Antipodal edge-colorings of hypercubes

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Revised July 2017

Abstract

Two vertices of the k-dimensional hypercube Q_k are *antipodal* if they differ in every coordinate. Edges uv and xy are *antipodal* if u is antipodal to x and v is antipodal to y. An *antipodal edge-coloring* of Q_k is a 2-edge-coloring such that antipodal edges always have different colors. Norine conjectured that for $k \geq 2$, in every antipodal edge-coloring of Q_k some two antipodal vertices are connected by a monochromatic path. Feder and Subi proved this for $k \leq 5$. We prove it for $k = 6$.

1 Introduction

The k-dimensionsal hypercube Q_k is the graph with vertex set $\{0,1\}^k$ in which vertices are adjacent if they differ in exactly one coordinate. Vertices in Q_k are *antipodal* if they differ in every coordinate. Edges uv and xy are *antipodal* if u is antipodal to x and v is antipodal to y. An *antipodal edge-coloring* of Q_k is a 2-edge-coloring in which antipodal edges have different colors. In an antipodal edge-coloring, the graphs formed by the two colors are isomorphic. Norine (see [1]) posed a conjecture about antipodal edge-colorings.

Conjecture 1.1 ([1])**.** *For $k \geq 2$, in every antipodal edge-coloring of Q_k there is a pair of antipodal vertices connected by a monochromatic path.*

A path whose endpoints are antipodal is an *antipodal path*; we seek a monochromatic antipodal path. Feder and Subi [2] proved a stronger version of Conjecture 1.1 for $k \leq 5$. A *geodesic* is a path P that is shortest among all paths having the same endpoints as P. A *k-geodesic* is a geodesic of length k. In Q_k, a geodesic changes each coordinate of the vertices at most once, so any geodesic antipodal path in Q_k is a k-geodesic. Feder and Subi showed for $k \leq 5$ that every antipodal edge-coloring of Q_k has a monochromatic antipodal geodesic.

Feder and Subi [2] also proved the conclusion for any 2-edge-coloring (not necessarily antipodal) in which the colors do not alternate along any 4-cycle. However, weakening the

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hypothesis to require only giving each color exactly half the edges in each dimension permits a counterexample. We show this in Figure 1 for Q_4, where each monochromatic subgraph consists of the same two isomorphic components. (In Q_3, having half the edges in each coordinate always yields monochromatic antipodal paths).

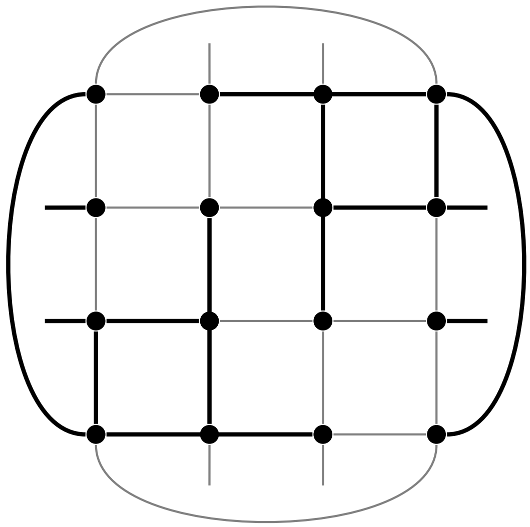

Figure 1: 2-edge coloring of Q_4 with eight edges in every dimension red.

A useful lemma from [2] (also proved in [3] and [4]) helps to simplify our arguments.

Lemma 1.2 ([2]). *If an antipodally edge-colored k-cube contains an antipodal geodesic that is the union of two monochromatic paths, then it contains a monochromatic geodesic from the common vertex of these paths to its antipode. In particular, a monochromatic $(k-1)$-geodesic guarantees a monochromatic antipodal geodesic.*

Proof. If an antipodal geodesic P consists of two monochromatic paths with common vertex u, then the union of P and the geodesic $\overline{P}$ using its antipodal edges consists of two monochromatic geodesics from u to $\overline{u}$. □

Feder and Subi [2] used Lemma 1.2 to show that a counterexample for Q_k yields a counterexample for Q_{k+1}. Given an antipodal edge-coloring f of Q_k with no monochromatic antipodal geodesic, use f on disjoint copies Q and Q' of Q_k in Q_{k+1}, complementing colors in one copy. Complete an antipodal edge-coloring g of Q_{k+1} by any antipodal coloring of the edges joining Q to Q'. For $v \in V(Q)$, let v' be its neighbor in Q', and let $\overline{v}$ be the vertex antipodal to v within Q. Suppose that g has a monochromatic antipodal geodesic P, which we may assume is red. Since P must cross the last direction once, it uses vv' for exactly one vertex v in Q. Let x and $\overline{x}'$ be the endpoints of P in Q and Q'. The edges in Q antipodal in Q_{k+1} to the part of P in Q' form a blue geodesic from $\overline{v}$ to x. Its union with the part of P in Q forms a geodesic from $\overline{v}$ to v in Q_k that changes color only once. By Lemma 1.2, f on Q_k contains a monochromatic antipodal geodesic, a contradiction.

