Edge-Colorings of Complete Graphs that Avoid Polychromatic Trees*

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Abstract

Given a positive integer n and a family $\mathcal{F}$ of graphs, let $R^*(n, \mathcal{F})$ denote the maximum number of colors in an edge-coloring of K_n such that no subgraph of K_n belonging to $\mathcal{F}$ has distinct colors on its edges. We determine $R^*(n, \mathcal{T}_k)$, where $\mathcal{T}_k$ is the family of trees with k edges. We derive general bounds for $R^*(n, T)$, where T is an arbitrary tree with k edges. Finally, we present a single tree T with k edges such that $R^*(n, T)$ is nearly as small as $R^*(n, \mathcal{T}_k)$.

Keywords: anti-Ramsey, edge-coloring, tree

1 Introduction

In classical Ramsey theory, we study monochromatic subgraphs that are forced by every coloring of the edges of K_n with a fixed number of colors. In anti-Ramsey theory, we study polychromatic subgraphs that are forced by edge-colorings of K_n using many colors. A subgraph in an edge-coloring is *polychromatic* if the colors on its edges are distinct. Given a positive integer n and a family $\mathcal{F}$ of graphs, the *anti-Ramsey number* $R^*(n, \mathcal{F})$ is the maximum number of colors in a coloring of $E(K_n)$ that has no polychromatic copy of any graph in $\mathcal{F}$. When $\mathcal{F}$ consists of a single graph H, we write $R^*(n, H)$ for $R^*(n, \{H\})$.

Anti-Ramsey numbers were introduced by Erdős, Simonovits and Sós [7]. They showed that these are closely related to Turán numbers. The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ of $\mathcal{F}$ is the maximum number of edges in an n-vertex simple graph having no member of $\mathcal{F}$ as a subgraph. When $\mathcal{F}$ consists of a single graph H, we write $\mathrm{ex}(n, H)$ for $\mathrm{ex}(n, \{H\})$.

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Taking one edge of each color in an edge-coloring of K_n shows immediately that $R^*(n,\mathcal{F}) \le \mathrm{ex}(n,\mathcal{F})$. For a lower bound, let $\mathcal{H}$ be the family of graphs obtainable by deleting one edge from a graph in $\mathcal{F}$, and let G be an n-vertex simple graph having no subgraph in $\mathcal{H}$. A coloring with a polychromatic copy of G and a single color on all remaining edges has no polychromatic subgraph in $\mathcal{F}$, and hence $R^*(n,\mathcal{F}) \ge \mathrm{ex}(n,\mathcal{H})$.

Using an earlier result of Erdős and Simonovits [6] on the asymptotics of Turán numbers (plus some additional observations), [7] showed also that $R^*(n,H) - ex(n,\mathcal{H}) = o(n^2)$ and hence that $R^*(n,H)/\binom{n}{2} \to 1-(1/d)$ as $n \to \infty$, where $d+1 = \min\{\chi(H-e) \colon e \in E(H)\}$ (the argument extends easily for an arbitrary family $\mathcal{F}$). These observations determine $R^*(n,H)$ asymptotically when $\min\{\chi(H-e) \colon e \in E(H)\} \ge 3$. When $\min\{\chi(H-e) \colon e \in E(H)\} = 2$, the problem is harder. Already it is nontrivial for a tree or a cycle.

Erdős, Simonovits, and Sós [7] began the study of $R^*(n,C_k)$. They noted that $R^*(n,C_3) = n$ and provided an edge-coloring of K_n having $n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1)$ colors and no polychromatic C_k. They conjectured that this value is optimal. Alon [1] proved the conjecture for $k \le 4$ and proved that $R^*(n,C_k) \le n(k-2) + \binom{k-1}{2}$ in general. In [9], we proved the conjecture for $k \le 6$ and improved the general upper bound to $n\left(\frac{k+1}{2} - \frac{2}{k-1}\right) - (k-2)$ for all k and to $nk/2 - (k-2)$ when k is even. The bipartite analogue (using many colors in a coloring of $E(K_{m,n})$ with no polychromatic C_{2k}) is solved in [3].

