Clearly $F_n(0) = 1$ for $n \geqslant 1$, and $F_1(k) = 0$ for $k \geqslant 1$. We show that $F_n(k) = (-1)^n\binom{n-1}{k}$ for $n \geqslant 1$, $k \geqslant 0$.

Now suppose $n \geqslant 2$. The set of permutations in $G_n(k)$ for which $\pi(n-1) \neq n$ and $\pi(n) \neq n$ is mapped onto itself if its elements are preceded by the transposition $(n-1, n)$, which interchanges even and odd permutations, so these permutations contribute 0 to $F_n(k)$. The permutations in $G_n(k)$ with $\pi(n) = n$ correspond to the permutations in $G_{n-1}(k)$ by restriction, without changing parity, so these contribute $F_{n-1}(k)$ to $F_n(k)$. Finally, any permutation in $G_n(k)$ with $\pi(n-1) = n$ can be obtained by following the transposition $(n-1, n)$ with a permutation in $G_{n-1}(k-1)$ extended to leave n fixed. In view of the change of parity, the permutations in $G_n(k)$ with $\pi(n-1) = n$ contribute $-F_{n-1}(k-1)$ to $F_n(k)$.

This yields the recurrence $F_n(k) = F_{n-1}(k) - F_{n-1}(k-1)$ for $n \geqslant 2$, $k \geqslant 1$, with boundary conditions $F_1(k) = \delta_{k0}$ and $F_n(0) = 1$. It follows by induction that $F_n(k) = (-1)^k\binom{n-1}{k}$.

The proposer gave a solution using generating functions. No other solutions were received.

Highly Asymetric Graphs

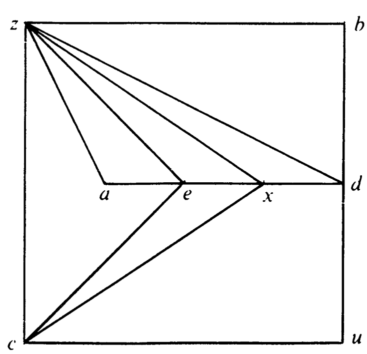
E 3187 [1987, 72]. *Proposed by Stephen J. Lipscomb, Mary Washington College, Fredericksburg, VA, and Allen J. Schwenk, Western Michigan University, Kalamazoo.*

A *vertex-deleted* subgraph $G - v$ of a graph G is formed by removing one vertex v and every edge incident with it. A graph is called *asymmetric* if it has no nontrivial automorphisms (symmetries).

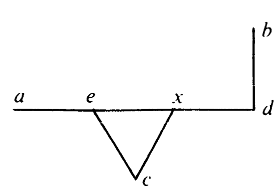
(a) Find a smallest possible asymmetric graph all of whose vertex-deleted subgraphs are also asymmetric.

(b) Same as (a) but also require that no pair of vertex-deleted subgraphs be isomorphic.

Solution by Douglas B. West, University of Illinois, Urbana. We interpret "smallest" to mean "fewest vertices". We show that the 8-vertex graph F below has the properties of both (a) and (b) and no smaller graph satisfies either. Note that the subgraph $H = F - u - z$ is an asymmetric graph with 6 vertices and 6 edges.



F



H

Call a graph *highly asymmetric* if it is asymmetric and all its vertex-deleted subgraphs are asymmetric. We first prove F is highly asymmetric. If a graph G has a unique vertex v of some degree, any symmetry must fix v, so asymmetry of $G - v$ is a sufficient (but not necessary) condition for asymmetry of G. Hence asymmetry of $F - u$ and $F - z$ follows from asymmetry of H. For any other $F - v$, z is the unique vertex of $F - v$ with maximum degree and is adjacent to all but u, so it suffices to show that no non-trivial symmetry of $F - v - z$ fixes u. This can be done quickly by examination. To see that F satisfies (b), note that the degree sequences of the vertex-deleted subgraphs are all distinct, except that $F - a$ and $F - b$ have a degree sequence 5444322, and $F - c$ and $F - e$ have degree sequence 5433221. In $F - b$, the vertices of degree 4 induce a triangle, while in $F - a$ they do not. Similarly, in $F - e$ the vertices of degree 1 and 5 are adjacent, while in $F - a$ they are not.

To show that no smaller graph than F is highly asymmetric, we need the following facts:

Let G be an asymmetric graph of smallest order. Then

(1) G and $\bar{G}$ are connected.

(2) G has six vertices.

(3) If G has at most 6 edges, then $G = H$ above.

(4) G has a vertex of degree at least 3.

(1) follows from minimality and the fact that G and $\bar{G}$ have the same symmetries; any non-trivial component of a disconnected G would have a non-trivial symmetry. Of G and $\bar{G}$, we may consider the one with fewer edges. Thus on four vertices we need only consider trees; on five vertices we need only consider trees or connected unicyclic graphs (5 edges); every one of these graphs has a symmetry of order 2. With the existence of H, this proves (2). On six vertices there are six (unlabeled) trees; each has a symmetry of order 2. Among the connected unicyclic 6-vertex graphs (six edges), there are $1, 1, 4, 7$, respectively, having a cycle of length $6, 5, 4, 3$. Every one of these has a symmetry of order 2 except the graph H above, which is asymmetric, proving (3). For (4), note that $\Delta(H) = 3$, where $\Delta(G)$ denotes the maximum vertex degree. The average degree of any 6-vertex graph with at least 7 edges is at least $2 \cdot 7/6 > 2$, so $\Delta(G) \geqslant 3$ even if $G \neq H$. (Note: there are also three asymmetric 6-vertex graphs with 7 edges, their complements with 8 edges, and $\bar{H}$ with 9 edges, but we need not determine the rest of the set to solve this problem.)

