Degree-associated reconstruction parameters of complete multipartite graphs and their complements

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Abstract

A vertex-deleted subgraph of a graph G is a *card*. A *dacard* consists of a card and the degree of the missing vertex. The *degree-associated reconstruction number* of a graph G, denoted $\mathrm{drn}(G)$, is the minimum number of dacards that suffice to reconstruct G. The *adversary degree-associated reconstruction number* $\mathrm{adrn}(G)$ is the least k such that every set of k dacards determines G. The analogous parameters for degree-associated edge reconstruction are $\mathrm{dern}(G)$ and $\mathrm{adern}(G)$. We determine these four parameters for all complete multipartite graphs and their complements. The answer is usually 2 for all four parameters, but there are exceptions in each case.

Key words: degree-associated reconstruction number, complete multipartite graph, adversary reconstruction.

1 Introduction

The Reconstruction Conjecture of Kelly [5, 6] and Ulam [13] has been open for more than 50 years. A subgraph of a graph G obtained by deleting one vertex is a *card*. Cards are unlabeled; that is, only the isomorphism class of a card is known. The multiset of cards is the *deck* of G. The Reconstruction Conjecture asserts that every graph with at least three vertices is uniquely determined by its deck. Such a graph is *reconstructible*.

For a reconstructible graph G, Harary and Plantholt [4] introduced the *reconstruction number*, denoted by $\mathrm{rn}(G)$; it is the least k such that some multiset of k cards from the deck of G determines G (meaning that every graph not isomorphic to G shares at most $\mathrm{rn}(G)-1$

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of these cards with G). Myrvold [11] proposed the *adversary reconstruction number*, denoted by $\mathrm{arn}(G)$; it is the least k such that every multiset of k cards G determines G. That is, when an adversary chooses the cards, we may need to request more cards to guarantee that G can be reconstructed no matter which ones are chosen.

A *degree-associated card* or *dacard* is a pair (C, d) consisting of a card and the degree of the missing vertex. The *dadeck* is the multiset of dacards. Given the full deck without degrees, it is easy to compute the degrees of the missing vertices; hence the Reconstruction Conjecture is equivalent to reconstructibility from the dadeck. Without having all cards, the degree of the vertex missing from a card is hard to compute, so the dacard provides more information. Hence graphs may be reconstructible using fewer dacards than cards.

If the Reconstruction Conjecture holds, then every graph is reconstructible from its dadeck. Ramachandran [12] defined the *degree-associated reconstruction number*, denoted $\mathrm{drn}(G)$, to be the minimum number of dacards that determine G. Barrus and West [1] proved $\mathrm{drn}(G) \geq 3$ for vertex-transitive graphs and $\mathrm{drn}(G) = 2$ for all caterpillars except stars (where the value is 1) and the one 6-vertex tree with vertex degrees $(3, 3, 1, 1, 1, 1)$. They also conjectured that the maximum of $\mathrm{drn}(G)$ over n-vertex graphs is $n/4 + 2$, achieved by the disjoint union of two copies of the complete bipartite graph $K_{n/4,n/4}$. Although $\mathrm{rn}(G) = 3$ for almost all graphs (Bollobás [2]), $\mathrm{drn}(G) = 2$ for almost all graphs [1].

Monikandan et al. [10] introduced the degree-associated analogue of Myrvold's adversary concept (attributing the definition to Ramachandran). For a graph G reconstructible from its dadeck, the *adversary degree-associated reconstruction number*, denoted $\mathrm{adrn}(G)$, is the least k such that every set of k dacards determines G. The definitions immediately yield $\mathrm{drn}(G) \leq \mathrm{adrn}(G)$ for every G. Equality holds when G is vertex-transitive, since G then has only one multiset of dacards of each size. The value of adrn is known for complete graphs, complete bipartite graphs, cycles, and wheels [10]. In a subsequent paper, Monikandan and Sundar Raj [8] determined adrn for double-stars (see also [7]), for subdivisions of stars, and for the disjoint union of t complete n-vertex graphs and s cycles of length m. The proof in [1] that yields $\mathrm{drn}(G) \leq \min\{r + 2, n - r + 1\}$ when G is r-regular with n vertices also implies $\mathrm{adrn}(G) \leq \min\{r + 2, n - r + 1\}$.

An *edge-card* of G is obtained by deleting one edge; the multiset of all edge-cards is the edge-deck. The Edge-Reconstruction Conjecture (Harary [3]) states that every graph with more than three edges is determined by its edge-deck (the claw $K_{1,3}$ and the disjoint union of a triangle and one vertex have the same edge-deck). The *degree* of an edge e, denoted by $d(e)$, is the number of edges incident to e; it is the degree of e in the line graph of G. The Edge-Reconstruction Conjecture is simply the statement that line graphs are reconstructable.

In reconstructing from edge-cards, the definitions of edge-reconstruction number and adversary edge-reconstruction number are analogous to the vertex setting. Similarly, we can associate the degree of the deleted edge with an edge-card to form a *decard*; the multiset of all decards is the *dedeck*. This leads to degree-associated edge-reconstruction parameters:

dern(G) is the minimum k such that some multiset of k decards determines G, and adern(G) is the minimum k such that every multiset of k decards determines G.