Anti-Ramsey numbers have also been studied for the path P_{k+1} and the star $K_{1,k}$ with k edges. Simonovits and Sós [10] proved that $R^*(n,P_{k+1}) = n\lfloor\frac{k-2}{2}\rfloor + c_k$ for large enough n, where c_k depends only on k. Jiang [8] proved for $n > k$ that $R^*(n,K_{1,k}) = \left\lfloor\frac{n(k-2)}{2}\right\rfloor + \left\lfloor\frac{n}{n-k+2}\right\rfloor$, except that when n, k, and $\left\lfloor\frac{2n}{n-k+2}\right\rfloor$ are all odd the value may exceed this by 1. Using the known asymptotics for $\mathrm{ex}(n,K_{2,t})$, Axenovich and Jiang [2] determined $R^*(n,K_{2,t})$ asymptotically.

Here we further study anti-Ramsey numbers for trees and for families of trees. Let $\mathcal{T}_k$ denote the family of all trees with k edges. For $n > k$, we prove that $R^*(n,\mathcal{T}_k) = l(n,k)+1$, where $l(n,k)$ is the maximum number of edges in a simple n-vertex graph such that every two components together have at most k vertices. Since the extremal graph for the latter problem must be a disjoint union of complete graphs, this reduces the computation of $R^*(n,\mathcal{T}_k)$ to the purely numerical question of computing $l(n,k)$. We compute this for all cases; when $n \ge 2k-1$, the extremal graph consists of one component of order $\lceil k/2\rceil$ plus components of order $\lfloor n/2\rfloor$ and perhaps one that is smaller. As a consequence, for $n \ge 2k-1$ we have $R^*(n,\mathcal{T}_k) = \frac{n}{2}\lfloor\frac{k-2}{2}\rfloor + \epsilon$, where $0 \le \epsilon \le \binom{k}{2}$. The full result for $n \ge 2k-1$ is

$$R^*(n,\mathcal{T}_k) - 1 = l(n,k) = \binom{\lceil k/2\rceil}{2} + r\binom{\lfloor k/2\rfloor}{2} + \binom{s}{2},$$

where $r = \left\lfloor\frac{n-\lceil k/2\rceil}{\lfloor k/2\rfloor}\right\rfloor$ and $s = n - \left\lceil\frac{k}{2}\right\rceil - r\left\lfloor\frac{k}{2}\right\rfloor$. For $k < n \le 2k-1$, it is $R^*(n,\mathcal{T}_k) - 1 = \binom{k-1}{2}$.

For any single tree T with k edges, $R^*(n,T) \le \mathrm{ex}(n,T) \le n(k-1)$, and the Erdős–Sós Conjecture (see [4]) would halve the upper bound. We construct trees with k edges whose anti-Ramsey numbers range between $R^*(n,\mathcal{T}_k)$ and $n(k-1)/2$. The *broom* $B_{s,t}$ is formed by identifying the center of a star with s edges and an endpoint of a path with t edges; the resulting tree has $k = s + t$ edges. We show that

$$\max\{R^*(n,K_{1,s+1}), R^*(n,P_{t+2})\} \le R^*(n,B_{s,t}) \le \max\{R^*(n,K_{1,s+1}), nt/2\}$$

when $n \ge (k-1)^2$. Thus for $s = t = k/2$, the broom $B_{s,t}$ is already forced when the number of colors is just slightly larger than $R^*(n,\mathcal{T}_k)$.

Throughout the paper, we use the following notion.

Definition 1 Let G be a graph and c be a coloring of $E(G)$. A *representing subgraph* of c is a spanning subgraph L of G having exactly one edge of each color of c (it may have isolated vertices).

As remarked earlier, a representing subgraph yields $R^*(n,H) \le \mathrm{ex}(n,H)$.

2 The Anti-Ramsey Number of $\mathcal{T}_k$

When $n \le k$, clearly $R^*(n,\mathcal{T}_k) = \binom{n}{2}$. We therefore assume henceforth that $n > k$.

Definition 2 Let G be an n-vertex graph. A coloring c of $E(K_n)$ is *induced* by G if c assigns distinct colors to the edges of G and assigns one additional color to all of $E(\bar{G})$.

