# Largest $r$-Regular Graphs with Equal Connectivity and Independence Number

Peter K. Kwok and Douglas B. West [1]
University of Illinois at Urbana-Champaign

**Abstract**

For $r \geq 3$, the maximum number of vertices in an $r$-regular graph with independence number equal to connectivity is $r^2$ if $r$ is even and $r^2 - 1$ if $r$ is odd. Furthermore, the bound is achieveable by a graph in which the independence number and connectivity both equal $r$.

The connectivity $\kappa(G)$ of a graph $G$ is at most its minimum vertex degree $\delta(G)$. Equality holds for many graphs, such as the complete bipartite graph $K_{r,r}$, which has independence number, connectivity and vertex degree all equal to $r$. This has only $2r$ vertices. In discussions with Stefan Brandt, the question arose of how large a regular graph $G$ could be if $\alpha(G) = \kappa(G) = \delta(G) = r$, where the independence number $\alpha(G)$ is the maximum number of pairwise nonadjacent vertices in $G$.

In this note, we solve this problem. Indeed, the upper bound holds more generally when we require only that $\alpha(G) = \kappa(G)$, without requiring that this equal the vertex degree. We use the notation $\Delta(G)$ for the maximum vertex degree, $\chi(G)$ for the chromatic number, and $n(G)$ for the number of vertices.

**Theorem 1** *Let $G$ a $r$-regular graph with $\alpha(G) = \kappa(G)$. Then*

$$n(G) \leq \begin{cases} r^2 + 1, & r = 1, 2; \\ r^2 - 1, & r \geq 3,\ r \text{ odd}; \\ r^2, & r \geq 3,\ r \text{ even}. \end{cases}$$

*Furthermore, equality is achieveable using graphs in which $\alpha(G) = \kappa(G) = r$.*

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**Proof:** Let $n$ be the order of a graph $G$ satisfying the conditions. When $r = 1$, the only graph that satisfies the conditions is $K_2$. When $r = 2$, the only graphs that satisfy the conditions are $C_4$ and $C_5$. Thus $n \le r^2 + 1$ for $r \le 2$.

If $r \ge 3$, then $G$ is not an odd cycle. Also $G$ is not a clique, because $\alpha(G) = \kappa(G)$. Thus, by Brook's Theorem, $\chi(G) \le \Delta(G)$. Also, $\kappa(G) \le \delta(G)$. Thus we have

$$n \le \chi(G)\,\alpha(G) = \chi(G)\,\kappa(G) \le \Delta(G)\,\delta(G) = r^2$$

For odd $r$, $r$-regular graphs of odd order do not exist. Therefore $n \le r^2 - 1$ for odd $r$.

We provide examples to achieve the upper bounds.

*Case 1:* $r = 3$. If $G$ is the 3-regular graph formed from an 8-cycle by adding a matching joining opposite vertices on the cycle, then $\alpha(G) = \kappa(G) = 3$.

*Case 2:* $r \ge 4$, *$r$ even.* Let $G_1, \ldots, G_r$ be cliques of order $r$, with each having $r/2$ vertices of type $A$ and $r/2$ vertices of type $B$. For $i = 1, \ldots, r$, add a matching of size $r/2$ between $G_i$ and $G_{i+1}$ that matches the vertices of Type $A$ in $G_i$ with the vertices of Type $B$ in $G_{i+1}$, where the indices are taken modulo $r$. The resulting graph $G$ is $r$-regular. Since it is covered by $r$ cliques, $\alpha(G) \le r$, and equality is achieved by picking one vertex of Type $A$ from each $G_i$. Given two vertices $x, y \in V(G)$, we find $r$ pairwise internally-disjoint $x, y$-paths in $G$ (if $x, y$ belong to distinct cliques, $r/2$ of the paths pass through cliques with increasing indices and $r/2$ through cliques with decreasing indices). By Menger's Theorem, $\kappa(G) = r$.

*Case 3:* $r \ge 5$, *$r$ odd.* Let $G_1, \ldots, G_{r-1}$ be cliques of order $r$, and let $W$ be a clique with vertices $\{w_1, \ldots, w_{r-1}\}$. From now on, all indices are taken modulo $r - 1$. In each $G_i$ for $1 \le i \le r - 1$, pick two vertices $u_i$ and $v_i$, and partition the remaining $r - 2$ vertices into two sets $A_i$ of size $(r - 1)/2$ and $B_i$ of size $(r - 3)/2$. Add edges $u_i w_i$ and $v_i w_{i+1}$ for $i = 1, \ldots, r - 1$. Add a matching of size $(r - 1)/2$ between $A_{2i}$ and $A_{2i+1}$ for $i = 1, \ldots, (r-1)/2$. Add a matching of size $(r-3)/2$ between $B_{2i-1}$ and $B_{2i}$ for $i = 1, \ldots, (r - 1)/2$.

We have constructed this graph $G$ to be $r$-regular. Again $G$ is covered by $r$ cliques, and it has an independent set of size $r$ consisting of one vertex of $A_i$ for each odd $i$, one vertex of $B_i$ for each even $i$, and one vertex of $W$. We find $r$ pairwise internally-disjoint $x, y$-paths for every pair of distinct vertices $x$ and $y$, as follows.

If $x \in V(G_i)$ and $y \in V(G_j)$ for some $1 \leq i < j \leq r-1$, then the first path passes through $u_i$, $w_i$, $w_j$ and $u_j$ and the second through $v_i$, $w_{i+1}$, $w_{j+1}$ and $v_j$. The third $x, y$-path requires special considerations. Suppose $k$ and $l$ are indices such that matchings were assigned between $A_i$ and $A_k$, and $A_j$ and $A_l$ in the construction. If $k = j$ and $l = i$, then the third path passes through $A_i$ and $A_j$; otherwise it passes through $A_k$, $w_{k+1}$, $w_{l+1}$, and $A_l$. The rest of the $x, y$-paths are divided into two groups: $(r-3)/2$ of them pass through cliques with increasing indices and $(r-3)/2$ through cliques with decreasing indices, where all paths are chosen to be pairwise internally disjoint from other existing paths.

If $x \in V(G_i)$ and $y \in V(W)$, then two of the paths pass through $u_i$ and $v_i$ and each of the others through some $u_k$ where $k = 1, \ldots, r-1$ and $k \neq i$. If $x, y \in V(W)$, then $r-2$ of the paths stay in $W$ and the other two pass through various $G_k$ with one via increasing $k$'s and the other via decreasing $k$'s. As in the previous case, Menger's Theorem applies. □

The extremal graph for even $r$ is not unique. After beginning with the cliques $G_1, \ldots, G_r$, we can alternatively add a perfect matching that includes at least one edge between each pair of cliques.