The study of dern and adern was initiated by Monikandan and Sundar Raj [9]. They determined dern(G) and adern(G) when G is a regular graph, a complete bipartite graph, a path, a wheel, or a double-star. They also proved that $\mathrm{dern}(G) \le 2$ and $\mathrm{adern}(G) \le 3$ when G is a complete 3-partite set whose part-sizes differ by at most 1.

In this paper, we determine drn, adrn, dern, and adern for all complete multipartite graphs and their complements. Let r be the number of parts, and let $\overline{n} = (n_1, \ldots, n_r)$; write $K_{\overline{n}}$ for the complete multipartite graph $K_{n_1,\ldots,n_r}$. A *clique-union* is a disjoint union of complete graphs; write $G_{\overline{n}}$ for the clique-union that is the complement of $K_{\overline{n}}$.

Trivially, a graph and its complement have the same value of drn [1] and the same value of adrn [8]. To determine $\mathrm{drn}(K_{\overline{n}})$ and $\mathrm{adrn}(K_{\overline{n}})$, we determine the values for $G_{\overline{n}}$. We show that in most cases $\mathrm{drn}(K_{\overline{n}}) = \mathrm{adrn}(K_{\overline{n}}) = 2$. With $n_1 \le \cdots \le n_r$, the exceptions for drn (Theorem 2.6) are $\mathrm{drn}(K_{\overline{n}}) = 1$ when $r = 1$ or $n_1 = 1$, and $\mathrm{drn}(K_{\overline{n}}) = 3$ when $r > 1$ and $n_1 = n_r \ge 2$. The exceptions for adrn (Theorem 2.7) are $\mathrm{adrn}(K_{\overline{n}}) = 1$ when $r = 1$ or when $n_r \le 2$ and $n_{r-1} = 1$, and $\mathrm{adrn}(K_{\overline{n}}) = 3$ when $n_j - n_i \in \{0, 2\}$ for some i and j with $n_j > 1$.

A significant difference between the vertex-based and edge-based parameters is that a graph and its complement need not have the same value of dern and adern. For example, consider the graph C_8' formed from an 8-cycle by adding a 4-cycle through the even-indexed vertices: $\mathrm{dern}(C_8') > 1$, but the complement of C_8' is determined by one decard.

Our results for degree-associated edge-reconstruction show that in most cases $\mathrm{dern}(G_{\overline{n}}) = \mathrm{adern}(G_{\overline{n}}) = \mathrm{dern}(K_{\overline{n}}) = \mathrm{adern}(K_{\overline{n}}) = 2$. We leave the descriptions of the exceptions to the statements of the individual theorems.

The main idea in the proofs is to obtain conditions on $\overline{n}$ under which any two dacards (or any two decards, respectively) determine the graph in question, $G_{\overline{n}}$ or $K_{\overline{n}}$. When these conditions fail, generally the value of the parameter is 3 or 4. However, we also note a simple condition implying $\mathrm{dern}(G) = 1$ sometimes holds for $G_{\overline{n}}$ or $K_{\overline{n}}$: If a graph G has an edge e such that $d(e) = 0$ or no two nonadjacent vertices in $G - e$ other than the endpoints of e have degree-sum $d(e)$, then the decard $(G - e, d(e))$ determines G (Lemma 3.1).

2 Vertex reconstruction for $G_{\overline{n}}$ and $K_{\overline{n}}$

In this section, we determine drn and adrn for $G_{\overline{n}}$ and $K_{\overline{n}}$. Our main tools are conditions under which two dacards determine $G_{\overline{n}}$. Let a *j-component* of a graph be a component isomorphic to K_j, the complete graph with j vertices.

Lemma 2.1. *If a graph H other than $G_{\overline{n}}$ shares at least two dacards with $G_{\overline{n}}$, then H arises from such a dacard (C, d) by adding a vertex with d neighbors in a $(d+1)$-component of C or with $d-1$ vertices in a $(d-1)$-component and one neighbor in another component.*

Proof. The graph H arises from C by adding a vertex v having d neighbors in C. In all cases other than those listed above, every card other than $H-v$ has a component that is not a complete graph. Hence the listed cases are the only graphs other than $G_{\overline{n}}$ that share at least two dacards with $G_{\overline{n}}$. The other cases to consider are (1) v has neighbors in at least three components of C, (2) v has at least two neighbors in each of two components of C, (3) v has $d-1$ neighbors in a j-component of C with $j > d-1$ and one neighbor in another component, and (4) v has all neighbors in a j-component of C with $j > d+1$. $\square$

Lemma 2.2. *Any two distinct dacards of $G_{\overline{n}}$ determine $G_{\overline{n}}$.*

Proof. Given $\overline{n} = (n_1, \ldots, n_r)$, let $\hat{n}_t$ denote the r-tuple obtained from n by decreasing the t-th entry by 1. Since the given dacards are distinct, we may assume that they are $(G_{\hat{n}_i}, n_i - 1)$ and $(G_{\hat{n}_j}, n_j - 1)$, where $n_i < n_j$.