The *size* of a graph G, written $e(G)$, is the number of edges it has. A coloring of K_n induced by G uses $e(G)+1$ colors (unless $G = K_n$).

Lemma 3 *If G is an n-vertex graph in which every two components together have at most k vertices, then a coloring of $E(K_n)$ induced by G has no polychromatic tree with k edges.*

Proof. Consider a tree T with k edges in K_n. Since T has $k+1$ vertices, T must contain vertices from three components of G. Therefore T has two edges in $E(\bar{G})$, which have the same color under every coloring induced by G. Thus such a coloring has no polychromatic tree with k edges. □

Let $\mathcal{L}(n,k)$ denote the family of n-vertex simple graphs in which every two components together have at most k vertices, and let $l(n,k)$ be the maximum size of a graph in $\mathcal{L}(n,k)$. Lemma 3 immediately yields $R^*(n,\mathcal{T}_k) \ge l(n,k)+1$ when $n > k$. We will show that this lower bound on $R^*(n,\mathcal{T}_k)$ is tight. We use $n(G)$ to denote the number of vertices (the order) of a graph G.

Lemma 4 *Every connected graph G contains a vertex w such that for all $e \in E(G)$, the component of $G - e$ containing w has at least $n(G)/2$ vertices.*

Proof. Let T be a spanning tree of G; it suffices to show that T contains such a vertex. For each edge $uv \in E(T)$, we orient it from u to v if in $T - uv$ the component containing v contains at least $\lceil n(T)/2 \rceil$ vertices (if an edge can be oriented either way, then we choose one direction). Let $D(T)$ denote the resulting oriented tree.

If $D(T)$ has a vertex x with outdegree at least 2, then $T - x$ has two disjoint subtrees each having at least $\lceil n(T)/2 \rceil$ vertices, which is impossible. Now, since T does not contain a cycle, $D(T)$ does not contain a directed cycle. Hence $D(T)$ contains a sink w (a vertex with outdegree 0). Since $D(T)$ has no vertex with outdegree at least two, every path in T with endpoint w is an oriented path with sink w in $D(T)$. Thus every edge uv points towards w, meaning that w is in a component of $T - uv$ with at least $\lceil n(T)/2 \rceil$ vertices. □

Theorem 5 *If $n > k$, then $R^*(n, \mathcal{T}_k) = l(n, k) + 1$.*

Proof. The lower bound follows from Lemma 3. It remains to show that $R^*(n, \mathcal{T}_k) \leq l(n, k) + 1$. Let c be a coloring of $E(K_n)$ that has no polychromatic tree with k edges. Let H be a representing subgraph of c that has a largest possible component. It suffices to show that $e(H) \leq l(n, k) + 1$.

By definition, H contains no tree with k edges; hence every component in H has order at most k. Let F denote a component of largest order in H. By Lemma 4, F contains a vertex u such that for all $e \in E(F)$, the component in $F - e$ containing u has at order at least $\lceil n(F)/2 \rceil$. Let v be a vertex of another component F' of H, which exists since $n > k$. Since H is a representing subgraph of c and $uv \notin E(H)$, there is an edge $e' \in E(H) - uv$ with color $c(uv)$, and $H' = H - e' + uv$ is also a representing subgraph of c. The edge e' must be a cut-edge of F, since otherwise H' has a component with larger order than F, contradicting our choice of H.

Let F_1 and F_2 denote the two components of $F - e'$, with $V(F_1)$ containing u. Since F_1 has at least half the vertices of F, we have $n(F_1) \geq n(F_2)$. Since $F_1 \cup uv \cup F'$ is a component of H', which has order at most that of F, we have $n(F') \leq n(F_2) \leq n(F_1)$. This implies that F_1 and F_2 are components of $H - e'$ with largest order. Since $n(F_1) + n(F_2) = n(F) \leq k$, the sum of the orders of any two components of $H - e'$ is at most k. Hence $H - e' \in \mathcal{L}(n, k)$. Thus $e(H - e') \leq l(n, k)$ and $e(H) \leq l(n, k) + 1$. □

In the remainder of this section, we complete the determination of $R^*(n, \mathcal{T}_k)$ by computing $l(n, k)$. The following fact is well known; we include a proof for completeness.

