

# BANDWIDTH AND DENSITY FOR BLOCK GRAPHS

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**Abstract.** The *bandwidth* of a graph  $G$  is the minimum of the maximum difference between adjacent labels when the vertices have distinct integer labels. We provide a polynomial algorithm to produce an optimal bandwidth labeling for graphs in a special class of block graphs (graphs in which every block is a clique), namely those where deleting the vertices of degree one produces a path of cliques. The result is best possible in various ways. Furthermore, for two classes of graphs that are “almost” caterpillars, the bandwidth problem is NP-complete.

## 1. INTRODUCTION

The *bandwidth problem* for a graph asks for a linear layout to minimize stretching of edges (see [10] for a VLSI circuit layout application). The *bandwidth* of an injection  $f: V(G) \rightarrow \mathbb{Z}$  is  $B(f) = \max_{uv \in E(G)} |f(u) - f(v)|$ . The *bandwidth*  $B(G)$  of a graph  $G$  is  $\min B(f)$  over all such injections; a numbering achieving the minimum is *optimal*. Surveys on bandwidth include [2] and [3].

Let  $n(G) = |V(G)|$ . Every numbering of  $G$  uses two labels differing by at least  $n(G) - 1$ , and the two corresponding vertices are connected by a path of length at most  $\text{diam } G$ ; thus  $B(G) \geq (n(G) - 1)/\text{diam } G$ . Considering all subgraphs, the *local density* is  $\beta(G) = \max_{H \subseteq G} [(n(H) - 1)/\text{diam } H]$ . Since every numbering of  $G$  includes a numbering of each subgraph,  $B(G) \geq \beta(G)$  (see Chung [3]). The local density bound is optimal for cliques, stars, and trees of diameter 3 (“double stars”). Sysło and Zak [17] and Miller [11] extended this to *caterpillars*, the trees in which deleting the vertices of degree one produces a path.

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Their proofs construct optimal numberings in polynomial time. This was further extended by Assmann, Peck, Sysłó, and Zak to *2-caterpillars*. (A  $k$ -caterpillar is a tree formed from a path by growing edge-disjoint paths of lengths at most  $k$  from its vertices. Commonly called “caterpillars with hairs of length at most  $k$ ”, these are not caterpillars when  $k \geq 2$ ). Among trees, bandwidth has also been computed for complete  $k$ -ary trees [15].

We extend the caterpillar result. A graph is a *block graph* if every block is a clique. This name arises because a graph  $G$  is the intersection graph of the blocks of some graph if and only if every block of  $G$  is a clique [8]. A *block path* is a block graph with  $k$  cutvertices and  $k + 1$  blocks in which the cutvertices induce a path. A *block caterpillar* is a block graph in which deleting the leaves (1-valent vertices) produces a block path. Fig. 1 illustrates a block caterpillar; the ellipses represent blocks that are cliques. We provide an algorithm to construct optimal numberings (with bandwidth  $\beta(G)$ ) for block caterpillars. Note that 2-caterpillars are not generally block caterpillars. We also demonstrate that the local density bound need not be optimal for block graphs of diameter 3 or for trees of diameter 4.

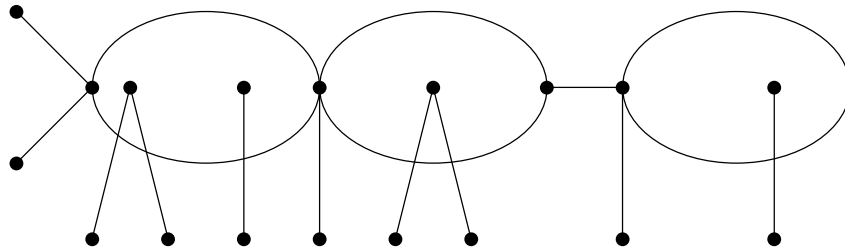


Fig. 1. Sketch of a block caterpillar

Computing bandwidth is NP-complete [14], even for trees with maximum degree 3 [4]; hence the interest in special classes. Slightly enlarging the classes of 2-caterpillars or block caterpillars yields classes on which bandwidth is NP-complete. Monien [12] proved that bandwidth is NP-complete for 3-caterpillars, although he needs paths of length 3 only at one vertex of the central path. We prove NP-completeness for two additional classes. One class consists of block graphs obtained from special block caterpillars by adding paths of length two from one vertex of the central path. The other class consists of trees that are almost caterpillars; they have a path containing all non-leaf vertices except one.

Because these trees are tolerance graphs, we conclude that bandwidth is NP-complete for tolerance graphs, answering a question posed by Kleitman. A graph is a *tolerance graph* if it is possible to assign each vertex  $v$  an interval  $I_v = [a_v, b_v]$  and a tolerance  $t_v$  such that vertices  $x, y$  are adjacent if and only if  $I_x \cap I_y$  has length at least  $\min\{t_x, t_y\}$ . The class of tolerance graphs (introduced in [6] and [7]) contains the class of interval graphs, on which there are polynomial time algorithms for bandwidth [9,13,16]. (Interval graphs are the tolerance graphs representable using the same tolerance for all vertices; more simply, they are the intersection graphs of families of real intervals.)

## 2. EXAMPLES

Before proving the main result, we exhibit examples where bandwidth does not equal local density. Equality holds for all trees of diameter 3, which are caterpillars, but this

does not extend to block graphs with diameter 3 or to trees with diameter 4.

**THEOREM 1.** There are block graphs of diameter 3 for which the bandwidth exceeds the local density bound.

**Proof:** Consider the block graph  $H_k$  with four blocks illustrated in Fig. 2. Three of the blocks are disjoint cliques  $X, Y, Z$  of order  $k$ . The fourth consists of  $x \in X, y \in Y, z \in Z$ , and one additional vertex  $w$  not in the other cliques. The largest subgraphs of  $H_k$  having diameter  $d$  have  $k, k+3, 3k+1$  vertices for  $d = 1, 2, 3$ , respectively (if  $k \geq 3$ ), so  $\beta(H_k) = k$  when  $k \geq 3$ . (For  $k = 2$  the graph is a block caterpillar, and  $\beta(H_2) = 3$ ).