In this paper, we first reprove the stronger version of Conjecture 1.1 for $k \in \{4, 5\}$ using a simpler approach than that of [2]. We then extend the method to prove the claim for $k = 6$.

Theorem 1.3. *For $2 \leq k \leq 6$, every antipodal edge-coloring of Q_k has a monochromatic antipodal geodesic.*

We hope that our approach leads to results for larger n. Meanwhile, there are some related results. Leader and Long [4] showed that every subgraph of Q_k having average degree d contains a geodesic of length at least d, which is sharp by the subgraph Q_d. Since both maximal monochromatic spanning subgraphs in an antipodal coloring of Q_k have average degree $k/2$, this result implies a monochromatic geodesic of length at least $k/2$. Also, Gandhi [3] establishes an upper bound on the number of monochromatic geodesics of length d in an antipodal coloring of Q_k and studies the maximum number of antipodal geodesics in a subgraph of Q_k containing a fixed proportion of the edges.

2 Smaller Cubes

The vertex antipodal to a vertex v will be denoted $\overline{v}$. All figures show red edges as bold and blue edges as dashed. Gray or thin edges have unspecified color. Some edges are omitted for clarity. An *alternating 4-cycle* is a 4-cycle $[a, b, c, d]$ whose edges alternate in color.

Lemma 2.1. *Every antipodal edge-coloring of Q_4 having an alternating 4-cycle contains a monochromatic antipodal geodesic.*

Proof. The edges antipodal to an alternating 4-cycle C also form an alternating 4-cycle, C'. They are connected by a path P of length 2 crossing the other two directions. If P is monochromatic, then adding the incident edges of that color in C and C' yields a monochromatic antipodal geodesic. Hence we may assume that P (with central vertex v) is not monochromatic (see Figure 2).

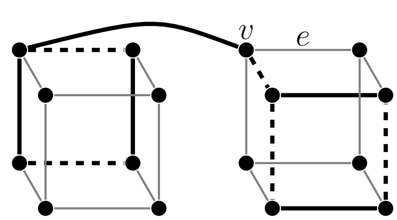

Figure 2: Finding a monochromatic 3-geodesic in Q_4.

Each edge of P forms a 3-geodesic with the incident edge of C or C' sharing its color and one of the two remaining edges incident to v. This third edge is the same for both edges of P, indicated by e in Figure 2. Hence either color on e completes a monochromatic 3-geodesic. By Lemma 1.2, the coloring then contains a monochromatic antipodal geodesic. □

Theorem 2.2. *Every antipodal edge-coloring of Q_4 has a monochromatic antipodal geodesic.*

Proof. By Lemma 2.1, we may forbid alternating 4-cycles. If some 4-cycle C has two edges of each color, then let v be a vertex where the color changes. Let e be an edge incident to v not on C. Given either color, e forms a monochromatic 3-geodesic with two edges from C. By Lemma 1.2, the coloring contains a monochromatic antipodal geodesic.

Thus we may assume that every 4-cycle has three edges of the same color. Let C be a 4-cycle with three red edges. All eight edges incident to C must be blue to avoid a red 3-geodesic. Similarly, all eight edges incident to the 4-cycle C' antipodal to C must be red. Now 4-cycles using no edges of C or C' have two edges of each color, a contradiction. □

Lemma 2.3. *For $k \geq 5$, every 2-edge-coloring of Q_k contains a monochromatic 3-geodesic.*

Proof. In a 2-edge-coloring, since $\delta(Q_k) \geq 3$, every vertex is the center of some monochromatic 2-geodesic. We may let $\langle u, v, w \rangle$ be a red 2-geodesic, as in Figure 3.

To avoid red 3-geodesics, the edges incident to u or w in other directions must be blue, as in Figure 3. These blue edges form blue 2-geodesics, extending to blue 3-geodesics unless the edges incident to their endpoints in other directions are red. Those red edges (extending from two blue 2-geodesics) include geodesics in the original two directions, and they extend by a third edge to form a red 3-geodesic, such as in the lower left of Figure 3. □

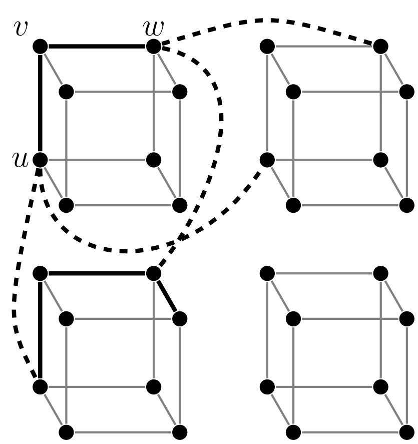

Figure 3: Finding a monochromatic 3-geodesic in Q_k.

