Rainbow spanning subgraphs of small diameter in edge-colored complete graphs

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Abstract

Let $s(n,t)$ be the maximum number of colors in an edge-coloring of the complete graph K_n that has no rainbow spanning subgraph with diameter at most t. We prove $s(n,t) = \binom{n-2}{2}+1$ for $n,t \geq 3$, while $s(n,2) = \binom{n-2}{2} + \lfloor \frac{n-1}{2} \rfloor$ for $n \neq 4$ (and $s(4,2) = 2$).

Keywords: spanning subgraph, rainbow subgraph, diameter, anti-Ramsey number
MSC code: 05C55, 05C35

1 Introduction

A *rainbow subgraph* of an edge-colored graph G is a subgraph whose edges have distinct colors. The general *anti-Ramsey problem* asks for the maximum number of colors in an edge-coloring of K_n having no rainbow copy of some graph in a class $\mathcal{F}$; this maximum number of colors is the *anti-Ramsey number* $\mathrm{AR}(n,\mathcal{F})$. (The *Ramsey* problem can be interpreted as asking for the minimum number of colors in an edge-coloring having no monochromatic copy of a graph in $\mathcal{F}$.) Early results considered the problem with $\mathcal{F}$ being a single graph: see [1] for a survey and [4] for the notable determination of $\mathrm{AR}(n, C_k)$. More recent work considers problems where $\mathcal{F}$ consists of spanning subgraphs of K_n; see [2] for a discussion of such problems involving spanning cycles, perfect matchings, and spanning trees.

In this paper, we study when rainbow spanning subgraphs of small diameter are forced. Let $s(n,t)$ be the maximum number of colors in an edge-coloring of K_n not having a rainbow spanning subgraph with diameter at most t. We compute $s(n,t)$. Previously, Montágh [3] obtained an upper bound on $s(n,3)$, showing that $\mathrm{AR}(n,\mathcal{D}_n) = \binom{n-2}{2} + 1$, where $\mathcal{D}_n$ is the

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family of n-vertex double-stars whose vertex of second largest degree is at most 4. That is, rainbow double stars that are "almost" stars are forced by having more than $\binom{n-2}{2}+1$ colors. To prove $s(n,t) \ge \binom{n-2}{2}+1$ when $t \ge 3$, we present an edge-coloring of K_n using $\binom{n-2}{2}+1$ colors that has no rainbow spanning subgraph with diameter at most t.

With more colors, such a subgraph is forced. Upper bounds for anti-Ramsey problems generally use the notion of *representing subgraph* of an edge-colored complete graph, which is a spanning subgraph containing one edge of each color. We give a short proof that any edge-coloring of K_n using at least $\binom{n-2}{2}+2$ colors has a representing subgraph with diameter at most 3. Thus when $t \ge 3$ the answer does not depend on t. The answer for the special case $t=2$ is somewhat different.

Theorem 1.1. $s(n,t) = \binom{n-2}{2} + \begin{cases} 1 & \textit{for } n,t \ge 3, \\ 2 & \textit{for } (n,t)=(4,2), \\ \left\lfloor \frac{n-1}{2} \right\rfloor & \textit{when } t=2 \textit{ and } n \ne 4. \end{cases}$

The aim is to show that when too many colors are used, every edge-coloring contains some representing subgraph having the forbidden property.

As noted by a referee, our work suggests the corresponding Turán-type extremal question: How many edges can an n-vertex graph have without having a spanning subgraph with diameter at most t? The answer is $\binom{n-1}{2}$, achieved by $K_{n-1}+K_1$.

2 Constructions and easy cases

For a vertex v in a graph G, let $N_G(v)$ be the set of neighbors of v, with $N_G[v] = N_G(v) \cup \{v\}$, and let the *degree* $d_G(v)$ of v be $|N_G(v)|$. An *x,y-path* in G is a path with endpoints x and y. The *distance* $d_G(x,y)$ between vertices x and y is the minimum length of an x,y-path in G, and the *diameter* $\operatorname{diam}(G)$ is $\max_{x,y\in V(G)} d_G(x,y)$. By convention, $d_G(x,y)$ is infinite when G has no x,y-path. When only one graph is under discussion, it is common to drop the subscripts in $N_G(v)$, $N_G[v]$, $d_G(v)$, and $d_G(x,y)$.

