Degree-associated Reconstruction of Graphs

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slides and papers on preprint page

Joint work with

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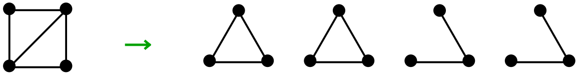
The Classical Problem

Def. A card of a graph G is an induced subgraph $G - v$. The deck of a graph is the multiset of its cards.



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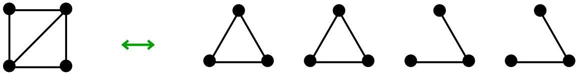

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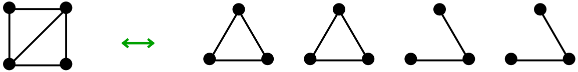

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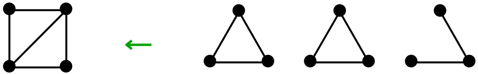

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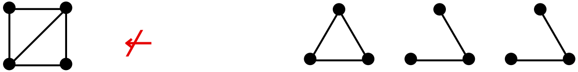

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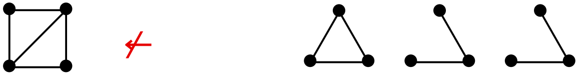

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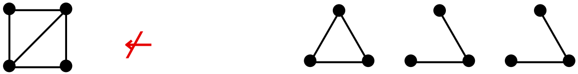

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- $\mathrm{rn}(G) = 3$ for all trees with ≥ 5 vertices (Myrvold ['90]).

Degree-associated Reconstruction

Obs. $|E(G)| = \frac{\sum_v |E(G-v)|}{n-2}$ when G has n vertices.

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Obs. Always $\mathrm{drn}(G) \le \mathrm{rn}(G)$.

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Ex. $\mathrm{rn}(tK_p) = p + 2$ (Myrvold [1989]),
but $\mathrm{drn}(tK_p) = 3$ (Ramachandran [2006]).

Results (Barrus–West [2010])

- **Random Graphs:** Almost always $\text{drn}(G) = 2$.

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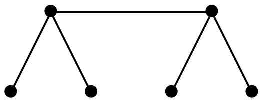
- **Trees:** If G is a caterpillar, then $\text{drn}(G) = 2$ unless G is a star or the graph $B_{2,2}$ below:



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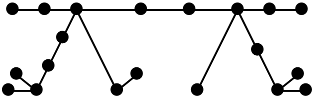

- **Stronger:** (Spinoza [2014+]) If G arises from a caterpillar by edge-subdivisions and then growing toes (pendant edges) at leaves, then $\mathrm{drn}(G) \le 2$.

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- Among vertex-transitive graphs, always $\alpha\text{drn}(G)=\text{drn}(G)$, so $\alpha\text{drn}(2K_{n/4,n/4})=(n/4)+2$.

Other Results on $\alpha\text{drn}(G)$

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- Always rn, drn, and αdrn are the same for G and $\overline{G}$, but not so for the edge versions ern, dern, and αdern.

Small Values

Thm. (Barrus–West [2010]) A single dacard (C, d) determines a graph only in the following cases.

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Cor. Almost always $\mathrm{rn}(G) = \alpha\mathrm{rn}(G) = 3$ and $\mathrm{drn}(G) = \alpha\mathrm{drn}(G) = 2$.

Regular Graphs

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$K_{m,m-1} + (t-2)K_{m,m} + K_{m,m+1}$ has $m+1$ such dacards. ■

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Ex. These conditions hold for the Petersen graph, hypercubes (dimension ≥ 3), and both $K_n \square K_2$ and $C_n \square K_2$ (for $n \geq 4$), but not for $C_3 \square K_2$.

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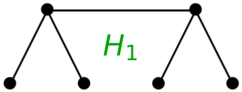

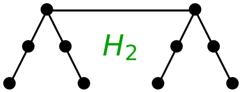

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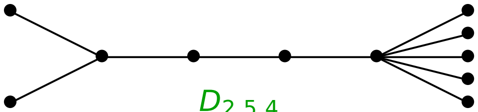

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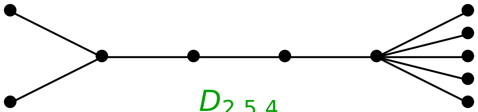

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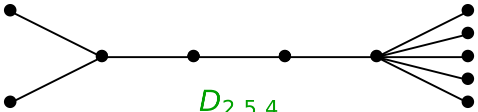

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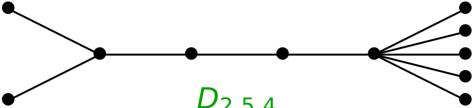

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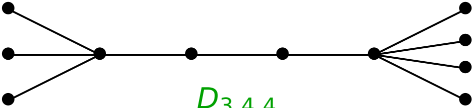

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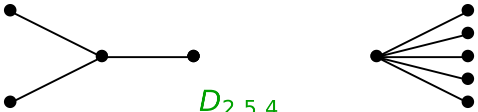

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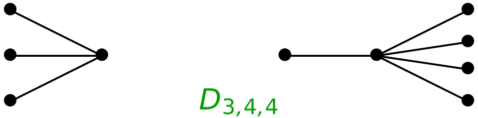

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When $p = 4$, another common dacard!

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Since $D_{m-1,n,p}$ (and G) has a vertex of degree $n + 1$, the missing leaf added to $D_{m,n-1,p}$ must be made adjacent to the vertex of degree n in $D_{m,n-1,p}$. ■

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