Suppose γ_n is a sequence such that $\gamma_{2n} \to \alpha$, $\gamma_{2n+1} \to \beta$. Show that

$$\frac{1}{n}\sum_{j=1}^{n}\frac{j}{j+1}\gamma_j \to \frac{1}{2}(\alpha+\beta).$$

Solution by Artin Boghossian, University of Petroleum and Minerals, Dhahran, Saudi Arabia. Let $x_j = (j/(j+1))\gamma_j$ and $y_j = (x_j + x_{j+1})/2$ and observe the identity

$$\frac{1}{n}\sum_{j=1}^{n}\frac{j}{j+1}\gamma_j = \frac{1}{n}\sum_{j=1}^{n}x_j = (x_1 - x_{n+1})/(2n) + \frac{1}{n}\sum_{j=1}^{n}y_j.$$

Since the x_n are bounded, $(x_1 - x_{n+1})/(2n) \to 0$. It follows from our assumptions that $y_{2j} \to (\alpha+\beta)/2$ and $y_{2j+1} \to (\alpha+\beta)/2$; therefore, $y_j \to (\alpha+\beta)/2$. But a theorem of Cauchy says that if $a_j \to a$, then $(1/n)\Sigma_{j=1}^{n}a_j \to a$. An application of this result to the sequence y_j completes the proof.

Editorial note. Many of those who submitted solutions remarked that the factor $j/(j+1)$ is superfluous. For any sequence

$$b_j \to b, \qquad \frac{1}{n}\sum_{j=1}^{n}b_j\gamma_j \to \frac{1}{2}(b\alpha + b\beta);$$

just replace the sequence x_j by $\hat{x}_j = b_j\gamma_j$ and replace α and β by $\hat{\alpha} = b\alpha$ and $\hat{\beta} = b\beta$ in the argument above.

Also solved by 45 other readers and the proposers.

Bandwidth of the Integer Simplex

E 3003 [1983, 400]. *Proposed by J. L. Brenner, Palo Alto, California.*

The 20 balls in a 4-layered pyramid (of balls) are labeled with the integers 1–20 (each integer being used). The maximum of the difference between the labels of two kissing balls is the "discrepancy." Find the *minimum possible discrepancy* (= bandwidth) under all 20! labelings. *What is the bandwidth for the pyramid of $k(k+1)(k+2)/6$ balls in k layers?

Solution by Douglas B. West, University of Illinois, Urbana, IL. For $k = 4$ the answer is 8, and in general $\left(\binom{k+2}{3} - 1\right)/(k-1)$ is a lower bound and $\lceil (k+3)/2 \rceil \lfloor (k+1)/2 \rfloor$ is an upper bound.

More generally, consider the problem of labeling the n vertices of a graph G with labels $1, \ldots, n$, attempting to minimize the maximum difference between adjacent labels. This minimum, over all possible labelings, is called the *bandwidth* $b(G)$; the term comes from the bandwidth of the adjacency matrix. The problem posed is to compute the bandwidth of the "adjacency graph" of the k-layered pyramid, which we denote G_k. The graph G_k can be described as follows. The vertices correspond to non-negative integer strings of length four whose sum is $k-1$; i.e., they can be viewed as integer points in a simplex in three dimensions. Two vertices x, y are adjacent if y is obtained from x by subtracting 1 from one coordinate and adding 1 to another. By this encoding the number of vertices is $\binom{k+2}{3}$, since the number of these strings is the number of compositions of $k-1$ with four parts. The vertices encoded using three 0's and $k-1$ are called the *extremes* of G_k.

For any connected graph G, the *distance* $d(v, w)$ between two vertices v, w is the number of edges in the shortest path between them, and the *diameter* $d(G)$ is the maximum distance between vertices. If G has n vertices, then $b(G) \geqslant (n-1)/d(G)$, since the shortest path between the vertices labeled 1 and n requires a step this large. For the integer simplex graph, $d(G_k) = k-1$; the distance between extremes is $k-1$, and other pairs belong to smaller simplexes and thus have shorter distance (in fact, $d(v, w) = k - 1 - (1/2)\Sigma\min\{v_i, w_i\}$). Hence $\left(\binom{k+2}{3} - 1\right)/(k-1)$ is a lower bound for $b(G_k)$. This yields 3, 5, 7 for $k = 2, 3, 4$, and in general a quadratic lower bound with leading coefficient $1/6$.

For upper bounds we need constructions. For $k = 2, 3$ it is easy to achieve the lower bound, and for $k = 4$ it is not too hard to find a labeling with maximum difference 8, such as that appearing below.

