RAMSEY THEORY AND BANDWIDTH OF GRAPHS

Zoltán Füredi[†]

Mathematical Institute of the Hungarian Academy of Science
and *University of Illinois, z-furedi@math.uiuc.edu*

Douglas B. West

University of Illinois, west@math.uiuc.edu

Abstract. The bandwidth of a graph is the minimum, over vertex labelings with distinct integers, of the maximum difference between labels on adjacent vertices. Kuang and McDiarmid proved that almost all n-vertex graphs have bandwidth $n-(2+\sqrt{2}+o(1))\lg n$. Thus the sum of the bandwidths of a graph and its complement is almost always at least $2n-(4+2\sqrt{2}+o(1))\lg n$; we prove that it is always at most $2n-4\log_2 n+o(\log n)$. The proofs involve improving the bounds on the Ramsey and Turán numbers of the "halfgraph".

1. THE PROBLEM

When the vertices of a graph are labeled injectively with integers, the *dilation* of an edge is the difference between the labels on its endpoints. The *bandwidth* $B(G)$ of the graph G is the minimum, over all such labelings, of the maximum edge dilation.

Chinn, Chung, Erdős, and Graham [3] investigated the sum $B(G)+B(\overline{G})$, where $\overline{G}$ denotes the complement of G. They proved that $B(G)+B(\overline{G})\geq n-2$ whenever G has n vertices (for $n\geq 4$). Equality holds when G is a 4-vertex path. They also established the existence of constants c_1,c_2 such that $B(G)+B(\overline{G})<2n-c_1\log n$ for every n-vertex G and $B(G)+B(\overline{G})>2n-c_2\log n$ for almost every n-vertex G.

Kuang and McDiarmid [17] improved the constant c_2. They proved that for the random graph generated with fixed edge probability p, almost all graphs have bandwidth $n-(2+\sqrt{2}+o(1))\log_{1/(1-p)} n$, where n denotes the number of vertices. With $p=1/2$, the complement of a random graph is also a random graph, and we obtain $c_2<4+2\sqrt{2}+\epsilon$ for any $\epsilon>0$ and sufficiently large n.

In this paper we increase the constant c_1 in the upper bound, proving that

---

[†]Research supported in part by the Hungarian National Science Foundation under grant OTKA 016389 and by the National Security Agency under grant MDA904-98-I-0022.
Running head: BANDWIDTH SUM
AMS codes: 05C35, 05C78, 05C80, 05D10
Keywords: bandwidth, Ramsey number, Turán number, halfgraph, random graph.
Completed August 28, 1998. Revised November 24, 1999.

THEOREM 1. If $f(n)$ is the maximum of $B(G)+B(\overline{G})$ over n-vertex graphs, then

$$2n-\left\lceil(4+2\sqrt{2})\log_2 n\right\rceil \le f(n) \le 2n-4\log_2 n+o(\log n).$$

2. RELATED EXTREMAL GRAPH PROBLEMS

The bandwidth problem can be expressed in terms of other classical extremal graph problems involving the occurrence of fixed subgraphs. We use $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G. We use $[n]$ to denote the set of the first n positive integers and P_n to denote the graph that is an n-vertex path.

The *halfgraph* H_r is the graph with $2r$ vertices defined by $V(H_r)=[2r]$ and $E(H_r)=\{ij\colon\ j-i\ge r\}$. When we let $a_i=i$ for $1\le i\le r$ and $b_j=r+j$ for $1\le j\le r$, we obtain the more common representation $E(H_r)=\{a_ib_j\colon\ i\le j\}$, which shows that H_r is bipartite.

The first definition of H_r leads to bandwidth via the observation that $\overline{H}_r=P_{2r}^{r-1}$, where P_{2r}^k has vertex set $[2r]$ and edge set $\{ij\colon\ 1\le j-i\le k\}$ (this is the kth *power* of P_{2r}). The bandwidth of P_n^k is k. In general, the bandwidth of an n-vertex graph is at most k if and only if the graph is contained in P_n^k, since the set of edges permitted in an optimal labeling forms P_n^k. For $n\ge 2r\ge 2$, this observation becomes

$$B(G)\le n-r-1 \text{ if and only if } H_r\subset\overline{G}. \tag{2.1}$$

Since the appearance of H_r requires $n\ge 2r$, one direction needs no restriction on n.