If $n_i = 1$, then $G_{\overline{n}}$ has an isolated vertex and is determined by the dacard $(G_{\hat{n}_i}, 0)$. Henceforth assume $n_i \geq 2$, so $n_j \geq 3$. Let H be a reconstruction from $(G_{\hat{n}_j}, n_j - 1)$ that also has $(G_{\hat{n}_i}, n_i - 1)$ as a dacard. By Lemma 2.1, when $H \ncong G_{\overline{n}}$ there are only two cases for the neighborhood of v that permit $G_{\hat{n}_i}$ to also be a card of H.

Case 1: All neighbors of v lie in one n_j-component of $G_{\hat{n}_j}$. In this case, the component C' of H containing v is not a complete graph and remains in any card obtained by deleting a vertex of degree $n_i - 1$, since $\delta(C') = n_j - 1$. Hence $(G_{\hat{n}_i}, n_i - 1)$ is not a dacard of H.

Case 2: v has $n_j - 2$ neighbors in an $(n_j - 2)$-component of $G_{\hat{n}_j}$ and one neighbor x in another component. In this case the only card of H distinct from $G_{\hat{n}_j}$ that is a clique-union is $H - x$. Since $(G_{\hat{n}_i}, n_i - 1)$ is a dacard of H, we have $d_H(x) = n_i - 1$. Now $H - x$ is obtained from $G_{\hat{n}_j}$ by adding v to turn an $(n_j - 2)$-component into an $(n_j - 1)$-component and deleting x from an $(n_i - 1)$-component; hence $H - x$ has the same number of n_j-components as $G_{\hat{n}_j}$, which is fewer than $G_{\overline{n}}$ has. Since $n_i < n_j$, in $G_{\hat{n}_i}$ the number of n_j-components is the same as in $G_{\overline{n}}$. However, the number of such components in $H - x$ is smaller. Hence $H - x$ cannot be $G_{\hat{n}_i}$, which eliminates this case. $\square$

Lemma 2.3. *Any three identical dacards of $G_{\overline{n}}$ determine $G_{\overline{n}}$.*

Proof. We may assume that the three dacards are copies of $(G_{\hat{n}_i}, n_i - 1)$. Let H be a graph having three copies of $(G_{\hat{n}_i}, n_i - 1)$ in its dadeck, obtained from $G_{\hat{n}_i}$ by adding a vertex v with degree $n_i - 1$ (in H).

If $n_i = 1$, then H is formed by adding an isolated vertex to $G_{\hat{n}_i}$. Hence $H \cong G_{\overline{n}}$.

If $n_i = 2$, then v has degree 1; let x be its neighbor in $G_{\hat{n}_i}$. Let x be in an n_j-component in $G_{\hat{n}_i}$. If $n_j \geq 3$, then $(G_{\hat{n}_i}, 1)$ occurs only once in the dadeck of H. If $n_j = 2$, then the component of H containing v is $K_{1,2}$, and H has two copies of $(G_{\hat{n}_i}, 1)$ in its dadeck. Having three copies of $(G_{\hat{n}_i}, 1)$ requires $n_j = 1$, which yields $H \cong G_{\overline{n}}$.

Hence we may assume $n_i \geq 3$. By Lemma 2.1, v has neighbors in at most two components of $G_{\hat{n}_i}$. If two components, then one is an $(n_i - 2)$-component and the other has just one neighbor of v. If one component, then it is an n_i-component or $H \cong G_{\overline{n}}$. Suppose $H \not\cong G_{\overline{n}}$.

Case 1: All neighbors of v lie in an n_i-component of $G_{\hat{n}_i}$. Here the component C' of H containing v is not complete. It remains in all but two cards where vertices of degree $n_i - 1$ are deleted. Since H has three copies of $(G_{\hat{n}_i}, n_i - 1)$ in its dadeck, this case does not arise.

Case 2: v has $n_i - 2$ neighbors in an $(n_i - 2)$-component of $G_{\hat{n}_i}$ and one neighbor x in another component. In this case, $H - u$ is a clique-union only for $u \in \{v, x\}$. Since H has at least three such cards, this case does not arise. □

With these lemmas, the results for drn and adrn on $G_{\overline{n}}$ and $K_{\overline{n}}$ do not require much more work. We will use two lemmas proved by Barrus and West [1] (the proofs are not difficult).

Lemma 2.4 ([1]). *A graph G satisfies* $\mathrm{drn}(G) = 1$ *if and only if G or its complement has an isolated vertex or has a leaf whose deletion leaves a vertex-transitive graph.*

Lemma 2.5 ([1]). *If G is vertex-transitive and is not complete or edgeless, then* $\mathrm{drn}(G) \geq 3$.

Since complete graphs and their complements are determined by any one dacard, we consider only $r \geq 2$ when studying $K_{\overline{n}}$.