Theorem 2.4. *Every antipodal edge-coloring of Q_5 has a monochromatic antipodal geodesic.*

Proof. View Q_5 as four copies of Q_3 in a 4-cycle. By Lemma 2.3, such a coloring has a monochromatic 3-geodesic P, as shown in red in the upper left copy of Q_3 in each case in Figure 4. The antipodal 3-geodesic $\overline{P}$ in blue is in the lower right. By Lemma 1.2, it suffices to find a monochromatic 4-geodesic. Consider a 4-cycle C through endpoints of P and $\overline{P}$. Let s and t be the vertices of C not in P or $\overline{P}$. To avoid a monochromatic 4-geodesic, edges of C must be colored oppositely from their incident edges on P and $\overline{P}$, as in Figure 4. Consider the edges incident to s and t in their copies of Q_3.

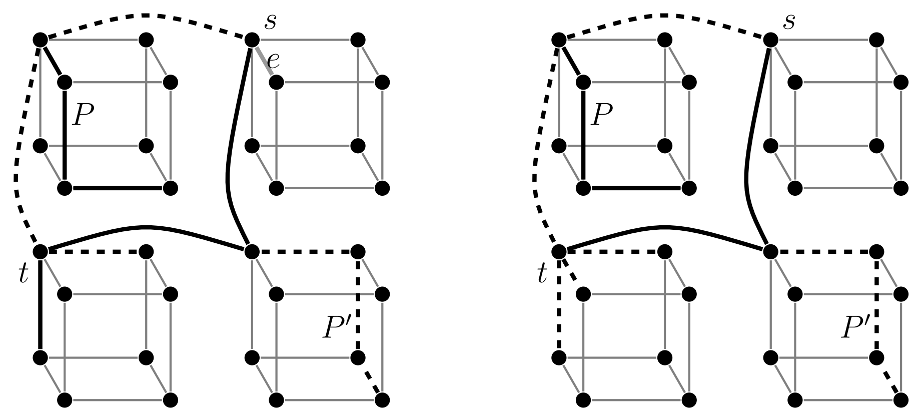

Figure 4: Case 1 and Case 2 for antipodal edge-colorings of Q_5.

Case 1: *Two edges incident to t in its 3-cube have different colors.* In the 3-cube containing s, either color on the edge e in the other direction completes a monochromatic 4-geodesic.
Case 2: *The three edges incident to t in its copy of Q_3 have the same color.* By symmetry, we may assume that these edges are blue. By the same reasoning, the three edges incident to s in its copy of Q_3 have the same color. If blue, then we have a blue 4-geodesic. Hence they are red, as in Figure 5.

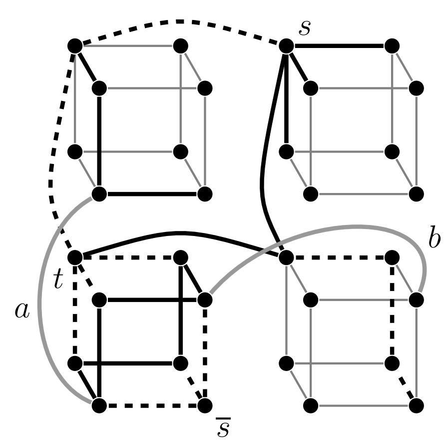

Figure 5: Case 2 for antipodal edge-colorings of Q_5, continued.

The edges at $\overline{s}$ antipodal to these are blue and lie in the copy of Q_3 containing t. Any additional blue edge in this copy of Q_3 would complete a blue antipodal geodesic, so the remaining edges form a red 6-cycle, as shown. Finally, consider the two edges a and b of Figure 5. If either is red, then we have a red 4-geodesic. If they are both blue, then we have a blue 4-geodesic through $\overline{s}$. □

3 The 6-Dimensional Cube

For clarity in discussing Q_6, we write the vertex names by collapsing six bits to two octal digits, with each digit representing the binary triple given by its binary expansion. Hence we view Q_6 as consisting of eight copies of Q_3 whose vertex sets are constant in the last three coordinates (see Figure 6). We write the edge joining ab and cd as $ab{:}cd$, extended to paths as $ab{:}cd{:}ef{:}gh$. Note that $\overline{ij} = (7-i)(7-j)$, which facilitates locating antipodal edges.

Lemma 3.1. *Every antipodal edge-coloring of Q_6 has a monochromatic 4-geodesic.*

Proof. By Lemma 2.3, every such coloring c has a monochromatic 3-geodesic P. By symmetry, we may assume that P is red with endpoints 06 and 76 (crossing the first three directions). If c has no monochromatic 4-geodesic, then the edges incident to 06 and 76 in the last three directions are blue. In particular, 06:02 and 06:07 are blue, as in Figure 6.