Thus $B(G)$ is large if and only if $\overline{G}$ contains no large halfgraph, and $B(\overline{G})$ is large if and only if G contains no large halfgraph. Studying the maximum of $B(G)+B(\overline{G})$ amounts to studying what combinations of halfgraphs must appear in G and $\overline{G}$; this is a problem of Ramsey graph theory.

Given graphs A,B, the *Ramsey number* $R(A,B)$ is the smallest integer n such that every red/blue-coloring of the edges of K_n yields a copy of A in red or a copy of B in blue. When $n<R(A,B)$, the clique K_n can be decomposed into a subgraph avoiding A and a subgraph avoiding B. When A and B are cliques, the numbers $R(k,l)=R(K_k,K_l)$ are the classical Ramsey numbers. The general definitions and results we use from Ramsey Theory appear in [14].

PROPOSITION 2. If $R(H_a,H_b)>n$ with $n\ge\max\{2a,2b\}$, then $f(n)\ge 2n-a-b$. If $f(n)\ge 2n-k$, then there exist a,b with sum k such that $R(H_a,H_b)>n$.

Proof: If $R(H_a,H_b)>n$, then there is an n-vertex graph G such that $H_a\not\subseteq\overline{G}$ and $H_b\not\subseteq G$. When $n\ge\max\{2a,2b\}$, (2.1) and the latter statement yield both $B(G)\ge n-a$ and $B(\overline{G})\ge n-b$. Thus $f(n)\ge 2n-a-b$.

If $f(n)\ge 2n-k$, then there exists an n-vertex graph G such that $B(G)+B(\overline{G})\ge 2n-k$. This requires the existence of a,b with sum k such that $B(G)\ge n-a$ and $B(\overline{G})\ge n-b$. By (2.1), this requires both $H_a\not\subseteq\overline{G}$ and $H_b\not\subseteq G$, which yields $R(H_a,H_b)>n$. ∎

For completeness, we sketch the proof of the lower bound on $f(n)$. We have noted that if $B(G)\ge n-r$ almost always, then $f(n)\ge 2n-2r$. We have $B(G)\ge n-r$ if and only if $H_r \not\subseteq \overline{G}$. Since H_r has $2r$ vertices and $r(r+1)/2$ edges, the standard Erdős-Rényi [4] random graph model with edge probability 1/2 yields $P(H_r \subseteq \overline{G}) \le n^{2r}2^{-r^2/2}$ (see [3, 18]). When $r = \lceil 4\log_2 n \rceil$, the probability is less than by 1/2, and the bound $f(n) \ge 2n - \lceil 4\log_2 n \rceil$ follows.

Considering the densest subgraph of H_r yields a better bound. Let $H_{r,m}$ be the subgraph obtained by discarding the $2m$ vertices of least degree (m from each partite set). Presence of H_r requires presence of $H_{r,m}$. Since $H_{r,m}$ has $2r - 2m$ vertices and $r(r+1)/2 - m(m+1)$ edges, in the random n-vertex graph G we have

$$P(H_r \subseteq \overline{G}) \le P(H_{r,m} \subseteq \overline{G}) \le n^{2r-2m}2^{-(r^2-2m^2)/2}.$$

The densest subgraph and best bound are obtained by setting $m = (1 - \sqrt{2}/2)r$. When $r = \left\lceil (2+\sqrt{2})\log_2 n \right\rceil$, the resulting bound on $P(H_r \subseteq \overline{G})$ is less than 1/2, which yields the lower bound in Theorem 1. Using the second moment method, Kuang and McDiarmid [17] obtained the precise threshold for the appearance of H_r, thus showing that $B(G) = (2 + \sqrt{2} + o(1))\log_2 n$ almost always.