Theorem 2.6. *If $r \geq 2$ and $n_1 \leq \cdots \leq n_r$, then*

$$\mathrm{drn}(K_{\overline{n}}) = \mathrm{drn}(G_{\overline{n}}) = \begin{cases} 1, & \text{if } n_1 = 1; \\ 3, & \text{if } n_1 = n_r \geq 2. \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Since $G_{\overline{n}}$ is the complement of $K_{\overline{n}}$, it suffices to determine $\mathrm{drn}(G_{\overline{n}})$, where $\overline{n} = (n_1, \ldots, n_r)$. By Lemma 2.4, $\mathrm{drn}(G_{\overline{n}}) = 1$ when $n_1 = 1$, and otherwise $\mathrm{drn}(G_{\overline{n}}) \geq 2$. If $n_1 < n_r$, then the dadeck of $G_{\overline{n}}$ has two distinct dacards. By Lemma 2.2, $\mathrm{drn}(G_{\overline{n}}) \leq 2$.

The only remaining case is $n_1 = n_r \geq 2$, where $G_{\overline{n}}$ is vertex-transitive and not complete or edgeless. By Lemma 2.5, $\mathrm{drn}(G_{\overline{n}}) \geq 3$. Since $r \geq 2$, in $G_{\overline{n}}$ there are at least four vertices, so $G_{\overline{n}}$ has at least three dacards, and Lemma 2.3 yields $\mathrm{drn}(G_{\overline{n}}) \leq 3$. □

Theorem 2.7. *For the complete r-partite graph $K_{\overline{n}}$ with $n_1 \leq \cdots \leq n_r$,*

$$\mathrm{adrn}(K_{\overline{n}}) = \mathrm{adrn}(G_{\overline{n}}) = \begin{cases} 1, & \text{if } n_1 = n_{r-1} = 1 \text{ and } n_r \in \{1,2\}; \\ 3, & \text{if } n_j - n_i \in \{0,2\} \text{ for some } i \text{ and } j \text{ with } n_j > 1; \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Again it suffices to determine $\mathrm{adrn}(G_{\overline{n}})$. By Lemmas 2.2 and 2.3, $\mathrm{adrn}(G_{\overline{n}}) \leq 3$. When $n_1 = n_{r-1} = 1$ and $n_r \in \{1, 2\}$, any dacard of $G_{\overline{n}}$ determines $G_{\overline{n}}$.

If $n_j - n_i \in \{0, 2\}$ with $n_j > 1$, then construct H from $G_{\hat{n}_j}$ by adding a vertex v as follows. If $n_j = n_i$, then let v have $n_j - 1$ neighbors in one n_j-component, which exists since $n_i = n_j$. If $n_j = n_i + 2$, then let v have $n_j - 2$ neighbors in an n_i-component and one neighbor x in an $n_j - 1$-component; note that $H - v \cong H - x \cong G_{\hat{n}_j}$. In either case, $H \ncong G_{\overline{n}}$ and $(G_{\hat{n}_j}, n_j - 1)$ appears at least twice in the dadecks of both H and $G_{\overline{n}}$. Thus $\mathrm{adrn}(G_{\overline{n}}) \geq 3$.

If $n_r \leq 2$, then the cases above yield $\mathrm{adrn}(G_{\overline{n}}) = 1$ if $n_{r-1} = 1$ and $\mathrm{adrn}(G_{\overline{n}}) = 3$ if $n_{r-1} = 2$. Therefore, we henceforth assume $n_r \geq 3$. Construct H from $G_{\hat{n}_r}$ by adding a vertex v having $n_r - 1$ neighbors not all in the same component of $G_{\hat{n}_r}$. Since $H \ncong G_{\overline{n}}$, and $(G_{\hat{n}_r}, n_r - 1)$ is a dacard of both H and $G_{\overline{n}}$, we have $\mathrm{adrn}(G_{\overline{n}}) \geq 2$.

For the upper bound, by Lemma 2.2 it suffices to prove that any two identical dacards determine $G_{\overline{n}}$. Let $(G_{\hat{n}_j}, n_j - 1)$ appear at least twice in the dadeck of $G_{\overline{n}}$ and also at least twice in the dadeck of a graph H obtained from $G_{\hat{n}_j}$ by adding v with $n_j - 1$ neighbors.

If $n_j = 1$, then H arises by adding an isolated vertex to $G_{\hat{n}_j}$, and $H \cong G_{\overline{n}}$.

If $n_j = 2$, then let x be the one neighbor of v in H, with x in an n_i-component of $G_{\hat{n}_j}$. If $n_i \geq 3$, then $(G_{\hat{n}_j}, 1)$ appears only once in the dadeck of H. If $n_i = 2$, then the earlier case $n_i = n_j$ applies. Thus $n_i = 1$, which yields $H \cong G_{\overline{n}}$.

Hence we may assume $n_j \geq 3$. If $H \ncong G_{\overline{n}}$, then Lemma 2.1 leaves two cases. In the case where v has neighbors in two components of $G_{\hat{n}_j}$, the component having $n_j - 2$ neighbors of v must be an $(n_j - 2)$-component of $G_{\hat{n}_j}$. Since $d_G(v) = n_j - 1$, the dacard $G_{\hat{n}_j}$ cannot be created from $G_{\overline{n}}$ by deleting v from an $(n_j - 1)$-component. Hence $G_{\overline{n}}$ must have an $(n_j - 2)$-component, which reduces to the earlier case $n_i = n_j - 2$.