Since c is antipodal, $\overline{P}$ from 71 to 01 is blue, and the edges from its endpoints crossing the last three directions are red. In particular, 01:05 and 01:00 are red. By symmetry, we may assume 02:00 is blue. Now 07:06:02:00 is a blue 3-geodesic. Thus the edges from 00 across the first three directions are red. Now 20:00:01:05 is a red 3-geodesic, so 05:07 and 05:45 are blue. This yields 45:05:07:06:02 as a blue 4-geodesic. □

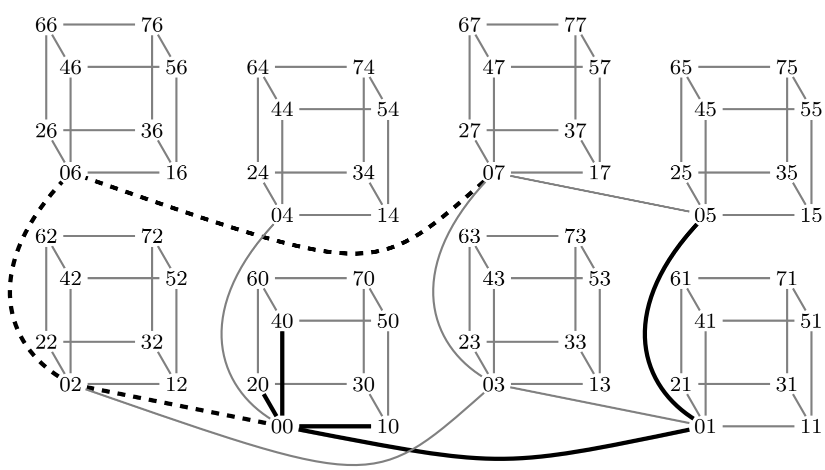

Figure 6: A partial antipodal edge-coloring of Q_6. Color of gray edges is undetermined.

By Lemma 1.2, we only need to get from a monochromatic 4-geodesic to a monochromatic 5-geodesic in order to find a monochromatic antipodal geodesic in an antipodal edge-coloring of Q_6. We capture part of the argument in a technical lemma in the hope that this will be useful for further work.

Lemma 3.2. *If P is a monochromatic 4-geodesic in an antipodal edge-coloring c of Q_6 having no monochromatic 5-geodesic, with endpoint v, then each neighbor of v in a direction not crossed by P has five incident edges of the same color, red at one and blue at the other.*

Proof. Suppose otherwise. By symmetry, we may assume that P is red and crosses the first four directions, with endpoints 02 and 76. We call edges in the first four directions *short edges*; the others are *long*. Thus P has four short edges. Since c has no monochromatic 5-geodesic, the long edges incident to 76 and 02 are blue. Since c is antipodal, $\overline{P}$ from 01 to 75 is blue, and the long edges incident to 01 and 75 are red, as in Figure 7. At this point the two colors are symmetric, as are vertices 77, 74, 03 and 00. Letting v be 76, our focus is on the neighbors 77 and 74. Each has a long incident edge of each color.

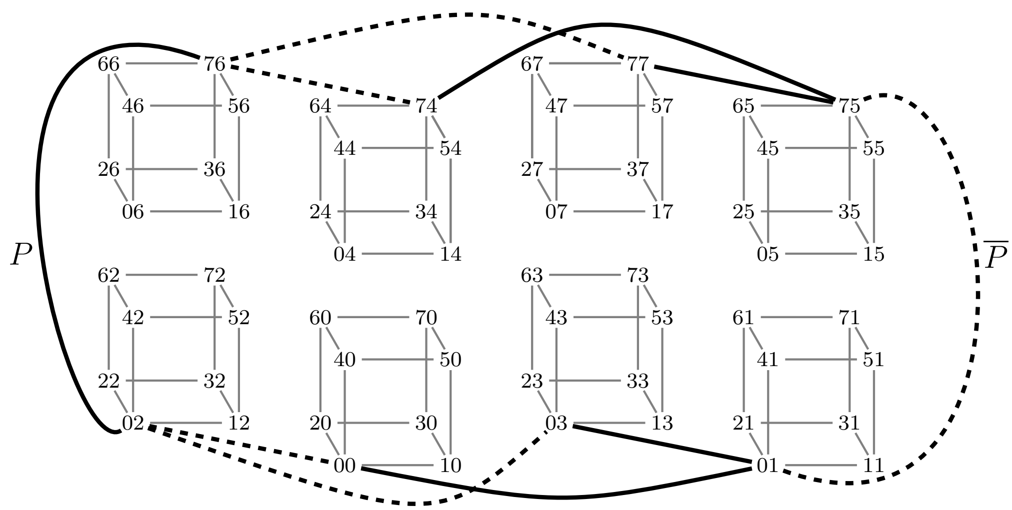

Figure 7: 4-geodesics not extending to 5-geodesics.