In terms of Ramsey numbers, the bound on $P(H_r \subseteq \overline{G})$ yields the following:

THEOREM 3. $R(H_r, H_r) > 2^{(r+1)/(2+\sqrt{2})}$. ∎

Proposition 2 implies that $f(n) = 2n - \min\{a+b\colon R(H_a, H_b) > n\}$ if $n \ge \max\{2a, 2b\}$ for a pair (a, b) where the minimum occurs. When we set $r = \left\lceil (2+\sqrt{2})\log_2 n \right\rceil - 1$, Theorem 3 yields $R(H_r, H_r) > n$. When $n \ge 34$, we have $n \ge 2\left\lceil (2+\sqrt{2})\log_2 n \right\rceil - 2$, and then Proposition 2 implies that $f(n) \ge 2n - 2\left\lceil (2+\sqrt{2})\log_2 n \right\rceil + 2$ when $n \ge 34$. With Theorem 3, we also have

COROLLARY 4. For $n \ge 34$, $f(n) = 2n - \min\{a+b\colon R(H_a, H_b) > n\}$. ∎

We henceforth assume that $n \ge 34$ (this threshold can be reduced by improving the lower bounds on Ramsey numbers for small halfgraphs).

Upper bounds on $R(H_a, H_b)$ that depend only on $a + b$ yield upper bounds on $f(n)$. An easy bound comes from the classical Erdős-Szekeres (see [14]) upper bound on Ramsey numbers: $R(p, q) \le \binom{p+q-2}{p-1} < 2^{p+q}$. This yields

$$R(H_a, H_b) \le R(2a, 2b) < 4^{a+b}. \tag{2.2}$$

By Proposition 2 and (2.2), we have $f(n) \le 2n - \frac{1}{2}\log_2 n$. We want to increase the coefficient on $\log_2 n$ to 4. To this end, we define the function g by

$$g(k) = \max_{a+b \le k} R(H_a, H_b). \tag{2.3}$$

Corollary 4 tells us that $g(k) \le n$ yields $f(n) < 2n - k$.

Let $\mathrm{ex}(n, F)$ denote the maximum number of edges in an F-free graph with n vertices. This is often called the *Turán number* of F. Kővári, T. Sós and Turán [16] showed that

$$\mathrm{ex}(n, K_{k,k}) \le \frac{1}{2}(k-1)^{1/k}n^{2-1/k} + \frac{1}{2}(k-1)n. \tag{2.4}$$

Let G be an n-vertex graph. If $\mathrm{ex}(n, K_{a,a}) + \mathrm{ex}(n, K_{b,b}) < \binom{n}{2}$, then G contains $K_{a,a}$ or $\overline{G}$ contains $K_{b,b}$. Since $H_r \subset K_{r,r}$, this yields $R(H_a, H_b) \le n$. A careful examination of (2.4) shows that $k = 2\log_2 n - O(\log_2\log_2 n)$ is small enough to yield $g(k) \le n$, and thus $f(n) < 2n - (2 - o(1))\log_2 n$. This technique of using the Turán number of a bipartite graph to prove an upper bound on its Ramsey number is now standard in extremal graph theory (see [10], for example).

To further improve this upper bound, we need a tighter bound on the Turán number of H_r. Observe that H_r is contained not only in $K_{r,r}$ but even in the subgraph obtained by removing $K_{\lfloor r/2\rfloor,\lfloor r/2\rfloor}$ from $K_{r,r}$. Due to its similarity to the halfgraph, we use the notation $H'(r,l)$ to denote the subgraph of the biclique $K_{r,r}$ obtained by deleting the edges of the biclique $K_{l,l}$. In the next section, we obtain an upper bound on $\mathrm{ex}(n, H'(r,l))$ implying that $g(k) \le n$ for k as large as about $4\log_2 n$. This yields the desired upper bound $f(n) < 2n - 4\log_2 n + o(\log_2 n)$.

3. TURÁN NUMBERS AND THE HALFGRAPH

Before proving our upper bound on $\mathrm{ex}(n, H'(r,l))$ in Theorem 6, we compare it with earlier results.