	$f(2010) = 5$	$f(1020) = 11$	
$f(3000) = 1$	$f(2001) = 6$	$f(1011) = 12$	$f(0030) = 17$
$f(2100) = 2$	$f(1110) = 7$	$f(1002) = 13$	$f(0021) = 18$
$f(1200) = 3$	$f(1101) = 8$	$f(0120) = 14$	$f(0012) = 19$
$f(0300) = 4$	$f(0210) = 9$	$f(0111) = 15$	$f(0003) = 20$
	$f(0201) = 10$	$f(0102) = 16$	

The presentation above suggests a general construction for an upper bound. Define a vertex to have *Type* j if the sum of its first two coordinates is j. Type i and Type j vertices can be adjacent only if $|i-j| \leqslant 1$. Hence if we assign labels in order to Type 1 vertices, then Type 2 vertices, and so on, the maximum difference between adjacent vertices is less than the maximum number of vertices of two consecutive Types. Since there are $(i+1)(k-i)$ vertices of Type i, this gives an upper bound whose leading term is $k^2/2$. With a bit more care, we can save another factor of 2. Within each Type, assign the labels to vertices in lexicographic order of their encoding. Then the largest neighbor of a vertex with non-zero first coordinate is obtained by shifting 1 from the first coordinate to the last. The resulting vertex has approximately the same position in the next Type; in particular, one cannot have adjacent vertices in the beginning of one Type and the end of the next. The resulting adjacent vertices with largest difference in labels are $(1, \lceil (k-3)/2 \rceil, 0, \lfloor (k-1)/2 \rfloor)$ and $(0, \lceil (k-3)/2 \rceil, 0, \lfloor (k+1)/2 \rfloor)$, and the difference in their labels is $\lceil (k+3)/2 \rceil \lfloor (k+1)/2 \rfloor - \varepsilon$, where $\varepsilon = 0$ if k is even and $\varepsilon = 1$ if k is odd.

Now we return to G_4. The lexicographic labeling above has two pairs of adjacent labels differing by 8, namely (7, 15) and (8, 16). Although there exist labelings with only one adjacent pair differing by 8, there is none with every adjacent pair differing by at most 7. In other words, $b(G_4) = 8$. To prove this, suppose that G_4 has a vertex labeling f with maximum difference 7; we will obtain a contradiction.

We want to show we may assume that f assigns labels in order by Types. Within distance two of labels 1 and 2 we have at most the 14 labels $2,\ldots,15$ and the 15 labels $\{1,\ldots,16\} - \{2\}$, respectively; similarly for labels 20 and 19. Let us compute how many vertices are within distance two for each of the three types of vertices (permutations of 3000, 2100, and 1110). Note that $d(v, w) = 3$ if and only if v, w are non-zero on disjoint sets of coordinates. Since there are $\binom{r+i-1}{i-1}$ compositions of r restricted to i coordinates, these vertices have 10, 4, 1 vertices at distance three, respectively, or 9, 15, 18 at distance one or two. Hence $f^{-1}(1)$ and $f^{-1}(20)$ must be extremes; we may assume they are 3000 and 0003, respectively. Similarly, $f^{-1}(2)$ and $f^{-1}(19)$ must be extremes or permutations of 2100. If $f^{-1}(2)$ is not an extreme, then the 15 vertices within distance two of $f^{-1}(2)$ must include $f^{-1}(1)$. By symmetry in the central coordinates, we may thus assume $f^{-1}(2) \in \{0300, 2100, 1200\}$ and $f^{-1}(19) \in \{0030, 0012, 0021\}$.

Now, vertices labeled 17, 18, 19, 20 must have distance three from both $f^{-1}(1)$ and $f^{-1}(2)$; hence these must be the four Type 0 vertices $\{0012, 0021, 0030, 0003\}$ that are non-zero only where both $f^{-1}(1)$ and $f^{-1}(2)$ are zero. Similarly, 1, 2, 3, 4 label the Type 3 vertices $\{3000, 0300, 2100, 0012\}$. Furthermore, if v is Type 2, then v has two Type 3 neighbors. Therefore, v has a neighbor w with $f(w) \leqslant 3$, and hence $f(v) \leqslant 10$. Similarly $f(v) \geqslant 11$ if v has Type 1, so Type 2 vertices have the labels $5,\ldots,10$ and Type 1 vertices have the labels $11,\ldots,16$.