Claim 1: 77 *and* 74 *do not together have three incident short edges of the same color in different directions that are not all at the same vertex.* Suppose otherwise. By symmetry, we may let this color be blue, with blue edges at 74 in directions 1 and 2 and a blue edge in direction 3 at 77. That is, we may assume that 74:34, 74:54, and 77:67 are blue. They complete blue 4-geodesics 67:77:76:74:34 and 67:77:76:74:54. To avoid blue 5-geodesics, the other short edges at 67 are red (to $27, 47, 63$), as are 14:54:50 and 14:34:30. Hence the edges 30:10:50 antipodal to 47:67:27 are blue (see Figure 8). We now consider cases based on the coloring of 34:24 and 54:44.

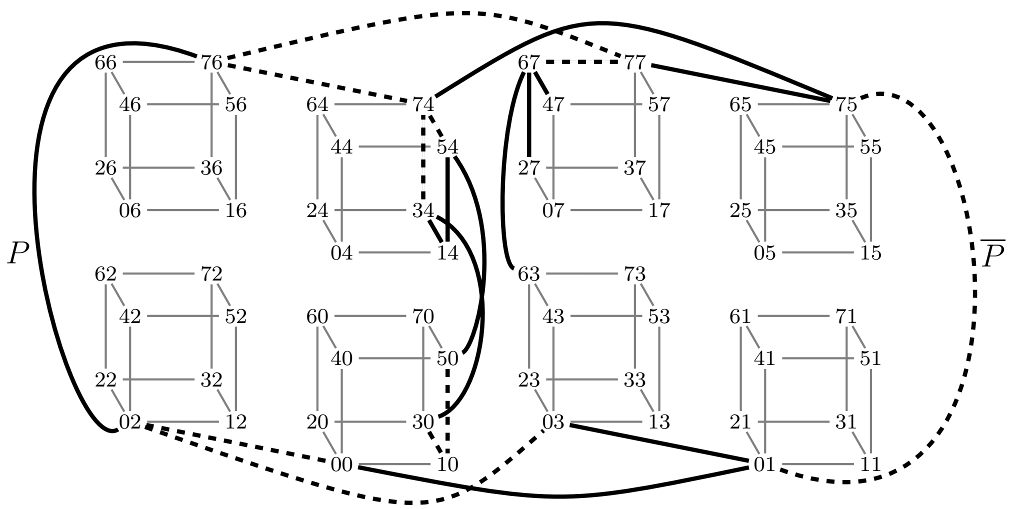

Figure 8: Setup for Claim 1 in Lemma 3.2.

Case A: 34:24 *or* 54:44 *is blue.* These edges are in direction 3. No choice has yet distinguished the first two directions (74:34 and 74:54 are in directions 1 and 2), so we may assume 34:24 is blue. Since 24:34:74:76:77 is blue, 04:24:20 and 57:77:73 are red, and their antipodal geodesics 73:53:57 and 20:00:04 are blue (see Figure 9). Hoping that the reader is now familiar with these types of arguments, we list successive implications leading to monochromatic 5-geodesics (and hence contradictions) in subcases.

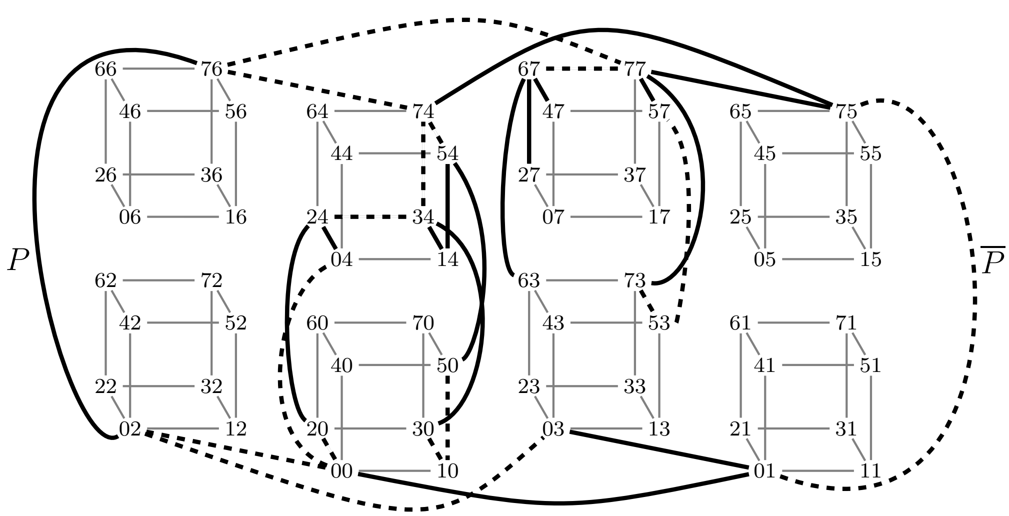

Figure 9: Case A in Lemma 3.2.