For every bipartite graph F that is not a forest, there is a positive constant $c(F)$ such that $\Omega(n^{1+c}) \le \mathrm{ex}(n,F) \le O(n^{2-c})$; this was observed by Erdős [unpublished] and appears in [16]. For a given F, the first problem in studying $\mathrm{ex}(n,F)$ is thus to find the right exponent (if such exists). It is conjectured ([7,9]) that $\mathrm{ex}(n, K_{k,k}) = \Theta(n^{2-1/k})$. This was proved for $K_{2,2}$ in Erdős-Rényi-Sós [5] and (simultaneously and independently) in Brown [2]. For $K_{3,3}$ it appears in Brown [2]. For $K_{k,l}$ with $k > l!$, results appear in Kollár-Rónyai-Szabó [15], later improved for $k > (l-1)!$ in Alon-Rónyai-Szabó [1].

Erdős [6] also proved that when $r > l \ge 1$, there exists a constant $c_{r,l}$ such that $\mathrm{ex}(n, H'(r,l)) < c_{r,l} n^{2-1/(r-l)}$. His proof is somewhat complicated and does not give a sufficiently good constant in the range where we need (when both r and l are about $\log_2 n$).

Another upper bound for $\mathrm{ex}(n, H'(r,l))$ follows from the method of Erdős and Simonovits [9]. They proved that $\mathrm{ex}(n,Q) \le O(n^{8/5})$, where Q is the 8-vertex 3-dimensional cube. Similarly, the correct order of growth for $\mathrm{ex}(n, H'(r,l))$ follows from the main result in Füredi [13]. However, in all of these articles the authors concentrated on large values of n compared to k. In the range we need, these results seem not to imply our bound.

In the proof, we extend $\binom{x}{t}$ to nonnegative real x for each nonnegative integer t. When $t = 0$, we take $\binom{x}{t} = 1$ for all real $x \ge 0$. When $t \ge 1$, we take $\binom{x}{t} = 0$ for $0 \le x < t-1$, and for $x \ge t-1$ we view $\binom{x}{t}$ as a real polynomial $x(x-1)\dots(x-t+1)/t!$ of degree t in x. The resulting functions are convex. Thus the mean of values of this function at several points is at least the value at the mean argument. In particular,

$$\sum_{i=1}^{m} \binom{x_i}{t} \ge m \binom{\Sigma x_i/m}{t}. \tag{3.1}$$

We will also use the following simple lemma, which was proved and applied in [12] to a related problem.

LEMMA 5 ([12]). If $v, t \ge 1$ are integers and $c, x_0, x_1, \ldots, x_t \ge 0$ are real numbers, then

$$\sum_{1\le i\le v} \binom{x_i}{t} \le c\binom{x_0}{t} \quad \text{implies} \quad \sum_{1\le i\le v} x_i \le x_0 c^{1/t} v^{1-1/t} + (t-1)v\,. \qquad \blacksquare$$

THEOREM 6. $\mathrm{ex}(n, H'(r,l)) \le \frac{1}{2}(r+l-1)^{1/(r-l)} n^{2-1/(r-l)} + \frac{1}{2}(r-l-1)n$.

Proof: Let G be an n-vertex graph not containing $H'(r,l)$; we bound $e = |E(G)|$. Let $d(x)$ denote the degree of a vertex x, and for $A \subset V(G)$ let $d(A)$ denote the number of common neighbors of A.

Let $t = r - l$, and let X be the number of copies of $K(t,t)$ in G. We form such a subgraph by choosing a t-set A and choosing t of its common neighbors. Each copy arises twice. Thus $X = \frac{1}{2}\sum_{|A|=t}\binom{d(A)}{t}$.