Next we restrict the values of $f^{-1}(16)$ and $f^{-1}(5)$. A Type 1 vertex v of the form $xy11$ has four neighbors of Type 2; hence it has a neighbor w with $f(w) \leqslant 7$. This implies $f(v) \leqslant 14$, so $f^{-1}(15)$ and $f^{-1}(16)$ must have the form $xy02$ or $xy20$. Since $d(f^{-1}(1), f^{-1}(16)) = 3$, we have $f^{-1}(16) \in \{0102, 0120\}$. In either case, $f^{-1}(16)$ has two neighbors of Type 2 (these are $\{1101, 0201\}$ or $\{1110, 0210\}$), which must have labels 9, 10. Either of these adjacent to $f^{-1}(2)$ must have label 9. Thus the choices for locating 2, 16 determine the locations for 9, 10; the four possible locations for 2, 9, 10, 16 are (0300, 0201, 1101, 0102), (0300, 0210, 1110, 0120), (2100, 1101, 0201, 0102), and (2100, 1110, 0210, 0120). Note that $f^{-1}(2) = 1200$ is no longer possible, since 1200 is adjacent to all potential Type 2 neighbors of $f^{-1}(16)$. By the complementary argument, the possible locations for 19, 12, 11, 5 are (0030, 1020, 1011, 2010), (0030, 0120, 0111, 0210), (0012, 1011, 1020, 2010), and (0012, 0111, 0120, 0210).

We have shown $f^{-1}(16) \in \{0102, 0120\}$ and $f^{-1}(5) \in \{2010, 0210\}$. By symmetry, we have three cases for $(f^{-1}(16), f^{-1}(5))$: (0120, 0210), (0102, 2010), and (0102, 0210). The first yields adjacent labels with difference 11. The second also falls quickly. Recalling the analysis above, $f(0102) = 16$ forces $\{f^{-1}(9), f^{-1}(10)\} = \{0201, 1101\}$ and $f(2010) = 5$ forces $\{f^{-1}(12), f^{-1}(11)\} = \{1020, 1011\}$. Because 15 and 6 must label a Type 2 and Type 1 vertex with one coordinate two, the remaining options are $f^{-1}(15) \in \{1002, 0120\}$ and $f^{-1}(6) \in \{2001, 0210\}$. Since these labels cannot be adjacent, we may assume by symmetry that $f(1002) = 15$ and $f(0210) = 6$. Unfortunately, the labels 13, 14 must go on the remaining Type 1 vertices 0120 and

0111, and 0210 is adjacent to both of them. (Putting 7, 8, 13, 14 on 1110, 2001, 0120, 0111 leads to a labeling with only one adjacent pair differing by 8.)

We are left with the possibility $f(0102) = 16$, $f(0210) = 5$. Consider 1011; its Type 2 neighbors are {1110, 1101, 2001, 2010}, which have labels in $\{6, \ldots, 10\}$. Because $f(0102) = 16$ forces $\{f^{-1}(9), f^{-1}(10)\} = \{0201, 1101\}$, one of {9, 10} does not appear on a neighbor of 1011, and 6 does appear. Hence $f(1011) \leqslant 13$, and equality must hold because $f^{-1}(5) = 0210$ implies $\{f^{-1}(12), f^{-1}(11)\} = \{0120, 0111\}$.

The fact that $f^{-1}(13)$ is not a neighbor of $f^{-1}(20)$ forces $d(f^{-1}(6), f^{-1}(20)) = 3$; i.e., $f^{-1}(6)$ has fourth coordinate 0. Also, $f(1110) > 6$, because 1110 has four Type 1 neighbors. This leaves $f^{-1}(6) = 2010$. The only remaining choices for the label on vertex 1020 are 14 and 15. Unfortunately, 1020 is adjacent to $2010 = f^{-1}(6)$, which completes the contradiction. (The labeling can be completed in eight ways with (2010, 1020) being the only adjacent pair differing by 8.)

Editorial comment. The problem of computing the bandwidth of an arbitrary input graph is "hard", in the sense that it is NP-complete (even if the graphs are restricted to be trees!). Nevertheless, this does not eliminate the possibility that there is a formula for the bandwidths of the graphs G_k. Indeed, it is possible that the lexicographic labeling is optimal.

A labeling of G_4 with maximum difference 8 was given jointly by D. P. Mehendale and M. R. Modak, and also by the proposer.

Regressing in L_1

E 3079 [1985, 215]. *Proposed by James Chew, North Carolina Agricultural and Technical State University.*

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three points in R^2, no two of which have the same x-coordinate. Say that $y = Ax + B$ is a *least absolute-value line* if the function

$$g(a, b) = \sum_{i=1}^{3} |ax_i + b - y_i|$$

attains a minimum at $(a, b) = (A, B)$. Must a least absolute-value line pass through two of the three points?

Solution I by O. P. Lossers, Eindhoven University of Technology, The Netherlands. The answer is yes.

Denote the point (x_i, y_i) by P_i and assume $x_1 < x_2 < x_3$. Each of the lines $y = ax + b$ divides the plane into an upper half plane $U_{a,b}$ and a lower half plane $L_{a,b}$. If a is fixed, $g(a, b)$ is a decreasing function of b when $U_{a,b}$ contains two or more of the points P_i in its interior, but $g(a, b)$ is an increasing function of b when $L_{a,b}$ contains two or more of the points P_i in its interior. Therefore, for fixed a, the