Determining $s(n,t)$ is easy when $t \ge 3$.

Lemma 2.1. *A connected n-vertex graph with at least $\binom{n-2}{2}+2$ edges has diameter at most 3. An n-vertex graph with at least $\binom{n-1}{2}+1$ edges is connected and has diameter at most 2.*

Proof. If G is connected and $\operatorname{diam}(G) > 3$, then there exist x and y with $d(x,y)=4$. Let $j=|N(x)|$ and $k=|N(y)|$, and let $l=|V(G)-N[x]-N[y]|$. To avoid having a shorter x,y-path, $N(x)$ and $N(y)$ must be disjoint and joined by no edge. Thus $j+k+l=n-2$; also $j,k \ge 1$. Hence $|E(\overline{G})| \ge j+k+2l+jk+1 = n-1+l+jk$. Since $l+j+k=n-2$ and $j,k \ge 1$, the quantity $l+jk$ is minimized when $l=n-4$ and $j=k=1$. Hence $|E(\overline{G})| \ge 2n-4$, yielding $|E(G)| \le \binom{n-2}{2}+1$.

If $\text{diam}(G) > 2$, then vertices x and y with $d(x,y) > 2$ are nonadjacent and have non-neighbors covering the remaining vertices, so $|E(\overline{G})| \geq n-1$, yielding $|E(G)| \leq \binom{n-1}{2}$. □

Lemma 2.2. *If $n, t \geq 3$, then $s(n,t) = \binom{n-2}{2} + 1$.*

Proof. An edge-colored complete graph in which all edges incident to two vertices have the same color has no connected rainbow spanning subgraph. Hence $s(n,t) \geq \binom{n-2}{2} + 1$.

For the upper bound, consider a coloring of $E(K_n)$ using at least $\binom{n-2}{2} + 2$ colors; we may assume equality. If a representing subgraph G has an isolated vertex, then the other vertices all have degree at least 2, since otherwise there are at most $\binom{n-2}{2} + 1$ edges. Now adding any edge e incident to an isolated vertex v and removing the edge in G with the same color as e yields a representing subgraph with no isolated vertex.

Hence we may assume that G has no isolated vertex. If G is disconnected, then $|E(\overline{G})| \geq 2(n-2)$, but $|E(\overline{G})| \leq 2n-5$. Hence G is connected, and Lemma 2.1 restricts G to diameter at most 3, as desired. □

The difficult case is $t = 2$, where we must force a rainbow spanning subgraph with diameter 2. For $s(3,2)$, two colors are clearly both necessary and sufficient.

Lemma 2.3. *$s(4,2) = 3$.*

Proof. For the lower bound, give colors 1 and 2 to two independent edges, and give color 3 to the remaining 4-cycle. The only spanning subgraph with three edges and diameter 2 is a star, but the coloring has no rainbow star. For the upper bound, every 4-vertex graph with at least four edges has diameter at most 2. □

For $t = 2$ with $n > 4$, we first provide the construction for the lower bound.

Lemma 2.4. *If $n > 4$, then $s(n,2) \geq \binom{n-2}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor$.*

Proof. In K_n, give one color to all edges incident to one vertex v and also to $\left\lceil \frac{n-1}{2} \right\rceil$ edges covering the remaining $n-1$ vertices. Give distinct other colors to the remaining edges. The total number of colors is $\binom{n-1}{2} - \left\lceil \frac{n-1}{2} \right\rceil + 1$, which equals $\binom{n-2}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor$. In any rainbow subgraph G, the degree of v is at most 1. Every vertex other than v has two incident edges of the same color and hence has degree at most $n-2$ in G. Therefore, not all vertices can be within distance 2 of v. □

3 Rainbow subgraphs with diameter 2

The upper bound for $n > 4$ when $t = 2$ is the difficult part. Let $\Gamma(v)$ denote the set of edges incident to v in G. Say that a color is *incident to* a vertex v if it is used on an edge in $\Gamma(v)$. A *clique* in a graph is a set of pairwise adjacent vertices.