Proposition 6 *Given positive integers n, q with $n \geq q$, let $J_{n,q} = rK_q + K_s$, where $r = \lfloor n/q \rfloor$ and $s = n - rq$. The graph $J_{n,q}$ is the unique graph of maximum size among the n-vertex graphs in which every component has at most q vertices.*

Proof. Let G be such a graph with maximum size. Clearly, each component of G is a complete graph. If G has two components with order less than q, then deleting a vertex from the smaller (or either if they have equal order) and adding it to the larger increases the number of edges. Hence at most one component has order less than q, and $G = J_{n,q}$. $\square$

For a graph G, we use $\rho(G)$ to denote $\sum_{u \in V(G)} d_G(u)$. Note that $\rho(G) = 2e(G)$.

Lemma 7 *If $n \geq a \geq b > 0$, then $\rho(J_{n,a}) - \rho(J_{n,b}) \geq r(a-b)(a-b+1)$, where $r = \lfloor n/a \rfloor$.*

Proof. We consider the change in degree for each vertex of $J_{n,a}$ when transforming to $J_{n,b}$. Since the maximum degree in $J_{n,b}$ is $b-1$, the ra vertices of maximum degree in $J_{n,a}$ lose degree $a - b$. If $s \geq b$, then the remaining s vertices do not gain degree, and hence $\rho(J_{n,a}) - \rho(J_{n,b}) \geq ra(a-b) \geq r(a-b)(a-b+1)$.

If $s \leq b-1$, then the remaining s vertices may gain degree. Since $a > b$, there are at least r components of order b in $J_{n,b}$ (trimming down components of order a), and at least s vertices are adjacent to no vertices in these components. Even if the trimmed and leftover vertices go into a single component, its size will be bounded by $s + r(a-b)$. Hence the s leftover vertices in $J_{n,a}$ have degree at most $s + r(a-b) - 1$ in $J_{n,b}$. Therefore, they gain degree at most $r(a-b)$ each. This yields $\rho(J_{n,a}) - \rho(J_{n,b}) \geq ra(a-b) - sr(a-b) \geq r(a-b)(a-s) \geq r(a-b)(a-b+1)$. $\square$

We use $G + H$ for the disjoint union of graphs G and H.

Lemma 8 *When $\lceil k/2 \rceil \leq m \leq k-1$ and $n \geq k$, the largest graph in $\mathcal{L}(n,k)$ that has a component of order m is $K_m + J_{n-m,k-m}$. Furthermore, $e(K_m + J_{n-m,k-m})$ is maximized when $m = k-1$ for $k+1 \leq n \leq 2k-1$ and when $m = \lceil k/2 \rceil$ for $n \geq 2k$.*

Proof. The first claim follows immediately from Proposition 6. Let $H_m = K_m + J_{n-m,k-m}$. To prove the second claim, we consider two cases.

Case 1: $k+1 \leq n \leq 2k-1$. In this case, we show that $e(H_m) \leq \binom{k-1}{2}$, with equality when $m = k-1$. Since $n > k$, H_m has a component of order $k - m$ and hence has at least $k - m$ vertices of degree $k - m - 1$. In transforming H_m into H_{k-1}, we increase the degree of the m vertices in the large clique by $k - 1 - m$ each, increase the degree of $k - m - 1$ vertices from smaller cliques by $(k-2) - (k-m-1)$ each, and decrease the degree of the remaining $n - k + 1$ vertices by at most $k - m - 1$ each. Thus

$$\rho(H_{k-1}) - \rho(H_m) \geq m(k-1-m) + (k-m-1)(m-1) - (n-k+1)(k-m-1) = (k-1-m)(2m-2+k-n).$$

Since $n \le 2k-1$, the factor $2m-2+k-n$ is nonnegative unless $m = k/2$ and $n = 2k-1$. In this remaining case, H_m has three components of order $k/2$ and one of order $k/2-1$. For each of these $k/2-1$ vertices, we overcounted the loss in degree by 1. The correction yields $e(H_{k/2}) = e(H_{k-1})$ when k is even and $n = 2k-1$.