Suppose that  $B(H_k) = k$ . We may assume that the optimal labeling  $f$  uses labels  $\{0, \dots, 3k\}$ . The distance between  $f^{-1}(0)$  and  $f^{-1}(3k)$  must be 3, so we may assume that  $f^{-1}(0) \in X$  and  $f^{-1}(3k) \in Z$ . Hence  $f(x) = k$  and  $f(z) = 2k$ . Since every vertex is within distance 2 of  $w$ , we have  $k \leq f(w) \leq 2k$ . Since  $|X - x| = k - 1$  and  $|Z - z| = k - 1$ , we must now have distinct vertices in  $Y$  with labels less than  $k$  and greater than  $2k$ . This yields adjacent vertices whose labels differ by more than  $k$ . (Note:  $w$  is needed in this construction, since  $B(H_k - w) = k$ .) ■

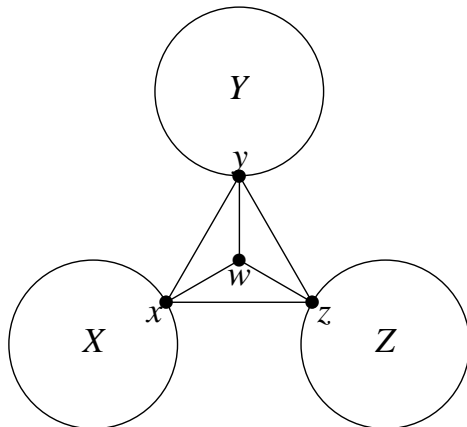


Fig. 2. The block graph  $H_k$  of diameter 3

**THEOREM 2.** There are trees of diameter 4 for which the bandwidth exceeds the local density bound.

**Proof:** Consider the tree  $T_k$  of diameter 4 illustrated in Fig. 3. Sets  $X, Y, Z$  each consist of  $k - 1$  leaves;  $W$  consists of  $k$  leaves. Sets  $X, Y, Z, W$  are adjacent to  $x, y, z, w$ , respectively, and the tree is completed by making  $w$  adjacent to  $\{x, y, z\}$ . The tree has  $4k + 1$  vertices and diameter 4, with the vertices of  $X, Y, Z$  being peripheral. The local density bound is  $k$  if  $k \geq 2$ , produced only by the full tree. (When  $k = 1$ , the tree is a star.)

Suppose that  $B(H_k) = k$ . We may assume that the optimal labeling  $f$  uses labels  $\{0, \dots, 4k\}$ . The distance between  $f^{-1}(0)$  and  $f^{-1}(4k)$  must be 4, so we may assume that  $f^{-1}(0) \in X$  and  $f^{-1}(4k) \in Z$ . Hence  $f(x) = k, f(w) = 2k$ , and  $f(z) = 3k$ . Since

$|X| = k - 1$  and  $|Z| = k - 1$ , the set  $Y \cup W$  has distinct vertices with labels less than  $k$  and greater than  $3k$ . Neither of these can be in  $W$ , since such labels differ by more than  $k$  from  $f(w)$ . This yields vertices of  $Y$  at distance 2 whose labels differ by more than  $2k$ . ■

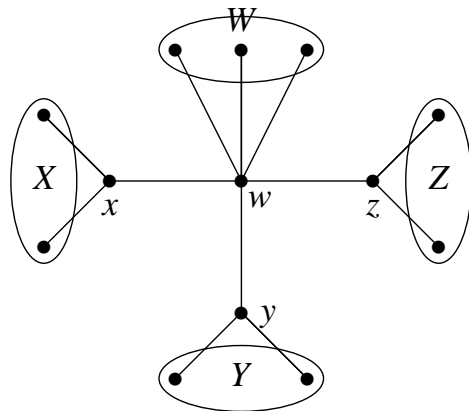


Fig. 3. The tree  $T_k$  of diameter 4

### 3. HUSIMI CATERPILLARS

We now construct optimal numberings of block caterpillars. We view the assignment  $f$  of distinct numbers to vertices as a placement of vertices in distinct positions; the *position* of  $x$  is  $f(x)$ . Our algorithm constructs a numbering with minimum bandwidth, but it generally does not assign consecutive numbers. Condensing the vertices to consecutive positions afterwards does not increase edge differences. An  $m$ -*representation* of a graph (or subgraph) is a numbering such that adjacent numbers differ by at most  $m$ . We use  $N(S) = \bigcup_{x \in S} N(x)$  to denote the set of vertices having a neighbor in  $S$ . A numbering  $f$  is *faithful* if  $f(x) < f(y)$  implies  $f(u) < f(v)$  whenever  $u, v$  are leaves adjacent to  $x, y$ , respectively. We begin with two elementary statements.

**LEMMA 1.** If a graph  $G$  has an  $m$ -representation, then  $G$  has a faithful  $m$ -representation.

**Proof:** When two leaves are mis-ordered in an  $m$ -representation, switching them decreases the maximum difference on their incident edges but changes no other edge difference. ■

We henceforth consider only faithful numberings. A faithful numbering of a block graph is determined by specifying the order and position of the non-leaves and the set of positions occupied by the leaves.

**LEMMA 2.** Suppose  $G$  is a block graph in which  $X$  is a set of vertices having leaf neighbors, and  $L$  is the set of their leaf neighbors. If  $X$  occupies consecutive positions  $\alpha, \dots, \beta$  in a faithful numbering, and the positions of  $L$  are between  $\alpha - m$  and  $\beta + m$ , then the differences on edges incident to  $L$  are at most  $m$ .

**Proof:** Let  $X = \{x_i\}$ , indexed by  $f(x_i) = i$  with  $\alpha \leq i \leq \beta$ . Let  $L = \{y_j\}$ , indexed by increasing position, with  $1 \leq j \leq |L|$ . Let  $u(j) = i$  if  $x_i$  is the neighbor of  $y_j$ . By faithfulness,  $u(1) = \alpha$  and  $u(|L|) = \beta$  and  $u(j) - u(j-1) \in \{0, 1\}$ . Hence  $f(y_j) - f(x_{u(j)})$  is a nondecreasing function bounded below by  $-m$  and above by  $m$ . ■

For a given block caterpillar  $G$ , let  $Q_1, \dots, Q_k$  be the consecutive blocks of the block path obtained by deleting the leaves of  $G$  (in the graph of Fig. 1,  $k = 4$ ). Let  $Q = \cup V(Q_i)$ . For  $v \in Q$ , let  $L(v)$  denote the set of leaves (in  $G$ ) adjacent to  $v$ , and let  $l(v) = |L(v)|$ .