Lemma 3.1. *If $n > 4$, then $s(n,2) \le \binom{n-2}{2} + \lfloor \frac{n-1}{2} \rfloor$.*

Proof. It suffices to study an edge-colored copy H of K_n using exactly $\binom{n-2}{2}+\lfloor\frac{n-1}{2}\rfloor+1$ colors. Suppose that H has no rainbow spanning subgraph with diameter 2. Thus a representing subgraph of H has no spanning complete bipartite subgraph (no copy of $K_{1,n-1}$ or $K_{2,n-2}$), and every vertex is incident to at most $n-2$ colors.

Given a vertex u, let G' be a subgraph of $H-u$ containing one edge of each color not incident to u. Avoiding rainbow spanning stars in H requires $\Delta(G') < n-2$, and hence $|E(G')| \le \binom{n-1}{2} - \lceil \frac{n-1}{2} \rceil = \binom{n-2}{2} + \lfloor \frac{n-1}{2} \rfloor - 1$. Therefore, at least two colors are incident to u. If equality holds, then $\overline{G'}$ is 1-regular (except for one vertex z of degree 2 when $n-1$ is odd). Hence $\overline{G'}$ is disconnected (since $n > 4$), which yields $\mathrm{diam}(G') = 2$. Choose $uv, uw \in \Gamma(u)$ with distinct colors so that either n is even or $\{v,w\} \ne N_{\overline{G'}}(z)$. Since $V(H) - \{u,v,w\} \subseteq N_{G'}(v) \cup N_{G'}(w)$, adding uv and uw to G' completes a rainbow spanning subgraph of H with diameter 2. We may therefore assume that every vertex has at least three incident colors.

Let G be any representing subgraph of H. Since $\mathrm{diam}(G) > 2$, we have $\Delta(G) \le n-2$. Since $K_{2,n-2} \not\subseteq G$, the vertices with degree $n-2$ form a clique. Pick $y,z \in V(G)$ such that $d_G(y,z) = 3$, which requires $d_G(y) + d_G(z) \le n-2$. Let $a = n-2-(d_G(y)+d_G(z))$. Since $|E(G)| = \binom{n-2}{2} + \lfloor \frac{n-1}{2} \rfloor + 1$,

$$\sum_{w \notin \{y,z\}} d_G(w) = 2|E(G)| - (n-2-a) = (n-2)(n-3) + (a+b), \qquad (*)$$

where $b = 2$ for even n and $b = 3$ for odd n. Thus $|X| \ge a+b$, where X is the set of vertices with degree $n-2$ in G, with equality only if all vertices outside $X \cup \{y,z\}$ have degree $n-3$. Indeed, with $W = V(G) - X - \{y,z\}$, we have $|X| = a+b+c$, where $c = |W|(n-3) - \sum_{w \in W} d_G(w)$. Since $a, c \ge 0$, we have $|X| \ge 2$. In particular, we have proved that if H has no rainbow spanning subgraph with diameter 2, then *every* representing subgraph of H has at least b vertices of degree $n-2$.

Suppose $\delta(G) \le 1$. Choose $u, v \in V(G)$ with u having smallest degree and v smallest among the others. Since $|E(G)| = \binom{n-2}{2} + \lfloor \frac{n+1}{2} \rfloor$, at least $\lfloor \frac{n+1}{2} \rfloor$ edges are incident to $\{u,v\}$ in G, with equality only when $G - \{u,v\}$ is complete. Recall that in H at least three colors are incident to u. If all vertices other than u have degree at least 3 in G (which includes all cases with $n > 6$), then we can bring in an edge of another color at u in place of the edge of that color in G, and this will increase the minimum degree. The remaining case is $n \in \{5,6\}$ with three edges incident to $\{u,v\}$ in G and $G - \{u,v\}$ being a complete graph. Again we can increase the minimum degree, since the three edges incident to $\{u,v\}$ can be chosen to be uv and edges of distinct other colors at u and v.

Hence we may choose a representing subgraph G with $\delta(G) \ge 2$. Define y, z, X, W, a, b, c as before. Since $d_G(y,z) = 3$, each vertex of X has y or z as its only nonneighbor and is adjacent to all of W; hence $\{y,z\}$ is the only pair with no common neighbor. By symmetry, we may assume $|X\cap N_G(z)|\ge|X\cap N_G(y)|$. Choose $x\in X\cap N_G(z)$. The color of xy in H appears on an edge xw in G, since otherwise H has a spanning rainbow star with center x. Let $G'=G-xw+xy$. If $w\ne z$, then $\mathrm{diam}(G')=2$; hence we may assume $w=z$. The exchange reduces the degree only at z; if $d_G(z)\ge 3$, then $\delta(G')\ge 2$.