We now show that no highly asymmetric graph G has 7 vertices. Suppose G is such a graph. We first claim each vertex of G has degree 2, 3, or 4. If v is a vertex of degree 1, delete its neighbor. If $d(v) = 5$, delete its non-neighbor. If $d(v) = 0$ or $d(v) = 6$, delete any vertex other than v. In each case, we obtain a graph $G - v$ with a vertex of degree 0 or 5. This implies $G - v$ or its complement is asymmetric and disconnected, contradicting (1).

Let w be a vertex of maximum degree in G. By (4), we have $\Delta(G - v) \geqslant 3$ for each v, so $d(w) \geqslant 3$. Since $G - w$ has at least 6 edges, G therefore has at least 9 edges. If exactly 9, then $d(w) = 3$ and $G - w = H$, since H is the only asymmetric 6-vertex graph with 6 edges. Morever, w must be adjacent to a, b and one of $\{c, d\}$. In either case, $G - a$ has a symmetry. Thus we may assume G has at least 10 edges.

As before, we may assume by complementation that G has no more edges than $\bar{G}$. Since $\binom{7}{2} = 21$, we may now assume G has exactly 10 edges. If $\Delta(G) = 3$, then G

has degree sequence 3333332, and deleting the vertex of minimum degree yields the sequence 333322. The complement of this graph, with degree sequence 332222, must be asymmetric. If the two 3-valent vertices u, v are not adjacent, then u, v have two common neighbors, and the remaining edge joins their remaining neighbors. If u, v are adjacent, then this degree sequence yields 2, 1, 0 graphs when u, v have 0, 1, 2 common neighbors, respectively. All four graphs with this degree sequence have symmetries of order 2.

Hence we may assume $\Delta(G) = 4$, so $G - w = H$. Since G has no vertex of degree 1, w must be joined to a, b, and two of $\{c, d, e, x\}$. We find a symmetry in $G - c$ unless the last two neighbors of w are x and one of $\{d, e\}$, But then G has a symmetry if w is adjacent to e, and $G + x$ has a symmetry if w is adjacent to d. This exhausts all possibilities for G.

No other solutions were received. The proposers supplied a 9-vertex highly asymmetric graph and a 10-vertex highly asymmetric graph with non-isomorphic vertex-deleted subgraphs.

Locating Corners from an Arbitrary Point in a Rectangle

E 3208 [1987, 456]. *Proposed by I. D. Berg R. L. Bishop, and H. G. Diamond, University of Illinois at Urbana-Champaign.*

Suppose that in the Euclidean plane, line segments of lengths a, b, c, d emanate from a given point P in clockwise order, where a, b, c, d are given positive numbers with

$$a^2 + c^2 = b^2 + d^2.$$

(i) Show that the four segments can be so placed that the endpoints determine a rectangle containing P, and show that this rectangle may have any specified area between 0 and some maximum value $M(a, b, c, d)$.

(ii) Find $M(a, b, c, d)$.

Solution by Carl Schoen, University of Wisconsin at Eau Claire. Without loss of generality, assume a is the smallest of a, b, c, d. Contrary to the assertion of the problem as printed, the minimum area is $(a + d)\sqrt{b^2 - a^2}$ if $b \leqslant d$ and $(a + b)\sqrt{d^2 - a^2}$ if $d \leqslant b$. The area can equal zero only if $a = \min\{b, d\}$. The maximum area is $ac + bd$.

We may orient any rectangle to have horizontal and vertical sides, P at the origin, and a in the first quadrant. Let α be the angle between a and the positive x-axis. For any choice of α with $0 \leqslant \alpha \leqslant \pi/2$, we construct such a rectangle. Having chosen a as the smallest of a, b, c, d, there is a unique placement for segment b so that the endpoints of a and b lie on a vertical line. Similarly, there is a unique placement for d so that the endpoints of a and d lie on a horizontal line. This determines a rectangle with fourth vertex C. If the rectangle meets the axes at $(-t, 0)$, $(u, 0)$, $(0, w)$, and $(0, -v)$, then $b^2 + d^2 = u^2 + v^2 + t^2 + w^2 = a^2 + PC^2$, which implies $PC = c$. Hence, the postulated rectangle can be constructed for arbitrary α.

If $A(\alpha)$ is the area of the resulting rectangle, than $A = (t + u)(v + w)$, where

$$w = a \sin \alpha, \qquad u = a \cos \alpha, \qquad t = \sqrt{d^2 - a^2 \sin^2 \alpha}, \qquad v = \sqrt{b^2 - a^2 \cos^2 \alpha}.$$