We first select special vertices  $\{v_i: 0 \leq i \leq k+2\}$ . If  $k \geq 2$  and  $1 < i \leq k$ , let  $v_i$  be the shared vertex between  $Q_{i-1}$  and  $Q_i$ . We may assume that  $Q_1$  contains a cutvertex  $v_1$  of  $G$  other than  $v_2$ . Otherwise, adding a leaf  $x$  adjacent to a non-cutvertex of  $Q_1$  yields a graph  $G'$  such that each  $H' \subseteq G'$  containing  $x$  has order and diameter one larger than  $H' - x \subseteq G$ . Thus  $\beta(G') = \beta(G)$ , and it suffices to study  $G'$  instead of  $G$ . Similarly, we may assume that  $Q_k$  contains a cutvertex  $v_{k+1}$  of  $G$  other than  $v_k$ . Select  $v_0 \in L(v_1)$  and  $v_{k+2} \in L(v_{k+1})$ .

By the same reasoning, if  $k = 1$  and  $|Q_1| \geq 2$ , we may assume existence of two cutvertices  $v_1, v_2$  with leaf neighbors  $v_0, v_3$ , respectively. For the degenerate case where  $G$  is a star, we let  $k = 0$  and set  $v_1$  to be the center and  $v_0, v_2$  to be arbitrary leaves. In all cases, set  $Q_0 = \{v_0, v_1\}$  and  $Q_{k+1} = \{v_{k+1}, v_{k+2}\}$ .

We will construct an  $m$ -representation of  $G$  such that  $f(v_i) = im$  for  $0 \leq i \leq k+2$ . This requires putting vertices of  $Q_i$  in positions  $\{im, \dots, (i+1)m\}$  and leaves adjacent to them in positions  $\{(i-1)m+1, \dots, (i+2)m-1\}$  (except  $\{v_0, v_{k+2}\}$ ). We impose additional special properties on the representation to facilitate the inductive argument.

**DEFINITION 1.** Let  $J_i = \{im+1, \dots, (i+1)m-1\}$ . For a block caterpillar  $G$  with distinguished vertices  $v_0, \dots, v_{k+2}$  as defined above, a *left-justified*  $m$ -representation is a faithful  $m$ -representation  $f$  such that the following properties hold for  $0 \leq i \leq k+1$ :

- 0)  $f(v_i) = im$ . (Also  $f(v_{k+2}) = (k+2)m$ .)
- 1) all filled positions in  $J_i$  precede all unfilled positions in  $J_i$ .
- 2) if  $J_i$  is not full, then  $f(N(Q_i)) \cap J_{i+1} = \emptyset$ .
- 3) all positions for  $Q_i - \{v_{i+1}\}$  precede all positions for  $L(v_{i+1})$ .

We construct a left-justified  $m$ -representation of  $G$  iteratively. The  $i$ th phase produces a left-justified  $m$ -representation of the graph  $G_i$  consisting of all edges incident to vertices of  $Q_1 \cup \dots \cup Q_i$ . The iteration uses the explicit algorithm for  $k = 1$ , so we present this as a lemma. A block caterpillar is a *clique-star* if the graph obtained by deleting all leaves is a clique; this corresponds to  $k = 1$  in the description of  $G$  as a block caterpillar. When numbering vertices, we say that an edge is *satisfied* if its endpoints are at most  $m$  apart. We use  $d(v)$  to denote the degree of a vertex  $v$  (number of incident edges).

**LEMMA 3.** Every clique-star  $G$  with local density at most  $m$  has a left-justified  $m$ -representation.

**Proof:** With vertices named as above, let  $X = \{x_0, \dots, x_t\}$  be the vertices of  $Q$  having leaf neighbors, with  $x_0 = v_1$  and  $x_t = v_2$ , and let  $Q' = Q - \{v_2\}$  and  $X' = X - \{v_2\}$ . Let

$f(v_i) = im$  for  $0 \leq i \leq 3$ . We will assign positions so that  $f(x_0) < \dots < f(x_t)$  and place the leaves faithfully in positions reserved for them.

Let  $l' = \sum_{v \in X'} l(v)$ ,  $N = n(G) - 1$ , and  $q = |Q'|$ . If  $l' \leq m$ , reserve positions  $0, \dots, l' - 1$  for leaves, and assign consecutive positions beginning with  $m$  in order to  $X'$ , then  $Q - X$ , then  $L(v_2) - \{v_3\}$ , skipping  $2m$  (assigned to  $v_2$ ) if  $q + l(v_2) > m$ . Lemma 2 and  $d(v_2) \leq 2m$  imply that every edge is satisfied, and by construction the left-justification condition holds.

If  $l' > m$ , let  $r$  be the least index such that  $\sum_{j=0}^r l(x_j) \geq m$ , and let  $p = \sum_{j=0}^{r-1} l(x_j)$ , so  $p + l(x_r) \geq m$ . Reserve all of  $J_0$  for leaves. Notice that  $r < t$ , by the definition of  $l'$ . We consider two cases. Each construction fills positions other than  $\{im\}$  from the left, and the left-justification condition holds.

If  $p + l(x_r) + q \leq 2m$ , assign positions for vertices of  $Q'$  as follows: put  $x_0, \dots, x_r$  at  $m, \dots, m + r$ , followed by  $Q - X$ , and put  $x_{r+1}, \dots, x_{t-1}$  at  $s - (t - r - 1), \dots, s - 1$ , where  $s = \min\{N, 2m\}$ . Place the remaining leaves in the lowest positions not yet filled. Since  $m \leq p + l(x_r) \leq 2m - q$ , applying Lemma 2 separately to  $x_0, \dots, x_r$  and to  $x_{r+1}, \dots, x_{t-1}$  guarantees that all edges are satisfied.