Case 2: $n \ge 2k$. In this case, we show that $e(H_m)$ is maximized when $m = \lceil k/2 \rceil$. In transforming $H_{\lceil k/2 \rceil + t}$ into $H_{\lceil k/2 \rceil}$, we transform $K_{\lceil k/2 \rceil + t} + K_{\lfloor k/2 \rfloor - t}$ into $K_{\lceil k/2 \rceil} + K_{\lfloor k/2 \rfloor}$ and $J_{n-k, \lfloor k/2 \rfloor - t}$ into $J_{n-k, \lfloor k/2 \rfloor}$. The first of these decreases the degree sum by $2t(\lceil k/2 \rceil - \lfloor k/2 \rfloor + t)$, which is at most $2t(t+1)$. On the other hand, Lemma 7 yields $\rho(J_{n-k,\lfloor k/2 \rfloor}) - \rho(J_{n-k,\lfloor k/2 \rfloor - t}) \ge rt(t+1)$, where $r = \left\lfloor \frac{n-k}{\lfloor k/2 \rfloor} \right\rfloor$. Since $n \ge 2k$, we have $r \ge 2$, and hence $\rho(H_{\lceil k/2 \rceil}) - \rho(H_{\lceil k/2 \rceil + t}) \ge -2t(t+1) + rt(t+1) \ge 0$. $\square$

These lemmas determine $l(n,k)$ and $R^*(n, \mathcal{T}_k)$. In the trivial case where $n \le k$, both equal $\binom{n}{2}$.

Theorem 9 *If $n > k$, then*

$$R^*(n, \mathcal{T}_k) - 1 = l(n,k) = \begin{cases} \binom{k-1}{2} & \text{if } k < n \le 2k-1 \\ \binom{\lceil k/2 \rceil}{2} + r\binom{\lfloor k/2 \rfloor}{2} + \binom{s}{2} & \text{if } n \ge 2k, \end{cases}$$

where $r = \left\lfloor \frac{n - \lceil k/2 \rceil}{\lfloor k/2 \rfloor} \right\rfloor$ and $s = n - \lceil k/2 \rceil - r \lfloor k/2 \rfloor$.

Proof. By Theorem 5, it suffices to determine $l(n,k)$, the maximum size of an n-vertex graph in which the sum of the orders of any two components is at most k. If a graph in $\mathcal{L}(n,k)$ has no component of order at least $\lceil k/2 \rceil$, then we can shift a vertex from the smallest component to the largest component to obtain a larger graph in $\mathcal{L}(n,k)$. Hence in an extremal graph the order of the largest component, m, satisfies $\lceil k/2 \rceil \le m \le k-1$. Now Lemma 8 completes the proof. $\square$

3 Bounds for Individual Trees

In this section, we obtain bounds on $R^*(n,T)$ when T is a tree with k edges. For any tree T with k edges, $R^*(n,T) \ge R^*(n,\mathcal{T}_k)$. Theorem 9 yields $R^*(n,\mathcal{T}_k) = \frac{n}{2}\left\lfloor \frac{k-2}{2} \right\rfloor + O(1)$ as $n \to \infty$. Thus $R^*(n,T) \ge \frac{n}{2}\left\lfloor \frac{k-2}{2} \right\rfloor + O(1)$ as $n \to \infty$. An easy upper bound follows from standard extremal results about trees.

Proposition 10 *If T is a tree with k edges and $n \geq 2k$, then*

$$\frac{n}{2}\left\lfloor\frac{k-2}{2}\right\rfloor + c_k \leq R^*(n,T) \leq n(k-1),$$

where c_k is independent of n.

Proof. As remarked above, the lower bound follows from our results on $R^*(n,\mathcal{T}_k)$. For the upper bound, we use $R^*(n,T) \leq \mathrm{ex}(n,T)$. It is well known that $\mathrm{ex}(n,T) \leq n(k-1)$; we include the argument for completeness.

Since a graph with minimum degree at least k contains every tree with k edges (by induction on k), it suffices to show that an n-vertex graph G with more than $n(k-1)$ edges has a subgraph with minimum degree at least k. Such a subgraph is obtained by taking a minimal subgraph H of G such that $e(H) \geq n(H)(k-1)$. □

Erdős and Sós [4] conjectured that $\mathrm{ex}(n,T) \leq \frac{1}{2}n(k-1)$ for every tree T with k edges, and the evidence for this is strong. If the conjecture holds, then the upper bound on $R^*(n,T)$ in Proposition 10 improves to $\frac{1}{2}n(k-1)$.