Subcase: 56:54 *is red.* Implications start from Figure 9.

geodesic	forces
56:54:14:34:30 red	46:56:57 blue
46:56:57:53:73 blue	46:44 and 73:33 red, antipodally 04:44 blue
03:02:00:04:44 blue	44:54 red
46:44:54:14:34:30 red	

Subcase: 56:54 *is blue.* Implications start from Figure 9.

geodesic	forces
56:54:74:34:24 blue	20:24:25 and 52:56:57 red, antipodally 25:21:20 blue
25:21:20:30:10:50 not blue	20:30 red
20:30:34:14:54 red	22:20 and 54:55 blue
25:24:04:14:54:50 not red	04:14 blue
14:04:00:20:22 blue	14:15 red
15:14:34:30:20 red	20:60 and 17:15:55 blue
17:15:55:54:74 blue	74:70 red
57:77:75:74:70 red	70:60 blue
70:60:20:00:02:03 blue	

Case B: 34:24 *and* 54:44 *are both red.* Starting from Figure 8, we again list implications leading to a monochromatic 5-geodesic (see Figure 10).

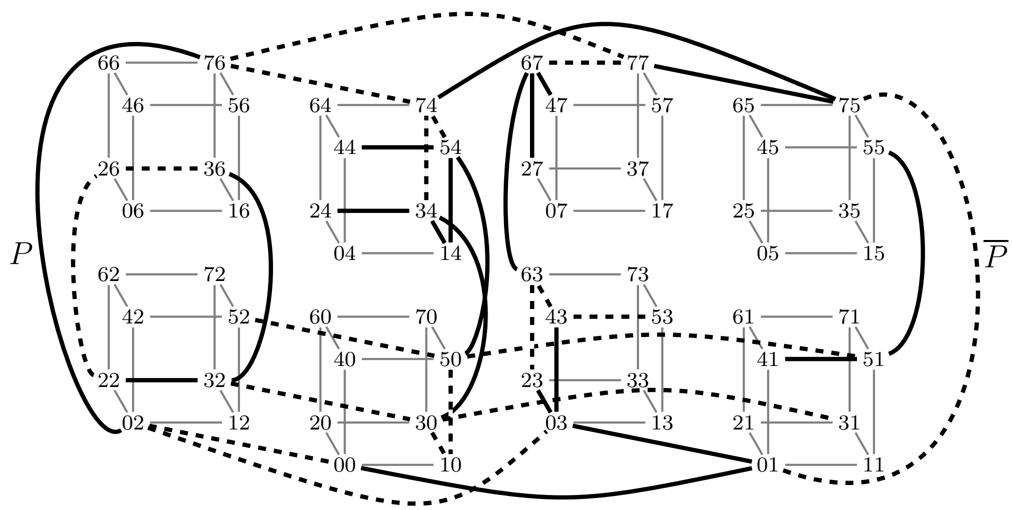

Figure 10: Case B in Lemma 3.2.

geodesic	forces
24:34:14:54:50 red	52:50:51 blue
44:54:14:34:30 red	32:30:31 blue
32:30:10:50:51 blue	22:32:36 and 41:51:55 red, antipodally 36:26:22 blue
24:34:14:54 red	53:43:63:23 blue, antipodally
53:43:63:23:22:26 not blue	22:23 red
54:74:34 blue	23:03:43 red, antipodally
36:32:22:23:03:43 red	

Claim 2: 74 *and* 77 *do not have incident short edges with the same color in different directions.* Suppose otherwise. We start again from Figure 7. By symmetry in color and in the first four directions, we may assume that 74:54 in direction 2 and 77:67 in direction 3 are blue. By Claim 1, 74:34 and 77:37 in direction 1 and 74:70 and 77:73 in direction 4 are red, which in turn requires 74:64 in direction 3 and 77:57 in direction 2 to be blue. Antipodally, 00:10, 00:20, 03:13, and 03:23 are all red (see Figure 11).