Finally, suppose that  $p + l(x_r) + q > 2m$ . This and  $p < m$  force  $x_r$  to have a leaf neighbor both below  $m$  and above  $2m$  (see Fig. 4). Place  $x_0, \dots, x_{r-1}$  at  $m, \dots, m + r - 1$ , and place  $x_{r+1}, \dots, x_{t-1}$  at  $2m - t + r + 1, \dots, 2m - 1$ . Above  $2m$ , reserve the next  $N - 2m - 1$  positions for leaves. Reserve the  $m + t - 1$  positions  $m + r, \dots, 2m - t + r$  for  $\{x_r\} \cup (Q - X)$  and  $m - q$  vertices of  $L(x_r)$ . If we can place  $x_r$  in this range to satisfy the edges to its extreme leaf neighbors, then Lemma 2 applied separately to  $x_0, \dots, x_{r-1}$  and to  $x_{r+1}, \dots, x_t$  guarantees that the other edges are satisfied and completes the proof.

To place  $x_r$ , observe that the lowest position in  $L(x_r)$  is  $p$  and the highest is  $N - p'$ , where  $p' = \sum_{i=r+1}^t l(x_i)$ . Hence we need  $\max\{m + r, N - p' - m\} \leq f(x_r) \leq \min\{2m - t + r, p + m\}$ , which requires four inequalities. Since  $t \leq q \leq m$ , we have  $m + r \leq 2m - t + r$ . Since each vertex in  $\{x_0, \dots, x_{r-1}\}$  has at least one leaf neighbor, we have  $m + r \leq p + m$ . The inequality  $N - p' - m \leq 2m - t + r$  follows from  $N \leq 3m$  and  $p' \geq t - r$ , which holds because each vertex in  $\{x_{r+1}, \dots, x_t\}$  has at least one leaf neighbor. Finally, the inequality  $N - p' - m \leq p + m$  follows from  $N - p' - p = d(x_r) \leq 2m$ , which holds because  $p + p'$  counts precisely the vertices nonadjacent to  $x_r$ . ■

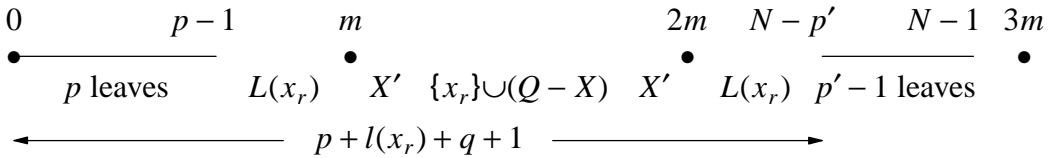


Fig. 4. Optimal numbering of a clique-star

**THEOREM 3.** For every block caterpillar  $G$ , the bandwidth  $B(G)$  equals the local density  $\beta(G)$ . Furthermore, if  $G$  is a block caterpillar and  $\beta(G) \leq m$ , then  $G$  has a left-justified  $m$ -representation produced by a linear-time algorithm.

**Proof:** Suppose that  $\text{diam } G = k + 2$ . As argued earlier, we may assume that deleting the 1-valent vertices of  $G$  (except for a pair  $v_0, v_{k+2}$  at maximum distance) produces a block path with cliques  $Q_0, \dots, Q_{k+1}$ . Furthermore,  $Q_0$  and  $Q_{k+1}$  have order 2, and  $v_0, \dots, v_{k+2}$  is a chordless path in  $G$  having maximum length, and  $v_i$  is the cut-vertex between  $Q_{i-1}$  and  $Q_i$  for  $1 \leq i \leq k + 1$ .

We consider a two-parameter family of subgraphs of  $G$ . Let  $G(h, i)$  be the subgraph consisting of edges incident to the vertices of  $Q_h \cup \dots \cup Q_i$ . For fixed  $i - h \geq 0$ , the subgraphs  $G(h, i)$  are the maximal subgraphs of diameter  $i - h + 3$ . Hence if  $\beta_1 = \max |Q_i| - 1$ ,  $\beta_2 = \max d(v_i)/2$ , and  $\beta' = \max_{h \leq i} \left\lceil \frac{n(G(h, i)) - 1}{i - h + 3} \right\rceil$ , then  $\max\{\beta_1, \beta_2, \beta'\} = \beta(G) \leq m$ .

Let  $G_i = G(1, i)$ . We produce a left-justified  $m$ -representation of each  $G_i$ , by induction on  $i$ , finishing with such a representation for  $G_k = G$ . Note that  $G_{i-1}$  contains all the vertices of  $Q_i - \{v_i\}$  as leaves if  $2 \leq i \leq k$ . For  $i > 1$ , the graph  $G_i$  is obtained from  $G_{i-1}$  by adding a clique on the vertices of  $Q_i - \{v_i\}$ , adding the pendant edges incident to these vertices, and adding the vertices of  $Q_{i+1} - \{v_i\}$  as leaves adjacent to  $v_i$ . Let  $Q'_i = Q_i - \{v_{i+1}\}$  for  $1 \leq i \leq k$ .

For  $i = 1$ , we apply Lemma 3, since  $G_1$  is a clique-star. For  $i > 1$ , assume that we have a left-justified  $m$ -representation  $f$  of  $G_{i-1}$ . Leaf neighbors of vertices in  $Q'_{i-1}$  may locate in  $J_i$  under  $f$ , but only if  $J_{i-1}$  is full. Let  $L'$  be the set of leaf neighbors of  $Q'_{i-1}$  in positions above  $im$ ; these vertices do not belong to  $G(i, i)$ . Nevertheless, let  $G'$  be the clique-star consisting of  $G(i, i)$  together with edges from  $v_i$  to  $L'$  (see Fig. 5). Because  $f$  is left-justified,  $L' = \emptyset$  if  $J_{i-1}$  is not filled by  $f$ .

We claim that  $\beta(G') \leq m$ . Since  $\beta(G(i, i)) \leq m$ , this fails only if the subgraphs involving  $L'$  are too big, meaning  $d_{G'}(v_i) > 2m$  or  $n(G') > 3m + 1$ . Since all neighbors of  $v_i$  in  $G'$  have labels between  $(i - 1)m$  and  $(i + 1)m$  in  $f$ , we have  $d_{G'}(v_i) \leq 2m$ . To bound  $n(G')$ , choose  $h$  to be the largest integer in  $\{1, \dots, i - 1\}$  such that  $J_h$  is not full, or  $h = 0$  if all these intervals are full. Since  $f$  is left-justified, every vertex of  $G_{i-1}$  in a position above  $hm$  belongs to  $G(h + 1, i)$ . Since  $J_{h+1}, \dots, J_{i-1}$  are full and contain the vertices of  $G(h + 1, i) - G'$ , we have  $n(G') = n(G(h + 1, i)) - (i - 1 - h)m$ . To bound  $n(G(h + 1, i))$ , we use  $\text{diam}(G(h + 1, i)) = i - h + 2$  and  $\beta(G(h + 1, i)) \leq m$  to obtain  $n(G') \leq m(i - h + 2) + 1 - (i - 1 - h)m = 3m + 1$ .

