MAXIMUM BANDWIDTH UNDER EDGE ADDITION

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Abstract

We determine how much the bandwidth $B(G)$ of a graph G can increase when a single edge is added. Let $g(b,n)$ be the maximum possible value of $B(G+e)$ when G has n vertices and bandwidth b. The problem of studying when $B(G+e) \leq B(G)+1$ was originally posed by Erdős. We determine

$$g(b,n) = \begin{cases} b+1 & \text{if } n \leq 3b+4 \\ \lceil (n-1)/3 \rceil & \text{if } 3b+5 \leq n \leq 6b-2 \\ 2b & \text{if } n \geq 6b-1. \end{cases}$$

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1. INTRODUCTION

Let G be a simple graph of order n. A bijection from $V(G)$ to $\{1,\ldots,n\}$ is a *numbering* of G. If f is a numbering of G, let $B_f(G) = \max\{|f(u)-f(v)|: uv \in E(G)\}$. The *bandwidth* of G is $B(G) = \min_{f\in F} B_f(G)$, where F is the set of all numberings of G. A numbering f such that $B_f(G) = B(G)$ is an *optimal* numbering of G. Surveys of results on bandwidth appear in [1] and [2].

We use $G+e$ to denote the graph obtained from G by adding the edge e. In 1971, Erdős (see [4]) asked whether $B(G+e) \le B(G)+1$ for every simple graph G and any $e \in E(\bar{G})$. Chvátalová [4] gave the counterexample shown in Fig. 1, where $B(G) = 2$ and $B(G+uv) \ge 4$. Chvátalová and Opatrny [5] determined the maximum of $B(G+uv)$ as a function of $B(G)$ and the distance $d_G(u,v)$.

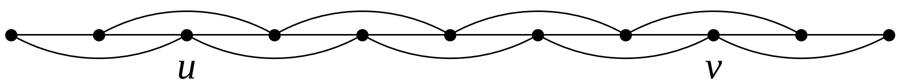

Fig. 1. A multiple increase in bandwidth.

In this paper, we determine the maximum value of $B(G+e)$ in terms of $B(G)$ and the number of vertices of G. Let $g(b,n)$ be the maximum value of $B(G+e)$ when G has n vertices and bandwidth b. We prove that $g(b,n)$ remains at $b+1$ when n is at most $3b+4$, reaches $2b$ when n is as large as $6b-1$, and has the value $\lceil (n-1)/3 \rceil$ for the intermediate values of n. Note that when $b \le 2$ the intermediate range is empty, and for $b = 1$ the value of $g(b,n)$ is always 2.

2. THE BOUNDS

We give two constructions for upper bounds, followed by a lower bound argument. The first construction shows that the addition of an edge cannot do worse than make the bandwidth double. This uniform upper bound, independent of the number of vertices $n(G)$, is optimal when $n(G) \ge 6B(G)-1$.

LEMMA 1. If $B(G) = b$, then $B(G+e) \le 2b$.

Proof: Let f be an optimal numbering of G, and let $V(G) = \{v_1,\ldots,v_n\}$, numbered so that $f(v_i) = i$. Let $v_l v_m$ be the added edge. We define a new numbering f' such that $|f'(x)-f'(y)| \le 2\,|f(x)-f(y)|$, and also $|f'(v_l)-f'(v_m)| = 1$. Let $r = \lfloor (l+m)/2 \rfloor$, and set $f'(v_r) = 1$ and $f'(v_{r+1}) = 2$. For every other v_i such that $|i-r| \le \min\{r-1, n-r\}$, let $f'(v_i) = f'(v_{i+1})+2$ if $i < r$ and $f'(v_i) = f'(v_{i-1})+2$ if $i > r$. This defines f' for all vertices except a set of the form $v_1,\ldots,v_k$ or $v_{n+1-k},\ldots,v_n$, depending on the sign of $r - \lfloor (n+1)/2 \rfloor$. In the first case, we assign $f'(v_i) = n+1-i$ for $i \le k$; in the second, we assign $f'(v_i) = i$ for $i > n-k$. The renumbering f' numbers the vertices outward from v_r to achieve $|f'(x)-f'(y)| \le 2\,|f(x)-f(y)|$. Since we begin midway between v_l and v_m, we also have $|f'(v_l)-f'(v_m)| = 1$. □

The next construction provides optimal renumberings for all the cases where $n(G) <$

$6B(G)$.

LEMMA 2. If $B(G) = b$ and $n(G) \leq 3(b+a)+1$, where $a \geq 1$, then $B(G+e) \leq b+a$ for any $e \in E(\bar{G})$.