The remaining case allowed by Lemma 2.1 is that all $n_j - 1$ neighbors of v lie in one n_j-component of $G_{\hat{n}_j}$. Since $G_{\hat{n}_j}$ is obtained from $G_{\overline{n}}$ by deleting v from an n_j-component, this requires a second n_j-component in $G_{\hat{n}_j}$, which reduces to the earlier case $n_j = n_i$. □

3 $\mathrm{dern}(G_{\overline{n}})$ and $\mathrm{adern}(G_{\overline{n}})$

We now consider degree-associated edge-reconstruction numbers. As noted in the introduction, a graph and its complement may have different values of dern. We begin with the clique-union $G_{\overline{n}}$ and later discuss $K_{\overline{n}}$.

Lemma 3.1. *If a graph G has an edge e such that $d(e) = 0$ or no two nonadjacent vertices in $G - e$ other than the endpoints of e have degree-sum $d(e)$, then the decard $(G - e, d(e))$ determines G.*

Proof. If $d(e) = 0$, then any reconstruction from the decard $(G - e, 0)$ must add an edge joining two isolated vertices in $G - e$, yielding a graph isomorphic to G. In the other case, since no other pair of non-adjacent vertices in $G - e$ has degree-sum $d(e)$, there is only one way to place the edge e to obtain a graph having $(G - e, d(e))$ as a decard. □

The condition in Lemma 3.1 is sufficient but not necessary. For example, if G consists of one edge uv joining two disjoint complete graphs, then the condition fails, but $\mathrm{dern}(G) = 1$. In general, $\mathrm{dern}(G) = 1$ if and only if G has an edge e such that all nonadjacent pairs in $G - e$ with degree-sum $d(e)$ are in the same edge-orbit in the complement of $G - e$.

Lemma 3.2. *For clique-unions,* $\mathrm{dern}(G_{\overline{n}}) = 1$ *if and only if (1)* $G_{\overline{n}}$ *has a* 2*-component, or (2) there exists* k *with* $k \geq 3$ *such that* $G_{\overline{n}}$ *has a* k*-component, has no* $(k-1)$*-component, and does not have both a* $(k+j)$*-component and a* $(k-j-2)$*-component for any* j *with* $0 \leq j \leq k-3$.

Proof. Sufficiency. If (1) holds, then Lemma 3.1 applies. Now suppose that $G_{\overline{n}}$ has a k-component and that $G_{\overline{n}}$ has neither a $(k-1)$-component nor both a $(k+j)$-component and a $(k-j-2)$-component with $0 \leq j \leq k-3$. For an edge uv in the k-component, $d(uv) = 2k-4$. In $G_{\overline{n}} - uv$, the only pair of non-adjacent vertices with degree-sum $2k-4$ is $\{u, v\}$, so Lemma 3.1 applies.

Necessity. We prove the contrapositive; assume that (1) and (2) fail. Since (1) fails, $G_{\overline{n}}$ has no 2-component. Consider any edge e in $G_{\overline{n}}$, it belongs to some k-component with $k \geq 3$. Since (2) fails, $G_{\overline{n}}$ also either has a $(k-1)$-component or has a $(k+j)$-component and a $(k-j-2)$-component for some j with $0 \leq j \leq k-3$. In both cases, we can add an edge with degree $2k-4$ to $G_{\overline{n}} - e$ that creates a graph not isomorphic to $G_{\overline{n}}$. □

Lemma 3.3. *Any two decards of* $G_{\overline{n}}$ *determine* $G_{\overline{n}}$*, except for two identical decards having degree* 2 *for the deleted edge.*

Proof. Let e_1 and e_2 be the edges deleted to form these decards, with $d(e_1) \leq d(e_2)$. By Lemma 3.1, we may assume $d(e_1) > 0$. Since all edges in $G_{\overline{n}}$ have even degree, we may assume $d(e_1) \geq 2$. We exclude the case $d(e_2) = 2$, so $d(e_2) \geq 4$.

Let H be a reconstruction from $(G_{\overline{n}} - e_2, d(e_2))$ that also has $(G_{\overline{n}} - e_1, d(e_1))$ as a decard; H is obtained from $G_{\overline{n}} - e_2$ by adding an edge e' of degree $d(e_2)$. If $H \not\cong G_{\overline{n}}$, then e' joins two vertices in distinct components of $G_{\overline{n}} - e_2$. Let H' be the component of H containing e'. Since $d(e_2) \geq 4$, there are at least six vertices in H'. For any edge e of degree $d(e_1)$ in H, the graph $H - e$ has a component of order at least 4 in which e' is a cut-edge. Since every component with at least four vertices in any decard of $G_{\overline{n}}$ has no cut-edge, $(G_{\overline{n}} - e_1, d(e_1))$ is not a decard of H. This contradiction yields $H \cong G_{\overline{n}}$. □

For $r \geq 2$, the disjoint union of $K_{1,3}$ with $r-2$ isolated vertices has the same dedecks as $G_{(1,\ldots,1,3)}$, so they are not reconstructible from their dedecks.