Now Lemma 3 yields a left-justified  $m$ -representation of  $G'$ . We shift each vertex  $(i - 1)m$  positions rightward to obtain an  $m$ -representation  $f'$  of  $G'$  using positions between  $(i - 1)m$  and  $(i + 2)m$ . Since the vertices of  $S = Q'_{i-1} \cup L'$  are leaf neighbors of  $v_i$  in  $G'$ , they occupy the lowest positions under  $f'$  (except that  $v_i$  itself may be among them). The only other vertices occupying positions in both  $f$  and  $f'$  are those of  $T = L(v_i) \cup Q_i$ , which are leaf neighbors of  $v_i$  in  $G_{i-1}$ . Since  $f$  is left-justified, the vertices of  $T$  also receive higher labels than those of  $S$  in  $f$ . Hence the positions assigned to  $S$  are the same in  $f'$  and  $f$  (those of  $T$  may have moved). We can make the vertices of  $S$  occur in the same order in  $f'$  as in  $f$ , since these vertices are leaves in  $G'$ .

We define the new  $m$ -representation  $f''$  by using  $f'$  to assign positions above  $(i - 1)m$  and  $f$  to assign positions below  $(i - 1)m$ . Since  $f$  and  $f'$  agree on  $S$  and there are no edges from  $T$  to vertices not in  $G(i, i)$ , we have satisfied all edges. The fact that  $f''$  is left-justified follows from  $f$  and  $f'$  being left-justified.

We comment on the complexity of the algorithm. The graph  $G$  is completely described by giving the set of vertices in each  $Q_i$  and the number of leaf neighbors of each clique vertex. The construction in Lemma 3 uses only these numbers, the number of additions and subtractions involving each one is bounded by a constant, and the information is not used further as we proceed in the iteration. Thus the algorithm runs in linear time. ■

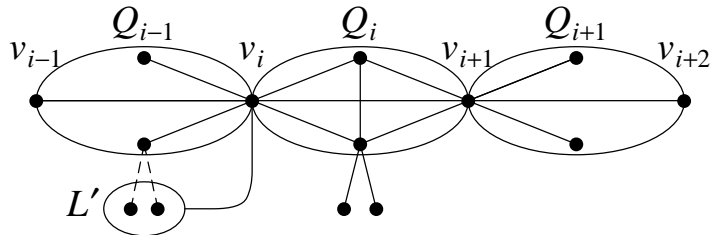


Fig. 5. The auxiliary graph  $G'$

#### 4. NP-COMPLETENESS RESULTS

Slightly enlarging the classes of 2-caterpillars or block caterpillars yields classes on which bandwidth is NP-complete. We prove this for two classes. The second class consists of trees that are almost caterpillars; they have a central path such that all other vertices except one are leaves. The first class consists of graphs that might be called “block 2-caterpillars”, but we use only a special subclass. For lack of a better name, we call the graphs in this special class “bugs”.

**DEFINITION 2.** A graph is a *bug* if it is obtained from a caterpillar with an edge  $xy$  on the spine by adding a (possibly empty) clique whose vertices are adjacent to  $\{x, y\}$  and growing a nonnegative number of paths of length 2 from  $x$ .

All caterpillars are bugs. Fig. 6 shows a special bug used in the NP-completeness proof. The *reflector*  $R_p$  of *thickness*  $p$  is the bug with  $5p + 1$  vertices obtained from the  $2p + 3$ -vertex caterpillar with degrees  $1, p, 2, 2, p, 2, 1$  along the spine by adding a clique of order  $p - 2$  adjacent to the third edge and growing  $p$  paths of length 2 from the central vertex.



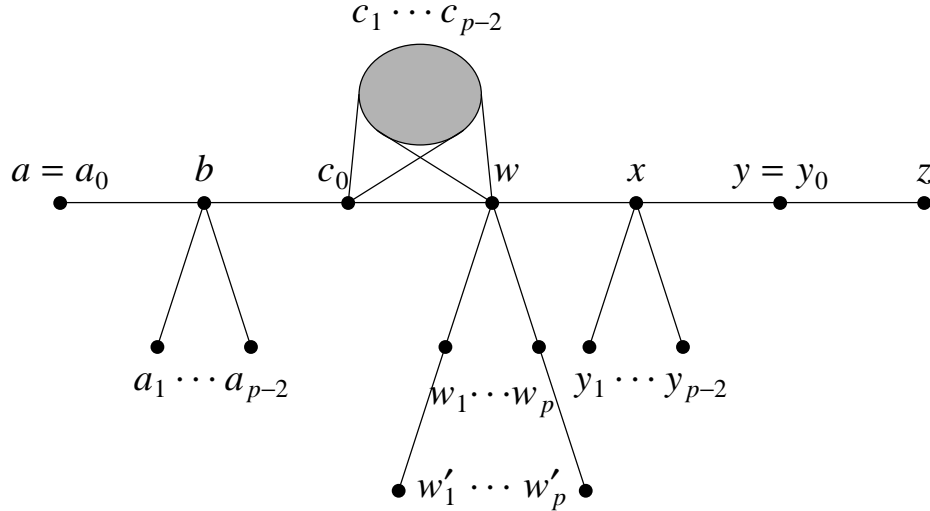


Fig. 6. The reflector  $R_p$  of thickness  $p$

We follow the method used by Monien [12] to prove that bandwidth is NP-complete for 3-caterpillars. A *peripheral vertex* of a graph is a vertex of maximum eccentricity, where the *eccentricity* of a vertex is its maximum distance from other vertices. Deleting the peripheral vertices of  $R_p$  yields a bug of diameter 4 with  $4p + 1$  vertices, so  $R_p$  has local density at least  $p$ , and the numbering in Fig. 7 shows that its bandwidth is  $p$ .