We first find a lower bound on X. By (3.1),

$$\frac{1}{2}\sum_{|A|=t}\binom{d(A)}{t} \ge \frac{1}{2}\binom{n}{t}\binom{\sum d(A)/\binom{n}{t}}{t}.$$

Since $d(A)$ counts the stars with leaf set A, the total $\sum d(A)$ is the number of stars with t edges in G. These can alternatively be counted by choosing t neighbors for each choice of the central vertex. Applying (3.1) to the resulting sum yields

$$\sum d(A) = \sum_{x\in V(G)}\binom{d(x)}{t} \ge n\binom{\sum d(x)/n}{t}.$$

Together, these computations yield

$$X \ge \frac{1}{2}\binom{n}{t}\binom{n\binom{2e/n}{t}/\binom{n}{t}}{t}. \tag{3.2}$$

We next find an upper bound on X. Let $\mathcal{A} = \{A \in \binom{V(G)}{t} \colon d(A) < r\}$. Consider copies of $K_{t,t}$ in which at least one of the partite sets belongs to $\mathcal{A}$. The number of these is at most $|\mathcal{A}|\binom{r-1}{t}$.

Now consider a copy of $K_{t,t}$ with partite sets A, B such that $A, B \notin \mathcal{A}$. Our main observation is that the prohibition of $H'(r,l)$ yields $d(A) \le 2r - t - 1$. If $d(A) \ge r + r - t = 2l + t$, then A has at least $2l$ common neighbors outside B, and after avoiding (at most) l of the r common neighbors of B there remain at least l common neighbors of A to complete a copy of $H'(r,l)$. The same argument applies to $d(B)$. Thus each such copy of $K_{t,t}$ is generated twice when we arbitrarily choose t common neighbors of a t-set outside $\mathcal{A}$. Thus the number of copies of $K_{t,t}$ with neither partite set in $\mathcal{A}$ is bounded by $\frac{1}{2}\left(\binom{n}{t} - |\mathcal{A}|\right)\binom{2r-t-1}{t}$.

Together, the two upper bounds yield

$$X \le \binom{n}{t}\max\left\{\binom{r-1}{t}, \frac{1}{2}\binom{2r-t-1}{t}\right\}. \tag{3.3}$$

Since the ratio of $\frac{1}{2}\binom{2r-t-1}{t}$ to $\binom{r-1}{t}$ is exactly $\binom{r-1+(r-t)}{t-1}/\binom{r-1}{t-1}$, always the second term in the maximization is larger.

Comparing (3.2) and (3.3) yields

$$n\binom{2e/n}{t}/\binom{n}{t} \le 2r-t-1.$$

Let $v = x_0 = n$, let $x_1 = \ldots = x_v = 2e/n$, and let $c = 2r-t-1$. Lemma 5 now yields

$$2e \le (2r-t-1)^{1/t} n^{2-1/t} + (t-1)n. \qquad \blacksquare$$

4. PROOF OF THE UPPER BOUND

The pair (r,r) is one instance of (a,b) such that $a+b=2r$, and thus $R(H_r,H_r) \le g(2r)$, where g is as defined in (2.3). Theorem 6 enables us to prove an upper bound on $g(2r)$ and thus an upper bound on $R(H_r,H_r)$ that differs from the lower bound in Theorem 3 by a factor of 2 in the exponent.

COROLLARY 7. If r is sufficiently large, then $R(H_r,H_r) \le g(2r) \le (3r+1)2^{r/2}$.
Proof: By the definition of $g(2r)$, there exist a,b with $a+b=2r$ such that $g(2r) = R(H_a,H_b)$. Let G be an n-vertex graph such that $H_a \not\subseteq G$ and $H_b \not\subseteq \overline{G}$. It suffices to show that $n < (3r+1)2^{r/2}$, when r is sufficiently large.