A graph with maximum degree $k-1$ has no star with k edges, so the bound on $\mathrm{ex}(n,T)$ in the Erdős–§ós Conjecture cannot be reduced below $\frac{1}{2}n(k-1)$ for all trees with k edges. As mentioned in the introduction, Simonovits and Sós [10] proved that $R^*(n,P_{k+1}) = n\lfloor\frac{k-2}{2}\rfloor + c_k$ when n is sufficiently large, where c_k depends only on k. Also, Jiang [8] proved for $n > k$ that $R^*(n,K_{1,k}) = \left\lfloor\frac{n(k-2)}{2}\right\rfloor + \lfloor\frac{n}{n-k+2}\rfloor$, except that when n, k, and $\lfloor\frac{2n}{n-k+2}\rfloor$ are all odd the value may be larger than this by 1. Note that for $n > 3k-6$, Jiang's result reduces to $R^*(n,K_{1,k}) = \left\lfloor\frac{n(k-2)}{2}\right\rfloor + 1$. These values are roughly $n(k-2)/2$ rather than the $n(k-1)/2$ of the Erdős–Sós Conjecture because the polychromatic part of the construction avoids trees with $k-1$ edges rather than k-edges.

If the Erdős–Sós Conjecture is true, then paths and stars have rather high anti-Ramsey number among trees of fixed sized. It is perhaps surprising then that combining a path and a star yields a specific tree whose anti-Ramsey number is near the lower bound in Proposition 10. We consider single trees in a family that includes both paths and stars and obtain anti-Ramsey numbers throughout the range from $nk/4$ to $nk/2$, asymptotically.

Definition 11 Given positive integers s,t, the *broom* $B_{s,t}$ is the tree with $s+t$ edges obtained by identifying the center of $K_{1,s}$ with an endpoint of P_{t+1}.

We give bounds for $R^*(n,B_{s,t})$. The numerical aspects use the constructions in [10] and [8] and the upper bound in [8], but our argument does not use the upper bound in [10]. We also need the following result.

Theorem 12 (Erdős and Gallai [5]) *A simple n-vertex graph having no cycle of length exceeding l has at most $l(n-1)/2$ edges.*

Theorem 13 *Let s and t be positive integers with $s+t=k$. If $n \geq (k-1)^2$, then*

$$\max\{R^*(n, K_{1,s+1}), R^*(n, P_{t+2})\} \leq R^*(n, B_{s,t}) \leq \max\{R^*(n, K_{1,s+1}), nt/2.\}$$

For sufficiently large n, this yields

$$\frac{1}{2}nr_1 + c_k \leq R^*(n, B_{s,t}) \leq \frac{1}{2}nr_2 + 1,$$

where $r_1 = \max\{s-1, 2\lfloor (t-1)/2 \rfloor\}$, $r_2 = \max\{s-1, t\}$, and c_k is independent of n.

Proof. Since $B_{s,t}$ contains both $K_{1,s+1}$ and P_{t+2}, the lower bounds follow immediately, with the numerical aspects of the lower bounds invoking the constructions of [10] and [8].

For the upper bound, we show that $R^*(n, B_{s,t}) \leq \max\{R^*(n, K_{1,s+1}), nt/2\}$. The numerical statement then follows from the upper bound of [8] on $R^*(n, K_{1,s+1})$.

Let c be a coloring of $E(K_n)$ with no polychromatic $B_{s,t}$, and let m be the number of colors used by c. We prove that if $m > R^*(n, K_{1,s+1})$, then $m \leq nt/2$. Since $m > R^*(n, K_{1,s+1})$, we have a polychromatic copy of $K_{1,s+1}$ under c. Let H be a representing subgraph of c with the largest possible maximum degree $\Delta(H)$; we have $\Delta(H) \geq s+1$.

Let u be a vertex of H with maximum degree; u has at least $s+1$ neighbors in H. Let F be the component of H containing u. We bound the size of each component separately.