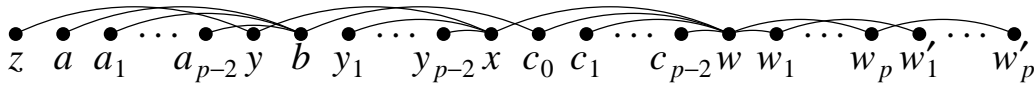


Fig. 7. An optimal numbering of  $R_p$

The property of  $R_p$  needed for the NP-completeness reduction is that in every optimal numbering of  $R_p$ , the peripheral vertices appear on the same end. As a result, placing the reflector in the middle of a caterpillar-like object forces its vertices to take positions on one end of a numbering that achieves a desired bandwidth. This motivates the term “reflector”.

**LEMMA 4.** Let  $R_p$  be the reflector of thickness  $p$ , where  $p \geq 4$ . In every optimal numbering of  $R_p$  with positions  $\{0, \dots, 5p\}$ , the positions of the peripheral vertices  $a$  and  $z$  are both below  $p$  or both above  $4p$ .

**Proof:** Let  $f$  be an optimal numbering of  $R_p$ , so  $B(f) = p$ . By symmetry, we may assume that  $f(x) < f(w)$ . Since  $R_p$  has  $4p + 1$  vertices with distance at most 2 from  $w$ , there are at least  $2p$  positions to each side of  $f(w)$ ; thus  $2p \leq f(w) \leq 3p$ . Since  $d(w) = 2p$ , vertices at distance 2 from  $w$  occupy positions more than  $p$  from  $f(w)$ . Since  $f(x) < f(w)$ , this forces  $f(y_0), \dots, f(y_{p-2})$  into the interval  $[f(w) - 2p, f(w) - p - 1]$ .

If this interval also contains  $f(b)$ , then  $f(a)$  and  $f(z)$  must both be below  $f(w) - 2p$ , since no other positions remain within  $p$  of  $f(b)$  or  $f(y)$ . Since  $f(w) \leq 3p$ , this will complete the proof.

Since  $d(b, w) = 2$ , it now suffices to show that  $f(b) < f(w)$ . For each vertex  $v$  at distance 2 from  $w$ ,  $p < |f(v) - f(w)| \leq 2p$ . With  $f(y_0), \dots, f(y_{p-2})$  below  $f(w)$ , this forces at least  $p-1$  of  $f(w'_1), \dots, f(w'_p)$  into  $[f(w) + p + 1, f(w) + 2p]$ . This in turn forces at least  $p-1$  of  $f(w_1), \dots, f(w_p)$  into  $[f(w) + 1, f(w) + p]$ . At most one position remains in this interval, so at least  $p-2$  of  $f(c_0), \dots, f(c_{p-2})$  lie in  $[f(w) - p, f(w) - 1]$ . All of these  $p-2$  vertices are within distance 2 from  $b$  in  $R_p$ , and thus  $f(b) \leq (f(w) - p + 2) + 2p$ .

If  $f(b) > f(w)$ , then  $f(b) \in \{f(w) + p + 1, f(w) + p + 2\}$ , because the neighbors of  $w$  occupy the positions within  $p$  of  $f(w)$ . This forces  $f(a_0), \dots, f(a_{p-2})$  to occupy positions above  $2p$ . When  $p \geq 4$ , there are at least 3 such values, and one of them is now too far above  $f(b)$ .

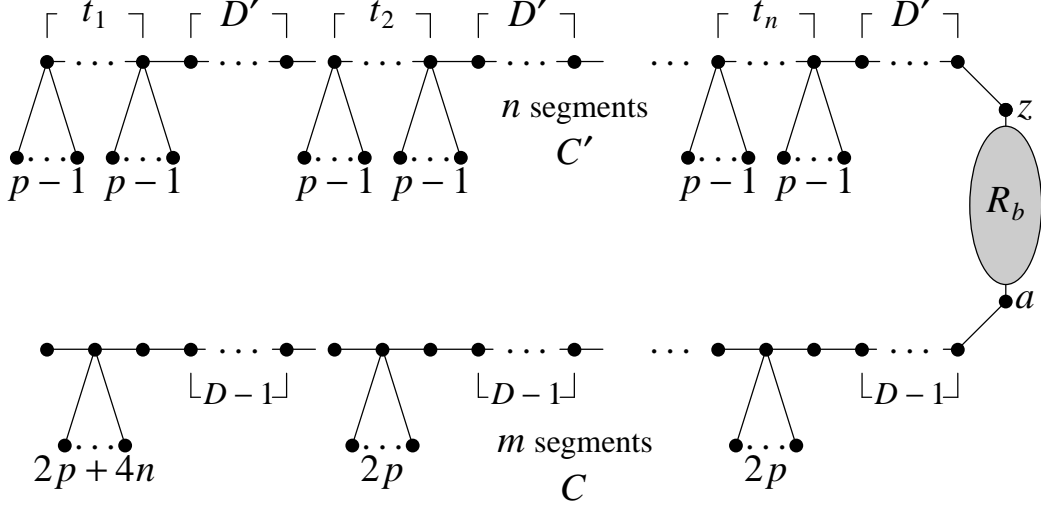
When  $f(x) > f(w)$ , the analogous argument locates  $a$  and  $z$  above  $4p$ . ■

We prove NP-completeness of the bandwidth problem for bugs by reduction from the Multiprocessor Scheduling Problem. An instance  $T$  of this problem consists of a number  $m$  of processors, a deadline  $D$ , and  $n$  tasks with integer execution times  $t_1, \dots, t_n$ . The decision problem asks whether the tasks can be assigned to the processors such that for each processor, the total execution time for the assigned tasks is at most  $D$ . When the answer is “Yes”, we say that the instance is *solvable*. As shown in [5, p95-106], Multiprocessor Scheduling is NP-complete in the strong sense, which means that we can consider the size of  $T$  to be  $m + n + \max t_i$ .