Let $k_G=|X\cap N_G(z)|-|X\cap N_G(y)|$. We have $k_{G'}<k_G$ if $k_G\ge 2$. Hence we may assume $k_G\in\{0,1\}$ or $d_G(z)=2=k_G$.

If $d_G(z)=2=k_G$, then $|X|=2$ and $N_G(z)=X$, and the color on x_iy appears also on x_iz, where $X=\{x_1,x_2\}$. Now $|X|=a+b+c$ and $b\ge 2$ requires $a=c=0$ and $b=2$. With $a=n-2-(d_G(y)+d_G(z))$ and $d_G(z)=2$, we also have $N_G(y)=W$. Let e be the edge in G having the same color as yz. In G, every vertex except y neighbors both vertices of X, so every nonadjacent pair not involving y has at least two common neighbors. Hence if e is not incident to y, then $\mathrm{diam}(G-e+zy)=2$. If $e=wy$ for some $w\in W$, then let $G'=G-\{x_1z,wy\}+\{x_1y,yz\}$, having the same colors as G. At each vertex, at most one edge was removed, so it suffices to check that $d_{G'}(y,v)\le 2$ for all v. Since y reaches z directly and all other vertices via x_1, we have $\mathrm{diam}(G')=2$.

We may therefore assume both that $\delta(G)\ge 2$ and that $N_G(y)\cap X$ and $N_G(z)\cap X$ partition X into sets differing in size by at most 1. Furthermore, since y and z each have a neighbor in X, any two vertices other than $\{y,z\}$ have a common neighbor in X.

Let e be the edge in G with the same color as yz, and let $\hat{G}=G-e+yz$. If e has neither endpoint in X, then deleting e yields all pairs except $\{y,z\}$ with a common neighbor, so $\mathrm{diam}(\hat{G})=2$. We may therefore assume that e has an endpoint in X. Note that since $\delta(G)\ge 2$ and $a=n-2-(d_G(y)+d_G(z))$, we have $d_G(y),d_G(z)\le n-4$.

Case 1: $|X|\ge 4$. From the argument above, y and z each have at least two neighbors in X. Hence any pair other than $\{y,z\}$ has at least two common neighbors in X. Thus deleting e does not increase their distance above 2, and $\mathrm{diam}(\hat{G})=2$.

Case 2: $|X|=2$, *or* $|X|=3$ *and n is odd.* Since $|X|=b$, we have $a=c=0$. Since $d_G(y),d_G(z)\le n-4$, we have $d_{\hat{G}}(y),d_{\hat{G}}(z)\le n-3$. Since also e has an endpoint in X, the representing subgraph $\hat{G}$ has fewer than b vertices of degree $n-2$ (which is forbidden).

Case 3: $|X|=3$ *and n is even.* Since $b=2$, we have $\{a,c\}=\{0,1\}$. When $n=6$, with $d_G(y),d_G(z)\ge 2$ we have $a=0$ and hence $c=1$, but then the one vertex of W has degree only 2, which prevents it from being adjacent to all of X. Hence we may assume $n\ge 8$.

Let x be an endpoint of e in X. Since z has two neighbors in X and y has one neighbor in X, we have $\mathrm{diam}(G-e)\le 2$ unless $e=xy$. Hence $\mathrm{diam}(\hat{G})>2$ requires $e=xy$, and any pair with distance 3 in $\hat{G}$ involves y. Since $d_G(y)\ge 2$, still x and y have a common neighbor $w\in W$. Since $yz\in E(\hat{G})$, a vertex w' with $d_{\hat{G}}(w',y)=3$ must lie outside $N_G(y)\cup N_G(z)\cup\{y,z\}$. If $a=0$, then there is no such vertex. Hence $(a,c)=(1,0)$ and there is one vertex w' in $W - (N_G(y) \cup N_G(z))$. However, $c = 0$ yields $d_G(w') = n - 3$. Since $y, z \notin N_G(w')$, we have $w \in N_G(w')$, and hence $\mathrm{diam}(\hat{G}) = 2$. $\square$

With Lemmas 2.2–2.4 and 3.1, the proof of Theorem 1.1 is now complete.

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