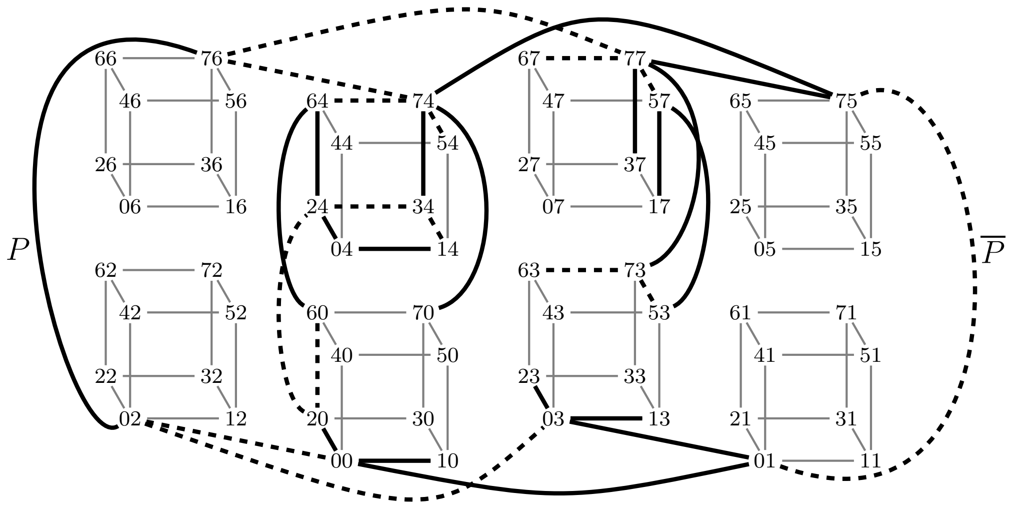

Figure 11: Claim 2 for Lemma 3.2.

Now 64:74:76:77:57 in blue forces 24:64:60 and 53:57:17 red (and 24:20:60 blue, antipodally). Symmetrically, 34:74:75:77:73 in red forces 14:34:24 and 63:73:53 blue (and 14:04:24 red, antipodally). We now have 4-geodesics from 14 to 60 in both colors using only short edges: 14:04:24:64:60 in red and 14:34:24:20:60 in blue. Hence either color on a long edge at 60 or 14 completes a monochromatic 5-geodesic, a contradiction.

Conclusion: Since there are four short edges incident to 77, some two of them have the same color, which by symmetry we may assume is blue. By Claim 2, all short edges at 74 are now red, and then also all short edges at 77 are blue. We have now proved the statement of the lemma: the two neighbors of v via long directions are 74 and 77; one has five incident edges in red, and the other has five incident edge in blue. □

Theorem 3.3. *Every antipodal edge-coloring of Q_6 has a monochromatic antipodal geodesic.*

Proof. By Lemma 1.2, it suffices to find a monochromatic 5-geodesic in such a coloring c. Lemma 3.1 provides a monochromatic 4-geodesic P, which by symmetry we may assume is red with endpoints 02 and 76, crossing the first four ("short") directions.

Suppose that c has no monochromatic 5-geodesic. By Lemma 3.2 (and symmetry) we may assume that the short edges at 74 are all red and those at 77 are all blue. Antipodally, the short edges at 00 all red, and those at 03 are all blue. Let T be the set of short edges at

74 and 00; they form two red stars. At present there is no distinction among short directions. We consider two cases.

Case A: *Some short edge incident to a leaf in T is red.* By symmetry, we may assume 04:14 is red. Antipodally, 73:63 is blue. The geodesic 74:76:77:73:63 is blue, so 43:63:23 must be red, and antipodally 34:14:54 is blue (see Figure 12).

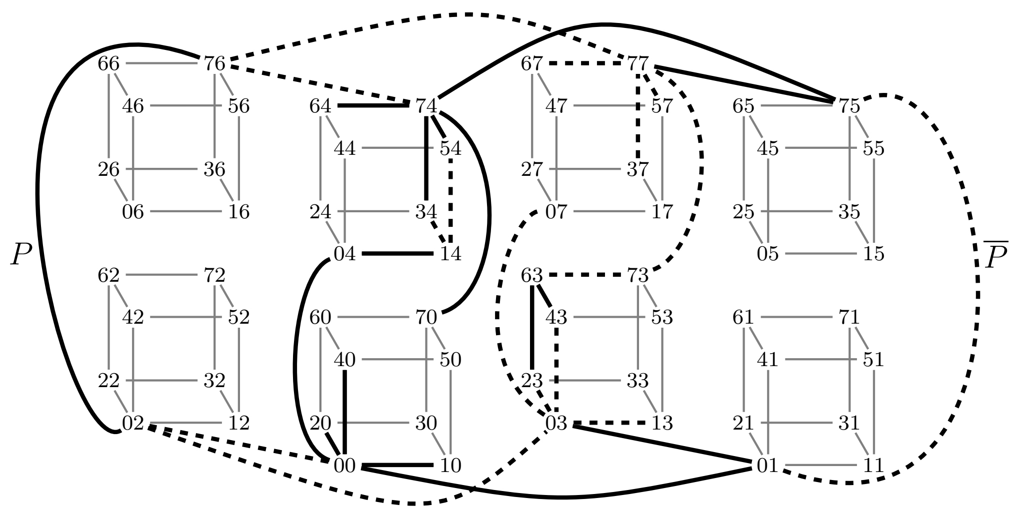

Figure 12: Case A for Theorem 3.3.

Say that a vertex is red if it has five incident red edges; blue if it has five incident blue edges. By Lemma 3.2, any neighbor of an endpoint of a monochromatic 4-geodesic R along the two directions not crossed by R is red or blue. Such vertices include 23 and 43 (neighboring 63 at the end of 74:76:77:73:63) and antipodally 54 and 34 (neighboring 14 at the end of 03:01:00:04:14). Each of the two pairs $\{23, 43\}$ and $\{54, 34\}$ has one red vertex and one blue vertex.