**THEOREM 4.** The bandwidth problem is NP-complete for bugs.

**Proof:** It suffices to show that for each instance  $T = (m, D, t_1, \dots, t_n)$  of the Multiprocessor Scheduling Problem, we can construct a bug  $G$  and an integer  $b$  such that  $T$  is solvable if and only if  $B(G) \leq b$ . Furthermore, the construction must run in time polynomial in  $m$ ,  $n$ , and  $\max t_i$ .

Given  $T$ , choose  $p$  such that  $p > 2n(D + 4)$ . Let  $b = p + 1 + 2n$ , and let  $D' = 2m(D + 2) - 4$ . We construct a bug  $G$  using two caterpillars and the reflector  $R_b$  of thickness  $b$  (see Fig. 8). Caterpillar  $C$  consists of  $m(D + 2 + 2p) + 4n$  vertices, with  $\lambda = m(D + 2)$  vertices on the spine. The  $2mp + 4n$  additional leaves of  $C$  are attached as follows:  $2p + 4n$  at the second vertex and  $2p$  at each vertex whose distance from the second vertex along the spine is a multiple of  $D + 2$ . Caterpillar  $C'$  consists of  $(p \sum_{i=1}^n t_i) + nD'$  vertices, with  $\sum_{i=1}^n t_i + nD'$  vertices on the spine. The  $i$ th *segment* of  $C'$  consists of  $t_i$  vertices each with  $p - 1$  leaves as neighbors followed by  $D'$  vertices with no leaf neighbors; the  $t_i p$  vertices appearing first are the *task vertices*. To complete the bug  $G$ , add two edges: one each from the peripheral vertices  $a$  and  $z$  of  $R_p$  to the last vertices on the spines of  $C$  and  $C'$ , respectively. Note that the number of vertices in  $G$  is given by a polynomial in  $n$ ,  $D$ , and  $\sum_{i=1}^n t_i$ .

Fig. 8. The bug  $G$  in the transformation

**Schedule yields numbering.** Suppose that  $T$  is solvable; we construct a numbering  $f$  of  $G$  such that  $B(f) = b$ . As in the proof of Theorem 3, we let  $J_i$  be the integer interval  $[ib+1, (i+1)b-1]$ . We first place the vertices from the spine of  $C$  in the positions  $\{ib: 0 \leq i < \lambda\}$ . For such a vertex in position  $ib$ , we place half its leaf neighbors in  $J_{i-1}$  and half in  $J_i$ , using the lowest  $p$  positions in each interval. For the reflector  $R_b$ , we use an optimal numbering in positions  $\lambda b, \dots, \lambda b + 5b$ , with  $f(a) = \lambda b$  and  $f(z) = \lambda b + 1$  (see Fig. 7).

Since  $T$  is solvable, there exist index sets  $\{I_j: 1 \leq j \leq m\}$  assigning jobs to processors such that  $\bigcup_{j=1}^m I_j = \{1, \dots, n\}$  and that  $\sum_{i \in I_j} t_i \leq D$  for each  $j$ . We may increase the task execution times to obtain  $\sum_{i \in I_j} t_i = D$  for each  $j$ , because this does not change  $b$  and the new corresponding bug contains the original bug as an induced subgraph. The segments of  $C'$  indexed by  $I_j$  thus have exactly  $(\sum_{i \in I_j} t_i)p = Dp$  vertices consisting of leaves and their neighbors; let  $S_j$  denote this set of vertices in  $C'$ .

We have already placed the vertices of the  $j$ th segment of  $C$  in some of the positions from  $(j-1)(D+2)b$  to  $(j(D+2)-1)b$ , using all the multiples of  $b$  and some positions in  $J_{(j-1)(D+2)}$  and  $J_{(j-1)(D+2)+1}$ . We now place the vertices of  $S_j$  in  $J_{(j-1)(D+2)+2}, \dots, J_{j(D+2)-1}$ , with  $J_i$  receiving one spine vertex at position  $ib+1$ , followed immediately by its  $p-1$  leaf neighbors in positions  $ib+2, \dots, ib+p$ .

In each such  $J_i$ , we have assigned  $p$  positions and still have  $b-1-p = 2n$  unassigned. This also holds for  $J_{j(D+2)}$  and  $J_{j(D+2)+1}$  such that  $1 \leq j < m$ , where we have placed leaves from  $C$  in the lowest positions (we have filled  $J_0$  and  $J_1$  completely). Thus there remain  $2n(\lambda-2)$  positions unassigned.

To complete the numbering, we must assign positions to the remaining vertices from the spine of  $C'$ . The remaining vertices consist of  $n$  paths, each of order  $D' = 2(\lambda-2)$ . We will place two vertices from each path into each  $J_i$  for  $2 \leq i < \lambda$ . For fixed  $k \in \{1, \dots, n\}$ , we place the  $k$ th path into  $L \cup U$ , where  $L = \{ib+p+2k-1: 2 \leq i < \lambda\}$  and  $U = \{ib+p+2k: 2 \leq i < \lambda\}$ .

We place the two endpoints of the path in the intervals  $J, J'$  containing their neighbors on the spine of  $C'$ . From the higher desired interval  $J'$ , the path moves up to  $J_{\lambda-1}$  via  $L$ . It then switches to  $U$  and moves down to  $J$  via  $U$ . From there down to its entrance to  $J$ , it uses the positions of  $U$  and  $L$  in each interval. It moves from  $J$  down to  $J_2$  via  $U$ , switches to  $L$  in  $J_2$ , and moves back up to  $J$  via  $L$ , where it ends (see Fig. 9). Because successive positions within  $L$  or within  $U$  differ by exactly  $b$ , we have completed a numbering showing that  $B(G) \leq b$ .

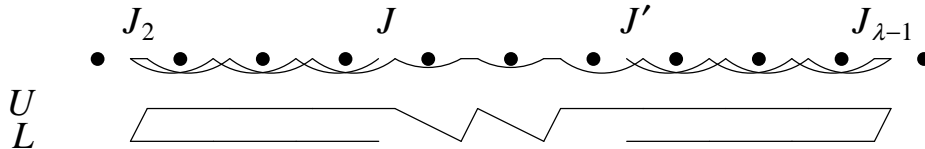


Fig. 9. Numbering a path

**Numbering yields schedule.** Conversely, suppose that  $G$  has a numbering  $f$  with  $B(f) = b$  (there is no better numbering, since  $B(R_b) = b$ ). We prove that an optimal numbering of  $G$  must have essentially the form described above, from which we obtain a positive solution for  $T$ .