Because $H_a \not\subseteq G$ and $H_a \subset H'(a,\lfloor a/2 \rfloor)$, Theorem 6 yields

$$e(G) \le \mathrm{ex}(n,H_a) \le \mathrm{ex}(n,H'(a,\lfloor a/2\rfloor)) \le \frac{n^2}{2}\left(\frac{a+\lfloor\frac{a}{2}\rfloor-1}{n}\right)^{1/\lceil\frac{a}{2}\rceil} + \frac{n}{2}(\lceil\frac{a}{2}\rceil-1).$$

The right side yields the bound

$$e(G) < \frac{n^2}{2}\left(\frac{3r}{n}\right)^{2/a} + \frac{na}{4}. \tag{4.1}$$

Summing (4.1) and its analogue for $\overline{G}$ yields

$$\binom{n}{2} = e(G) + e(\overline{G}) \le \frac{n^2}{2}\left(\frac{3r}{n}\right)^{2/a} + \frac{na}{4} + \frac{n^2}{2}\left(\frac{3r}{n}\right)^{2/b} + \frac{nb}{4}.$$

This simplifies to

$$1 - \frac{r+1}{n} < c^{2/a} + c^{2/b}, \tag{4.2}$$

where $c = 3r/n$.

We may assume that $r \le 2\log_2(n/3r)$, since otherwise the desired inequality holds. Since $c<1$, differentiating the function f defined by $f(x) = c^{2/x}$ shows that f is concave for $x > -\ln c$. If $\min\{a,b\} > \ln(n/3r)$, then concavity implies that the upper bound in (4.2) is at most $2c^{4/(a+b)}$. We obtain $1 - \frac{r+1}{n} < 2\left(\frac{3r}{n}\right)^{2/r}$, which rearranges to yield $n < 3r2^{r/2}(\frac{n}{n-r-1})^{r/2} < (3r+1)2^{r/2}$.

It remains only to eliminate the case $a = \min\{a, b\} < \ln(n/3r)$. We obtain a contradiction to (4.2) by showing that the right side is too small. Since $c^{2/x}$ is a monotone increasing function, we obtain an upper bound by using $a \le \ln(n/3r)$ and $b < 2r$. We also use our restriction to $r \le 2\log_2(n/3r)$. These yield

$$c^{2/b} < c^{2/2r} = e^{-(1/r)\ln(n/3r)} = e^{-(1/r)\log_2(n/3r)\ln 2} \le e^{-(1/2)\ln 2} = \frac{1}{\sqrt{2}} < .7072$$

$$c^{2/a} < c^{2/\ln(n/3r)} = e^{-2\ln(n/3r)/\ln(n/3r)} = e^{-2} < .1354$$

When $n \ge (3r+1)2^{r/2}$, this contradicts (4.2) for $r \ge 3$, which eliminates this case and completes the proof. ■

As observed in Section 2, $g(k) \le n$ yields $f(n) < 2n - k$. Thus Corollary 7 yields the desired upper bound $f(n) < 2n - 4\log_2 n + O(\log_2\log_2 n)$.

5. CONCLUSION AND OPEN PROBLEMS

Our upper and lower bounds for $R(H_r, H_r)$ are much closer together than the best known upper and lower bounds for $R(K_r, K_r)$. One might expect that tighter bounds on $R(H_r, H_r)$ are hopeless without improving the bounds for $R(K_r, K_r)$. It would be interesting to find a direct relationship between these two functions.

Our method can be generalized for decompositions of K_n into k edge-disjoint n-vertex graphs, with k fixed. Over such decompositions, $\max(B(G_1) + B(G_2) + \ldots + B(G_k)) = kn - \Theta(\log n)$. It would be interesting to narrow the gap in the coefficient of $\log n$.

If we can allow the number of pieces in the decomposition to grow arbitrarily with n, then the maximum sum is $\binom{n}{2}$, achieved by decomposition into $\binom{n}{2}$ individual edges. The maximum can reach $O(n^2)$ when the number of pieces is linear, which is not surprising given the result for fixed k. For example, when we decompose K_n into stars of sizes 1 through $n-1$, the sum of the bandwidths is $\lceil \frac{n}{2} \rceil \lceil \frac{n+1}{2} \rceil$. How small can k be in terms of n to achieve various growth rates for the bandwidth sum?