We show first that no component of H other than F has a cycle of length at least t. Let F' be such a component containing such a cycle C, and let v be a vertex on C. Since H is a representing subgraph of c and $uv \notin E(H)$, there exists an edge e' in H with color $c(uv)$. Since $H' = H - e' + uv$ is another representing subgraph of c, the choice of H implies that e' is incident to u. Thus $e' = uw$ for some $w \in N_H(u)$. We obtain a copy of $B_{s,t}$ in H' by combining the edge uv, a path of length $t-1$ within C starting from v, and s edges from u to $N_H(u) - \{w\}$. This contradiction eliminates the possibility of such a cycle in F'. By Theorem 12, $e(F') \leq (t-1)[n(F')-1]/2 < n(F')(t-1)/2$.

It remains to show that $e(F) \leq n(F)t/2$; we consider two cases.

Case 1: $\Delta(H) \geq k$. We show that F does not contain a cycle of length at least $t+1$; Theorem 12 then yields $e(F) \leq [n(F)-1]t/2 < n(F)t/2$. If F has such a cycle, then F contains a path P of length t starting at u. Since u has degree at least k in H, and $k = s+t$, we conclude that u has at least s neighbors in H outside $V(P)$. Combining P with s edges from u to such neighbors forms a forbidden $B_{s,t}$ in H.

Case 2: $s+1 \leq \Delta(H) \leq k-1$. We first show that $F - N_H[u]$ has no cycle of length at least $t-1$. Given such a cycle C, let P be a shortest path in F from u to C. Since

$V(C) \cap N_H[u] = \emptyset$, P has length at least 2. Using part of C, we extend P to a path of length t starting at u on which u has exactly one neighbor. Adding s edges incident to u completes a forbidden copy of $B_{s,t}$ in H. Hence $F - N_H[u]$ has no such cycle.

Theorem 12 now yields $e(F - N_H[u]) \le (t-2)[n(F - N_H(u)) - 1]/2 < n(F)(t-2)/2$. The only edges in F that we have not considered are those incident to $N_H[u]$. Since $\Delta(H) \le k-1$, there are at most $(k-1)^2$ such edges. Hence $e(F) \le n(F)(t-2)/2 + (k-1)^2 \le n(F)t/2$ when $n \ge (k-1)^2$. □

We use the restriction $n \ge (k-1)^2$ only in the last line; most likely this threshold can be reduced.

When $s = 1$, the broom $B_{s,t}$ is P_{t+2}. Simonovits and Sòs [10] showed that $R^*(n, P_{t+2})$ is asymptotic to $n\lfloor (t-1)/2 \rfloor$. This relates to Theorem 13 in two ways. First, our argument gives a short proof of a slightly weaker bound; their lengthier argument is needed to prove the optimal asymptotic bound. Second, their bound suggests that with more effort it may be possible to replace $nt/2$ with $n(t-1)/2$ in our upper bound, or perhaps to show that the lower bound in terms of paths and stars is tight. When the star dominates, already Theorem 13 yields the exact answer.

Corollary 14 *If s, t are positive integers with $s + t = k$ and $s \ge t + 1$, then $R^*(n, B_{s,t}) = R^*(n, K_{1,s+1}) = \lfloor n(s-1)/2 \rfloor + 1$ when $n \ge (k-1)^2$.*

Proof. When $s \ge t+1$, we have $R^*(n, K_{1,s+1}) \ge R^*(n, P_{t+2})$ and $R^*(n, K_{s+1}) \ge nt/2$. Hence the claim follows from the bounds in Theorem 13. □

For large k and larger n, $R^*(n, B_{s,k-s})/nk$ varies between roughly 1/4 and roughly 1/2 as s varies between 2 and $k-2$, Thus the bounds that result from Proposition 10 and from the Erdős–Sós Conjecture cannot be much improved in general. Motivated by this, we suggest two questions.

Conjecture 15 *If T is a tree with k edges and n is sufficiently large, then $R^*(n, T) \le n(k-2)/2 + c_k$, where c_k is independent of n.*

Conjecture 16 $\max\{R^*(n, T)\colon\ T \in \mathcal{T}_k\} = R^*(n, K_{1,k})$ *when n is sufficiently large.*

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