The positions within  $b$  of the vertex  $x$  of degree  $2 + 2p + 4n = 2b$  on the spine of  $C$  must be occupied by the neighbors of  $x$ , and no edge can stretch across this interval. Thus  $x$  and these vertices occupy positions at one end of the numbering; by symmetry, we may assume that these positions are  $0, \dots, 2b$ . Similarly, the reflector  $R_b$  must occupy  $5b + 1$  consecutive positions, with its peripheral vertices within  $b$  of the end, and no edge can stretch across these positions. Thus we may assume that  $R - b$  occupies positions  $\lambda b, \dots, (\lambda + 5)b$  and that  $\{f(a), f(z)\} = \{\lambda b, \lambda b + 1\}$ .

Among the remaining positions, which must be filled since  $G$  has  $(\lambda + 5)b + 1$  vertices, there is a path of length  $\lambda$  from a leaf neighbor of  $x$  to  $a$ . Since there is a leaf neighbor of  $x$  in position 0 and  $a$  has position at least  $\lambda b$ , the vertices of this path must occupy the positions  $\{ib: 0 \leq i \leq \lambda\}$ .

It remains to assign tasks to sets  $I_j$  such that  $\sum_{i \in I_j} t_i \leq D$ . Let  $z_j = [1 + j(D + 2)]b$  for  $0 \leq j \leq m$ . We say that task  $i$  belongs to set  $I_j$  if and only if  $z_{j-1} < f(v) < z_j$  for some non-leaf task vertex  $v$  in the  $i$ th segment of  $C'$ . We show first that task  $i$  belongs to only one set; suppose not. Since the non-leaf task vertices in the  $i$ th segment of  $C'$  induce a connected subgraph (a path), there must exist adjacent non-leaf task vertices  $u, v$  such that  $f(u) < z_j < f(v)$  for some  $j$ . Since  $u$  and  $v$  are adjacent, we have  $f(v) - f(u) \leq b$ . Now  $u, v, f^{-1}(z_j)$  and their neighbors all have positions in the interval  $[f(u) - b, f(v) + b]$ . There are  $4p + 3$  of these vertices, but at most  $3b + 1$  positions in the interval. From  $b = p + 1 + 2n$  and  $p > 8n$  we obtain  $4p + 3 > 3b + 1$ , and we cannot place  $4p + 3$  vertices into  $3b + 1$  positions.

The positions outside  $[z_0, z_m]$  are already filled, so we have assigned each task to exactly one set  $I_j$ . We must show that  $\sum_{j \in I_j} t_i \leq D$  for each  $j$ . Into the interval  $[z_{j-1} - b, z_j + b]$ , we have now placed all the task vertices for tasks in  $I_j$ ,  $D + 5$  vertices from

the spine of  $C$ , and  $4p$  leaves of  $C$  (when  $j = m$ , some of this count is replaced by vertices of  $R_b$ ). The number of vertices is  $(\sum_{j \in I_j} t_i)p + D + 5 + 4p$ , and the number of positions in the interval is  $(D + 2)b + 1 + 2b$ . Thus  $\sum_{j \in I_j} t_i \leq [(D + 4)(p + 1 + 2n) - (D + 5 + 4p)]/p = D + [2n(D + 4) - 1]/p$ . Since  $p > 2n(D + 4)$ , we conclude that  $\sum_{j \in I_j} t_i \leq D$ . ■

The paradigm in the proof of Theorem 4 applies more generally. All we need is a graph to play the role of the reflector. This graph  $R'_b$  will have bandwidth  $b$  and two special vertices such that every optimal numbering puts those two vertices in positions near one end. We can then use  $R'_b$  in place of  $R_b$  to form a bug-like graph and follow the proof of Theorem 4.

For example, we use this approach to prove that bandwidth is NP-complete on a class of trees that are tolerance graphs. A *near-caterpillar* is a tree having a single path that includes all but one of the non-leaf vertices. For our reflector with bandwidth  $b$  we use a near-caterpillar with  $4b + 1$  vertices very similar to the near-caterpillar  $T_b$  of Fig. 3. Define  $R'_b$  for  $b$  even to be the same as  $T_b$  in Fig. 3 except that the  $4b - 3$  leaves are redistributed among the sets  $X, Y, Z, W$  so that the sizes of  $X, Y, Z, W$  are  $b/2, b, b/2, 2b - 3$ , respectively. The bandwidth of  $T_b$  exceeds  $b$ ; the bandwidth of  $R'_b$  is its local density  $b$ , but this is achievable only by putting specified vertices near one end.

**LEMMA 5.** In every optimal numbering  $f$  of the near-caterpillar  $R'_b$  using positions  $0, \dots, 4b$ , the vertices of  $X \cup Z$  are all below  $b$  or all above  $3b$ .

**Proof:** The local density of  $R'_b$  is  $b$ , and we have a numbering achieving bandwidth  $b$  in which  $x, z, w, y$ , are numbered  $b, b + 1, 2b, 3b$ , respectively. Now consider an arbitrary optimal numbering  $f$ . By symmetry, we may assume that  $f(w) < f(y)$ . Since  $w$  has degree  $2b$ , its neighbors fill the positions within  $b$  to each side of  $f(w)$ . Hence all of  $Y$  is outside this interval, and no edges stretch across, so  $f(y) = f(w) + b$  and  $Y$  occupies  $[f(w) + b + 1, f(w) + 2b]$ . Since no edge involving  $x$  or  $z$  can stretch across this interval, all of  $X \cup Z$  must be on the other end, below  $f(w) - b$ . We thus have  $2b$  vertices to each side of  $f(w)$ , and all of  $X \cup Z$  is below  $b$ . ■

By following the argument of Theorem 4, bandwidth is NP-complete for near-caterpillars. It is immediate that every near-caterpillar is a tolerance graph, since a tree is a tolerance graph if and only if it does not contain the tree obtained from the claw  $K_{1,3}$  by subdividing each edge twice [7]. This tree is forbidden from near-caterpillars, since every path in it misses at least two non-leaf vertices. Thus bandwidth is NP-complete for a subclass of tolerance graphs.

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