New lower bounds for matching numbers of general and bipartite graphs

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Abstract

Let $\alpha'(G)$ denote the maximum size of a matching in a graph G, and let $\delta(G)$ denote the minimum vertex degree. Classical lower bounds are $\alpha'(G) \ge \min\{\delta(G), \lfloor n/2 \rfloor\}$ when G has n vertices and $\alpha'(G) \ge \min\{2\delta(G), |X|, |Y|\}$ when G is a bipartite graph with partite sets X and Y. Let $d_1, \ldots, d_n$ be the vertex degrees in nonincreasing order. We strengthen both bounds by proving $\alpha'(G) \ge \max_k \min\{d_{n-k}, \lfloor (n-k)/2 \rfloor\}$ when G has n vertices and $\alpha'(G) \ge \max_k \min\{2d_{n-k-1}, |X| - k, |Y| - k\}$ when G is bipartite.

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A *matching* in a graph G is a set of edges such that no two have a common vertex. Finding the maximum size of a matching in G, written $\alpha'(G)$, is a fundamental optimization problem. Various lower bounds on $\alpha'(G)$ in terms of the minimum degree are standard exercises in graph theory.

Theorem 1. $\alpha'(G) \ge \min\{\delta(G), \lfloor n/2 \rfloor\}$ *when G is an n-vertex graph.*

Brandt [1] attributes this result to Erdős and Pósa [3]. Another well-known result is that a graph G contains every tree with at most $\delta(G)$ edges. Brandt [1] proved a common generalization of these facts, showing that G contains every forest having at most $\delta(G)$ edges and at most $|V(G)|$ vertices. We will strengthen Theorem 1 in a different direction.

Theorem 1 has a simple bipartite analogue. An X, Y*-bigraph* is a bipartite graph with partite sets X and Y.

Theorem 2. *If G is an X, Y-bigraph, then* $\alpha'(G) \ge \min\{2\delta(G), |X|, |Y|\}$.

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Theorem 2 has a simple direct proof: If a maximum matching M is smaller, then it omits vertices $x \in X$ and $y \in Y$. These vertices must have as neighbors both endpoints of some edge in M (since $|M| < 2\delta(G)$ and the set of vertices not covered by M is independent), yielding an augmenting path of length 3. Theorem 1 has a similar short proof.

Confirming a conjecture made by the computer program Graffiti, DeLaViña and Gramajo [2] strengthened Theorem 2 as follows.

Theorem 3 ([2])**.** *Let G be a connected X,Y-bigraph. If $d_1, \ldots, d_n$ are the vertex degrees in nonincreasing order, then $\alpha'(G) \geq \min\{2d_{n-1}, |X|, |Y|\}$.*

In this short note we further strengthen Theorem 3. Also, we first prove an analogous strengthening of Theorem 1.

Theorem 4. *Let G be an n-vertex graph. If $d_1, \ldots, d_n$ are the vertex degrees in nonincreasing order, then $\alpha'(G) \geq \max_k \min\{d_{n-k}, \lfloor (n-k)/2 \rfloor\}$.*

Proof. We prove that $\alpha'(G) \geq \min\{d_{n-k}, \lfloor n/2 \rfloor\}$ for $0 \leq k < n$. Suppose this fails for some k, so $\alpha'(G) < d_{n-k}$. Let A be the set of vertices covered by a maximum matching M. Vertices outside A are independent. If for some $x, y \in V(G) - A$ there are three edges joining $\{x, y\}$ to the endpoints of one edge of M, then there is a larger matching.

Hence we may assume $d(x) + d(y) \leq 2\alpha'(G)$ whenever $x, y \in V(G) - A$. Therefore all vertices outside A except possibly one have degree at most $\alpha'(G)$. Since $\alpha'(G) < d_{n-k}$, at most k vertices have degree at most $\alpha'(G)$. Thus $n - |A| \leq k + 1$, so $\alpha'(G) \geq (n-k-1)/2$.

We conclude that $\alpha'(G) < d_{n-k}$ yields $\alpha'(G) \geq \lceil n-k-1 \rceil /2\}$, since $\alpha'(G)$ is an integer. Finally, $\lceil (n-k-1)/2 \rceil = \lfloor (n-k)/2 \rfloor$. □

Setting $k = 0$ yields Theorem 1 as a special case. Similarly, setting $k = 0$ in the next theorem yields Theorem 2. We slightly generalize and simplify the approach in [2]; the proof expands on the short argument given above for Theorem 2 and has the same spirit as the proof of Theorem 4.

Theorem 5. *Let G be a connected X,Y-bigraph. If $d_1, \ldots, d_n$ are the vertex degrees in nonincreasing order, then $\alpha'(G) \geq \max_k \min\{2d_{n-k-1}, |X| - k, |Y| - k\}$.*

Proof. Suppose that $\alpha'(G) < \min\{2d_{n-k-1}, |X|-k, |Y|-k\}$ for some k. Let M be a maximum matching in G. Let X' and Y' be the subsets of X and Y covered by M. The hypothesis implies $X - X' \neq \varnothing$ and $Y - Y' \neq \varnothing$. Since M is maximal, no edges join $X - X'$ and $Y - Y'$. If there exist $u \in X - X'$ and $v \in Y - Y'$ with $d(u) + d(v) > \alpha'(G)$, then by the pigeonhole principle some edge of M has endpoints in $N(u)$ and $N(v)$, and G has an M-augmenting path from u to v. Hence no such u and v exist.

Hence by symmetry in X and Y we may assume that vertices of $X - X'$ have degree at most $\frac{\alpha'(G)}{2}$. Since $\alpha'(G) < 2d_{n-k-1}$, there are at most $k+1$ such vertices. Hence $|X - X'| \leq k + 1$. Since $|X'| = \alpha'(G)$, assuming $\alpha'(G) < |X| - k$ yields $|X - X'| = k + 1$.

Now $d(u) \le \frac{\alpha'(G)}{2}$ for $u \in X - X'$ and $d(v) > \frac{\alpha'(G)}{2}$ for $v \notin X - X'$. Choose $u \in X - X'$. Since G is connected, u has a neighbor y in Y'. Let xy be the edge of M covering y. Choose $v \in Y - Y'$. Now $d(x), d(v) > \frac{\alpha'(G)}{2}$. Since $N(x) \subseteq Y'$ and $N(v) \subseteq X'$, there now exists an edge of M whose endpoints lie in $N(x)$ and $N(v)$. This yields an M-augmenting path of length 5 from u to v (through $u, y, x, N(x), N(v), v$), which is a contradiction. □

References

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