Theorem 3.4. *For the graph* $G_{\overline{n}}$ *with* $n_1 \leq \cdots \leq n_r$ *and* $\overline{n} \neq (1, \ldots, 1, 3)$,

$$\mathrm{dern}(G_{\overline{n}}) = \begin{cases} 1, & \text{if } \overline{n} \text{ satisfies the condition of Lemma 3.2;} \\ 4, & \text{if } \overline{n} = (1, \ldots, 1, 3, \ldots, 3) \text{ with } n_1 = 1 \text{ and } n_{r-1} = 3; \\ 2, & \text{otherwise.} \end{cases}$$

Proof. We have $\mathrm{dern}(G_{\overline{n}}) = 1$ when Lemma 3.2 applies (including all cases with $n_1 > 1$), and $\mathrm{dern}(G_{\overline{n}}) \geq 2$ otherwise. If $n_r \geq 4$, then Lemma 3.3 yields $\mathrm{dern}(G_{\overline{n}}) \leq 2$.

In the remaining case, $G_{\overline{n}}$ has at least two 3-components, at least one 1-component, and no components of other sizes. All decards of $G_{\overline{n}}$ are copies of $(G', 2)$, where G' is obtained from $G_{\overline{n}}$ by deleting one edge. The graph obtained from $G_{\overline{n}}$ by replacing a 1-component and a 3-component with $K_{1,3}$ also has three copies of $(G', 2)$ in its dedeck, so $\mathrm{dern}(G_{\overline{n}}) \geq 4$.

Let H be a graph having four copies of $(G', 2)$ in its dedeck; H is obtained from G' by adding an edge e' joining two vertices with degree-sum 2. If e' joins the two vertices of degree 1 in G', then $H \cong G_{\overline{n}}$. Otherwise, e' joins an isolated vertex with a vertex v of degree 2 in G'. The component of G' containing v may be a triangle or a path of length 2. Each resulting graph shares at most three decards with $G_{\overline{n}}$. Hence $H \cong G_{\overline{n}}$, yielding $\mathrm{dern}(G_{\overline{n}}) \leq 4$. □

Let $F + F'$ denote the disjoint union of graphs F and F'.

Theorem 3.5. *For the graph $G_{\overline{n}}$ with $n_1 \leq \cdots \leq n_r$ and $\overline{n} \neq (1, \ldots, 1, 3)$,*

$$\mathrm{adern}(G_{\overline{n}}) = \begin{cases} 1, & \textit{if every edge of } G_{\overline{n}} \textit{ satisfies the condition of Lemma 3.1;} \\ 3, & \textit{if } n_1 = 2 \textit{ and } G_{\overline{n}} \textit{ has a 3-component;} \\ 4, & \textit{if } n_1 = 1 \textit{ and } G_{\overline{n}} \textit{ has a 3-component;} \\ 2, & \textit{otherwise.} \end{cases}$$

Proof. Case 1: Every edge of $G_{\overline{n}}$ satisfies the condition of Lemma 3.1. By Lemma 3.1, every decard $(G_{\overline{n}} - e, d(e))$ determines $G_{\overline{n}}$.

Cases 2 and 3: $n_1 \leq 2$ and $G_{\overline{n}}$ has a 3-component. Let $c = 5 - n_1$; we prove $\mathrm{adern}(G_{\overline{n}}) = c$. The graph $K_2 + K_3$ shares two decards with a 5-vertex path, and the graph $K_1 + K_3$ shares three decards with $K_{1,3}$. Hence $\mathrm{adern}(G_{\overline{n}}) \geq c$.

For the upper bound, we show that any c decards determine $G_{\overline{n}}$. By Lemma 3.3, we may assume that all are copies of $(G', 2)$, where G' arises by deleting an edge of a 3-component in $G_{\overline{n}}$. Let H be a graph having c copies of $(G', 2)$ in its dedeck; H arises from G' by adding an edge e' joining vertices with degree-sum 2.

When e' joins vertices of degree 1 in G', there are three possible graphs; one is $G_{\overline{n}}$. The others have a 5-vertex path or 3- and 4-vertex paths as components. The dedecks have at most two copies of $(G', 2)$, since G' has only one path component with at least three vertices.

When $n_1 = 1$ and $c = 4$, also e' may join a vertex of degree 2 to an isolated vertex, creating a component that is $K_{1,3}$ or $K_{1,3}$ plus an edge. Since G' has no such component, decards shared with $G_{\overline{n}}$ must arise by deleting an edge of this component. Three edges yield G' when the component is $K_{1,3}$, but only one in the other case. Hence $\mathrm{adern}(G_{\overline{n}}) \leq c$.