Proof: Again let f be an optimal numbering of G, with $V(G) = \{v_1, \ldots, v_n\}$ such that $f(v_i) = i$. Let $e = v_l v_m$, with $l < m$. If $m - l \leq b + a$, there is nothing to prove, so we may assume $m - l > b + a$. We will change the numbering to make the new values on v_l and v_m differ by $b + a$. We choose values r and s with $l \leq r < s \leq m$ and define f' by $f'(v_l) = r$, $f'(v_m) = s$, $f'(v_i) = i - 1$ for $l < i \leq r$, $f'(v_i) = i + 1$ for $s \leq i < m$, and otherwise $f'(v_i) = i = f(v_i)$. See Fig. 2.

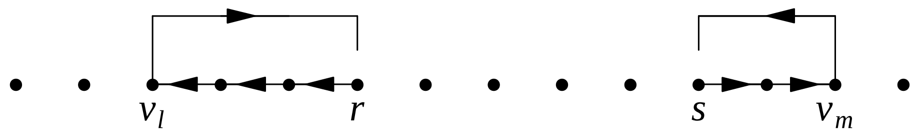

Fig. 2. A bound on increase in bandwidth.

We want to choose r, s in the model above so that $B_{f'}(G+e) \leq b+a$. If $s - r = b+a$, then any two vertices other than $v_l v_m$ whose numbers both change are not adjacent in G. Since $a \geq 1$ and the vertices other than v_l, v_m shift by at most 1, we need only verify that the edges involving v_l and v_m have satisfactory differences under f'. In order to guarantee this, it suffices to have $r \leq \max\{l + a, 1 + b + a\}$ and $s \geq \min\{m - a, n - b - a\}$.

If $l \geq n - 2(b+a)$, we set $r = l$ and $s = l + b + a$. We have $s = l + b + a \geq n - (b+a)$, which suffices. If $m \leq 1 + 2(b+a)$, then symmetrically we set $s = m$ and $r = m - b - a$, yielding $r \leq 1 + b + a$. In the remaining case, with $l \leq n - 1 - 2(b+a)$ and $m \geq 2 + 2(b+a)$, we set $r = 1 + b + a$ and $s = 1 + 2(b+a)$. Now the condition $n \leq 1 + 3(b+a)$ yields $l \leq b + a < r$ and $s \geq n - b - a$, and all the conditions that together suffice are satisfied. □

We have a uniform argument for the lower bound. One of the common lower bounds for bandwidth is the "density" bound, originally observed in [3] and applied more generally in [2]: if G has a subgraph with order m and diameter d, then $B(G) \geq (m-1)/d$.

LEMMA 3. If $n \geq 6b - 1 - 3k$ with $0 \leq k \leq b - 1$, then there is an n-vertex graph G with bandwidth b and an edge $e \in E(\bar{G})$ such that $B(G+e) \geq 2b - k$.

Proof: Since the bandwidth of a graph is as large as the bandwidth of any subgraph, we may assume G is the bth "power" of P_n. In other words, given an optimal numbering f of G such that $f(v_i) = i$ for all i, we may assume $v_i v_j \in E(G)$ if and only if $|i - j| \leq b$. Let $e = v_{1+b} v_{n-b}$; it suffices to show $B(G+e) \geq 2b - k$.

The vertices $S = \{v_i\colon i \leq b+1 \text{ or } i \geq n - b - 1\}$ induce a subgraph of $G + e$ having diameter 3. Let f' be an optimal numbering of $G + e$. If the vertices assigned numbers 1 and n by f' are both in S, then the subgraph of $G + e$ induced by S contains an edge uv with $|f'(u) - f'(v)| \geq \lceil (n-1)/3 \rceil \geq 2b - k$. If the vertex x assigned number 1 or number n by f' belongs to $G - S$, then all neighbors of x have numbers lying to one side of $f'(x)$;

since each vertex of $G - S$ has $2b$ neighbors in G, this implies $B_{f'}(G + e) \geq 2b$. In either case, we have $B(G + e) \geq 2b - k$. □

Together, these lemmas yield the main result.

THEOREM 1. If $g(b,n)$ denotes the maximum value of $B(G+e)$ when G has n vertices and bandwidth b, then

$$g(b,n) = \begin{cases} b+1 & \text{if } n \leq 3b+4 \\ \lceil (n-1)/3 \rceil & \text{if } 3b+5 \leq n \leq 6b-2 \\ 2b & \text{if } n \geq 6b-1. \end{cases}$$

Proof: Lemmas 1 and 3 provide the upper and lower bounds when $n \geq 6b - 1$ (set $k = 0$). Lemma 2 provides the upper bound when $n \leq 3b + 4$ (set $a = 1$). If $3b + 5 \leq n \leq 6b - 2$, there is a unique positive integer value of a such that $3(b+a) - 1 \leq n \leq 3(b+a)+1$. Now Lemma 2 establishes $g(b,n) \leq b + a = \lceil (n-1)/3 \rceil$, and Lemma 3 establishes $g(b,n) \geq 2b - (b - a) = b + a$, since $3(b + a) - 1 = 6b - 1 - 3(b - a)$. □

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