Since the edge 04:14 we assumed red is in direction 3, still directions 1 and 2 remain symmetric. One of $\{43, 23\}$ has its solo color in direction 1, the other in direction 2. Hence by symmetry we may say that 43 is blue and 23 is red. Antipodally, 34 is red and 54 is blue.

Subcase: 51:53 *is red.* From 43 and 54 being blue and 23 and 34 being red, we use that 34:24 is red and all of 03:43:47, 56:54:50, 44:54:56 are blue.

geodesic	forces
00:02:03:43:47 blue	67:47:57 red
77:75:74:34:24 red	20:24:04 blue, antipodally 57:53 red
67:47:57:53:51 red	50:51:11 blue
56:54:50:51:11 blue	31:11:01 red
31:11:01:00:04 red	04:44 blue
20:24:04:44:54:56 blue	

Subcase: 51:53 *is blue.* Consider the blue 3-geodesic 51:53:43:03 (using that 43 is blue). It extends to blue 4-geodesics in three ways, by adding 03:07, 03:23, or 03:02. Avoiding blue 5-geodesics makes 02:22:23:27:07 red, putting two red edges at both 22 and 27. Now applying Lemma 3.2 to 51:53:43:02:23 makes one of $\{22, 27\}$ red and the other blue, a contradiction.

Case B: *All short edges incident to leaves of T are blue.* In this case all edges of the copy of Q_4 on vertices with second coordinate 0 or 4 that are not incident to 74 or 00 are blue, as in Figure 13, where for clarity the six blue edges in direction 4 (vertical) are not drawn in the figure. To avoid a blue 5-geodesic, all edges leaving this copy of Q_4 must be red except the one edge shown at each of 74 and 00.

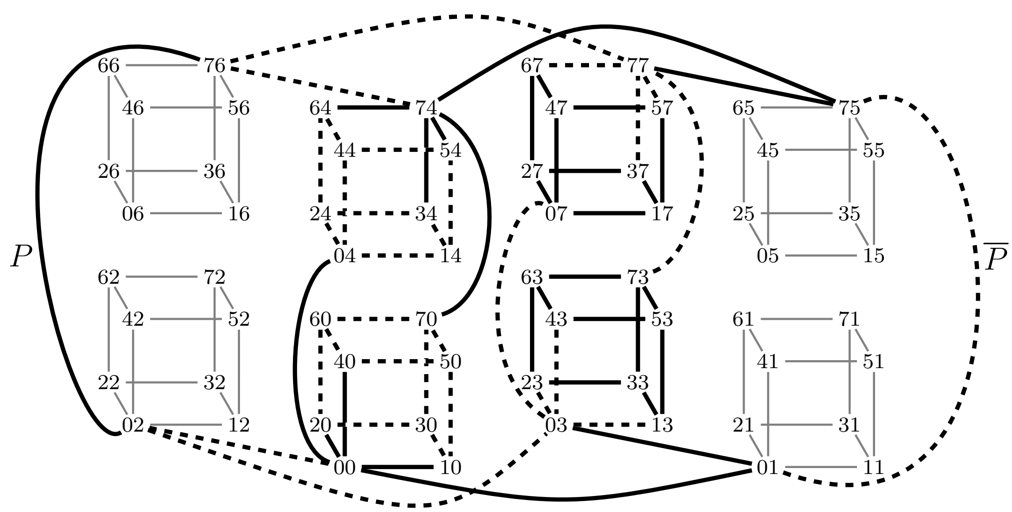

Figure 13: Case B for Theorem 3.3.

Similarly, all edges of the copy of Q_4 on vertices with second coordinate 3 or 7 that are not incident to 03 or 77 are red (again six vertical red edges are not drawn), and all edges leaving it are blue except the one edge shown at each of 03 and 77.

We list the remaining implications for this case.

geodesic	forces
55:54:74:64:66 red	62:66:26 blue
71:70:74:34:36 red	26:36:16 blue
62:66:26:36:16:17 blue	

This completes the proof. □

References

[1] M. DeVos and S. Norine, Edge-antipodal colorings of cubes. The Open Problem Garden, http://garden.irmacs.sfu.ca/?q=op/edge_antipodal_colorings_of_cubes.

[2] T. Feder and C. Subi, On hypercube labellings and antipodal monochromatic paths, *Discrete Appl. Math.* 161 (2013), 1421–1426.

[3] K. Gandhi, Maximal monochromatic geodesics in an antipodal coloring of hypercube, 2015 manuscript, http://math.mit.edu/research/highschool/primes/materials/2014/Gandhi.pdf.

[4] I. Leader and E. Long, Long geodesics in subgraphs of the cube, *Discrete Math.* 326 (2014), 29–33.