Case 4: Some edge fails the condition of Lemma 3.1, and $G_{\overline{n}}$ does not have both a 3-component and a smaller component. Since some edge fails the condition of Lemma 3.1, $\mathrm{adern}(G_{\overline{n}}) > 1$ and $n_r \geq 4$. Also, $n_1 = 3$ or $G_{\overline{n}}$ has no 3-component. By Lemmas 3.1 and 3.3, any two decards determine $G_{\overline{n}}$. □

4 $\mathrm{dern}(K_{\overline{n}})$ and $\mathrm{adern}(K_{\overline{n}})$

Since Monikandan and Sundar Raj [9] determined these parameters for complete bipartite graphs, we henceforth assume $r \geq 3$. In the complete multipartite graph $K_{\overline{n}}$, let a *k-part* be a partite set of size k (analogous to k-component in $G_{\overline{n}}$). Let $m = \sum n_i$, so any vertex in an n_j-part has degree $m - n_j$.

Lemma 4.1. *A decard $(K_{\overline{n}} - e, d(e))$ determines $K_{\overline{n}}$ if and only if (1) e joins a 1-part and a 2-part, or (2) e joins a k-part and an ℓ-part such that $|k - \ell| \notin \{1, 2\}$ and $K_{\overline{n}}$ has no $(\frac{k+\ell}{2} + 1)$-part.*

Proof. Sufficiency. Under (1), $d(e) = 2m-5$. In $K_{\overline{n}} - e$, the only nonadjacent pairs of vertices with degree-sum $2m - 5$ are the endpoints of e and the vertices of the 2-part containing an endpoint of e. Adding either edge yields $K_{\overline{n}}$. Under (2), $d(e) = 2m - (k + \ell) - 2$. Since $|k - \ell| \notin \{1, 2\}$, no two vertices within one of these parts has degree-sum $d(e)$ in $K_{\overline{n}} - e$. Since also $K_{\overline{n}}$ has no $(\frac{k+\ell}{2} + 1)$-part, the only nonadjacent vertices in $K_{\overline{n}} - e$ with degree-sum $d(e)$ are the endpoints of e.

Necessity. An edge e failing the condition joins a k-part and an ℓ-part such that (1) $\{k, \ell\} \neq \{1, 2\}$ and (2) $|k - \ell| \in \{1, 2\}$ or $K_{\overline{n}}$ has a $(\frac{k+\ell}{2} + 1)$-part. In either case, we can add an edge with degree $2m - (k + \ell) - 2$ to $K_{\overline{n}} - e$ to create a graph not isomorphic to $K_{\overline{n}}$. $\square$

Lemma 4.2. *Any two decards of $K_{\overline{n}}$ determine $K_{\overline{n}}$, except when the two decards are the same and are generated by deleting an edge joining a 3-part to a smaller part.*

Proof. Let $(K_{\overline{n}} - e_1, d(e_1))$ and $(K_{\overline{n}} - e_2, d(e_2))$ be two decards. Suppose that e_1 joins an i-part and a j-part, with $i \leq j$, and e_2 joins a k-part and an ℓ-part, with $k \leq \ell$. We may assume that neither edge satisfies the condition of Lemma 4.1. Thus $(i, j), (k, \ell) \notin \{(1, 1), (1, 2)\}$. Hence, $i + j \geq 4$ and $k + \ell \geq 4$.

The complement of $K_{\overline{n}}$ is $G_{\overline{n}}$, with r components, all complete. The complement of $K_{\overline{n}} - e_1$ is $G_{\overline{n}} + e_1$, where e_1 joins an i-component and a j-component of $G_{\overline{n}}$. Hence $G_{\overline{n}} + e_1$ has $r - 1$ components.

Let H be a reconstruction from $(K_{\overline{n}} - e_1, d(e_1))$ that also has $(K_{\overline{n}} - e_2, d(e_2))$ as a decard. Let e' be the added edge, so $H = K_{\overline{n}} - e_1 + e'$. Let e be the edge deleted from H to obtain $K_{\overline{n}} - e_2$; note that e has degree $d(e_2)$ in H. We may assume $e \neq e'$, even if $K_{\overline{n}} - e_1 \cong K_{\overline{n}} - e_2$. Let $H' = K_{\overline{n}} - e_1 + e' - e$; the complement of H' is $G_{\overline{n}} + e_1 - e' + e$.

If $e' = e_1$, then $H = K_{\overline{n}}$, so we may assume $e' \neq e_1$. Our task is to show that in this case, H' cannot be isomorphic to $K_{\overline{n}} - e_2$, and hence no graph other than $K_{\overline{n}}$ shares these two decards with $K_{\overline{n}}$.

Since $e' \neq e_1$, the edge e' is not a cut-edge of $G_{\overline{n}} + e_1$. Hence $G_{\overline{n}} + e_1 - e'$ has $r - 1$ components. If the endpoints of e lie in different components of $G_{\overline{n}} + e_1 - e'$, then the complement of H' has $r - 2$ components. Since the complement of $K_{\overline{n}} - e_2$ has $r - 1$

components, H' cannot be a decard of $K_{\overline{n}}$ and hence cannot be $K_{\overline{n}} - e_2$. Hence we may assume that the endpoints of e lie in one component of $G_{\overline{n}} + e_1 - e'$.

Let G' denote the non-complete component of $G_{\overline{n}} + e_1$, in which e_1 is a cut-edge. Since e_1 does not satisfy the condition of Lemma 4.1, we have that $j - i \in \{1, 2\}$ or that $K_{\overline{n}}$ has an $(\frac{i+j}{2} + 1)$-part, or both. We distinguish cases based on the location of e'.