When $l = r-1$, the graph $H'(r,l)$ is a double star with two adjacent vertices of degree r. It is easy to see that $\mathrm{ex}(n, H'(r, r-1)) = (r-1)(n-r+1)$ (for $r > r_0$), so in this case the bound in Theorem 6 is asymptotically optimal within a factor of 2. For $r - l = 2$, Theorem 6 gives

$$\mathrm{ex}(n, H'(r, r-2)) \le \frac{1}{2}(2r-3)^{1/2}n^{3/2} + O(n).$$

The best lower bound from [13] gives

$$\mathrm{ex}(n, H'(r, r-2)) \ge \mathrm{ex}(n, K_{2,r}) = (1 + o(1))\frac{1}{2}(r-1)^{1/2}n^{3/2},$$

so in this case Theorem 6 is asymptotically optimal within a factor of $\sqrt{2}$. It would be interesting to find the correct asymptotic behavior for each $r - l$.

It is easy to see that $\mathrm{ex}(n, H_3) = \mathrm{ex}(n, K_{2,2}) + O(1)$; we expect that equality holds for sufficiently large n. For larger fixed r, we conjecture that $\mathrm{ex}(n, H_r) \sim \mathrm{ex}(n, K_{\lceil r/2 \rceil, \lceil r/2 \rceil})$. In other words, is the Turán number of H_r about the same as the Turán number of its largest complete bipartite subgraph?

References

[1] N. Alon, L. Rónyai, and T. Szabó, Norm-graphs: variations and applications, *J. Comb. Th. Ser. B* **76** (1999), 280–290.

[2] W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9** (1966), 281–289.

[3] P. Z. Chinn, F. R. K. Chung, P. Erdős, and R. L. Graham, On the bandwidths of a graph and its complement, *The Theory and Appl. of Graphs*, 4th Conf., Kalamazoo, Mich. 1980. (1981), pp. 243–253.

[4] P. Erdős and A. Rényi, On the evolution of random graphs, *Magyar Tud. Akad. Mat. Kut. Int. Közl.* **5** (1960), 17–61. Also see [8], 574–617.

[5] P. Erdős, A. Rényi, and V. T. Sós, On a problem of graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 215–235. Also see [8], 201–221.

[6] P. Erdős, On an extremal problem in graph theory, *Colloq. Math.* **13** (1965), 251–254. Also see [8], 182–185.

[7] P. Erdős, Some recent results on extremal problems in graph theory, *Theory of Graphs* (Internat. Sympos., Rome, 1966), Gordon and Breach, New York, 1967, 117–130. Also see [8], 222–228.

[8] P. Erdős, *The Art of Counting*, Selected writings of P. Erdős, (J. Spencer ed.), The MIT Press, Cambridge, 1973.

[9] P. Erdős and M. Simonovits, Some extremal problems in graph theory, *Combinatorial Theory and its Appl. I (Proc. Colloq., Balatonfüred, 1969)*, North Holland, Amsterdam, 1970, pp. 377–390. Also see [8], 246–259.

[10] R. J. Faudree and M. Simonovits, Ramsey problems and their connection to Turán-type extremal problems, *J. Graph Theory* **16** (1992), 25–50.

[11] Z. Füredi, On a Turán type problem of Erdős, *Combinatorica* **11** (1991), 75–79.

[12] Z. Füredi, An upper bound on Zarankiewicz' problem, *Combinatorics, Probability and Computing* **5** (1996), 29–33.

[13] Z. Füredi, New asymptotics for bipartite Turán numbers. *Journal of Combinatorial Theory, Ser. A* **75** (1996), 141–144.

[14] R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey Theory*, second edition, (Wiley-Interscience, 1990).

[15] J. Kollár, L. Rónyai, and T. Szabó, Norm-graphs and bipartite Turán numbers, *Combinatorica* **16** (1996), 399–406.

[16] T. Kővári, V. T. Sós, and P. Turán, On a problem of K. Zarankiewicz, *Colloq. Math.* **3** (1954), 50–57.

[17] Y. Kuang and C. McDiarmid, On the bandwidth of a random graph, *Ars Combinatoria* **20A** (1985), 29–36.

[18] J. Turner, Probabilistic analysis of bandwidth minimization algorithms, *Proc. 15th*

ACM Symp. Thy. Comput. (1983), 467–476.