t 0$ , where the intervals  $I_j$  contain no ends of  $q$ -good 0-intervals, and the  $t$  elements indicated by 0's are the ends of the  $q$ -good 0-intervals (some of the  $I_j$ 's may be empty). The location of the first  $q$ -good 0-interval implies that  $w_0(I_1) = qw_1(I_1)$ . The location of each successive  $q$ -good 0-interval implies that  $w_0(I_j) = qw_1(I_j) - 1$  for each  $j > 1$ . To form  $b'$ , we insert a 0 in some  $I_r$ , obtaining  $I'_r$ . Each 0 ending a  $q$ -good 0-interval in  $b$  does so also in  $b'$ . In addition, the last 0 in  $I'_r$  (which may or may not be the added 0) also ends a  $q$ -good 0-interval in  $b'$ .

We now prove the theorem by induction on  $p$ . When  $p = 1$ , the desired statement is a weakening of the Strong Cycle Lemma. For  $p > 1$ , we begin by finding a 0-linearization in which every 0-interval is  $q$ -good. To do this, delete  $p - 1$  0's arbitrarily to obtain a  $(k, qk + 1)$ -arrangement  $\tilde{a}$ . By the Strong Cycle Lemma,  $\tilde{a}$  has a unique 0-linearization in which

every 0-interval is  $q$ -good. Each time we replace one of the deleted 0's, the augmentation property implies that again every 0-interval is  $q$ -good. After replacing all the deleted 0's, we have a 0-linearization  $b$  of  $a$  in which every 0-interval is  $q$ -good.

Let  $a'$  be the  $(k, qk + p - 1)$ -arrangement obtained by deleting the last element of  $b$  from  $a$ . Consider  $i$  such that  $p \leq i \leq qk + p$ ; we have  $p - 1 \leq i - 1 \leq qk + p - 1$  and  $qk + 2(p - 1) - (i - 1) = qk + 2p - i - 1$ . By the induction hypothesis,  $a'$  has at least  $qk + 2p - i - 1$  0-linearizations in which at least  $i - 1$  0-intervals are  $q$ -good. By the augmentation property, the replacement of the missing 0 converts these to 0-linearizations of  $a$  in which at least  $i$  0-intervals are  $q$ -good. Since every 0-interval in  $b$  is  $q$ -good,  $b$  provides the additional needed 0-linearization. ■

The extended Strong Cycle Lemma is best possible in the sense that all its lower bounds may hold with equality simultaneously. This is achieved by the  $(k, qk + p)$ -arrangement in which all the 1's appear together and all the 0's appear together, which has exactly  $p$  0-linearizations in which all 0-intervals are  $q$ -good and one 0-linearization in which exactly  $i$  0-intervals are  $q$ -good for each  $p \leq i < qk + p$ .

On the other hand, there may be more 0-linearizations with at least  $i$  0-intervals that are  $q$ -good than guaranteed by the extended Strong Cycle Lemma, so its inequalities cannot be replaced by equalities. When  $p = tk$ , consider the periodic  $(k, qk + p)$ -arrangement  $a$  in which each 1 is followed by a string of exactly  $q + t$  0's before the next 1. This arrangement has exactly  $q + t$  "types" of 0-linearizations. When the first 1 in a 0-linearization of  $a$  appears after position  $q$ , every one of the  $(q + t)k$  0-intervals is  $q$ -good; there are  $(t + 1)k$  such 0-linearizations. This already is  $k$  more than guaranteed by the Lemma, so the guarantee is exceeded when the desired number of  $q$ -good 0-intervals is  $i > (q - 1)k + p$ . When the first 1 appears in position  $j + 1$  for some  $0 \leq j \leq q$ , the number of 0-intervals that are not  $q$ -good is  $\sum_{i=0}^r q - j - it$ , where  $r = \min\{k - 1, \lfloor (q - j)/t \rfloor\}$ . There are  $k$  such 0-linearizations for each  $j$ . When  $k > 1$ , in this class of  $(k, qk + p)$ -arrangements every 0-linearization has more than  $p$  0-intervals that are  $q$ -good.

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