Case 1: $j - i \in \{1, 2\}$ and e' joins two vertices in the j-part having an endpoint of e_1. Since $(i, j) \neq (1, 2)$, we have $j \geq 3$. Since $e' \in E(G')$ in this case, $G' - e'$ is the only component of $G_{\overline{n}} + e_1 - e'$ that is not complete. Hence $G' - e'$ is the component of $G_{\overline{n}} + e_1 - e'$ containing both endpoints of e (recall that $K_{\overline{n}} - e_2 = H - e$).

If $j \geq 4$, then $G' - e' + e$ has no cut-edge. Thus no component with at least four vertices in the complement of H' has a cut-edge. However, since $k + \ell \geq 4$, the complement of $K_{\overline{n}} - e_2$ has a component with at least four vertices in which e_2 is a cut-edge. Hence $H' \ncong K_{\overline{n}} - e_2$.

If $j = 3$ and $i = 1$, then $G' - e' \cong K_{1,3}$. We have $G' - e' + e \cong G'$ and $G_{\overline{n}} + e_1 - e' + e \cong G_{\overline{n}} + e_1$. Hence, $H' \cong K_{\overline{n}} - e_1$. In this case we have assumed that the two decards are not the same. That is, $K_{\overline{n}} - e_2 \ncong K_{\overline{n}} - e_1$, which yields $H' \ncong K_{\overline{n}} - e_2$, as desired.

If $j = 3$ and $i = 2$, then $G' - e'$ is a 5-vertex path. Up to isomorphism, there are four ways to add e to form $G' - e' + e$. One of them creates G'; as in the previous subcase, this is forbidden by assuming $K_{\overline{n}} - e_2 \ncong K_{\overline{n}} - e_1$. In the other three cases (5-cycle, 4-cycle plus pendant edge, 3-cycle plus two pendant edges), $G' - e' + e$ cannot be produced by adding an edge joining two complete graphs. That is, $H - e$ is not a decard of $K_{\overline{n}}$.

Case 2: e' joins two vertices of $K_{\overline{n}}$ in an $(\frac{i+j}{2} + 1)$-part. In this case, $i + j$ is even. By Case 1, we may assume that the part containing the endpoints of e' does not contain an endpoint of e_1. Hence the complement $G_{\overline{n}} + e_1 - e'$ of H has two non-complete components, one of which contains both endpoints of e.

Since we have assumed $e \neq e'$, and e' is the only edge missing from the component of $G_{\overline{n}} + e_1 - e'$ containing its endpoints, again e must join two vertices of G', as in Case 1. Since G' has at least four vertices, $G' + e$ is not a complete graph. Hence the complement of H' has two non-complete components. This implies that H' cannot be a decard of a complete multipartite graph and hence cannot be $K_{\overline{n}} - e_2$. □

The problem is simpler for $r = 2$, because $K_{m,n}$ is edge-transitive. Monikandan and Sundar Raj [9] solved it; we include the statement for completeness.

Theorem 4.3 ([9])**.** *For $1 \leq m \leq n$ and $(m, n) \neq (1, 3)$,*

$$\mathrm{adern}(K_{m,n}) = \mathrm{dern}(K_{m,n}) = \begin{cases} 3, & \textit{if } (m,n) = (2,3); \\ 2, & \textit{if } n \geq 4 \textit{ and } m \in \{n-1, n-2\}; \\ 1, & \textit{otherwise }; \end{cases}$$

Theorem 4.4. *If $r \geq 3$ and $n_1 \leq \cdots \leq n_r$, then*

$$\mathrm{dern}(K_{\overline{n}}) = \begin{cases} 1, & \text{if } (n_1,\ldots,n_r) \text{ satisfies the condition of Lemma 4.1;} \\ 2, & \text{otherwise ;} \end{cases}$$

Proof. We have $\mathrm{dern}(K_{\overline{n}}) = 1$ when $(n_1,\ldots,n_r)$ satisfies the condition of Lemma 4.1. Otherwise, $\mathrm{dern}(K_{\overline{n}}) > 1$ and $n_1 \neq n_r$. Since also $r \geq 3$, there are distinct decards of $K_{\overline{n}}$, and Lemma 4.2 yields $\mathrm{dern}(K_{\overline{n}}) \leq 2$. □

Theorem 4.5. *If $r \geq 3$ and $n_1 \leq \cdots \leq n_r$, then*

$$\mathrm{adern}(K_{\overline{n}}) = \begin{cases} 1, & \text{if every edge of } K_{\overline{n}} \text{ satisfies the condition of Lemma 4.1;} \\ 3, & \text{if } n_1 = 2 \text{ and } K_{\overline{n}} \text{ has a 3-part;} \\ 4, & \text{if } n_1 = 1 \text{ and } K_{\overline{n}} \text{ has a 3-part;} \\ 2, & \text{otherwise ;} \end{cases}$$

Proof. Case 1: Every edge of $K_{\overline{n}}$ satisfies the condition of Lemma 4.1. By Lemma 4.1, every decard $(G_{\overline{n}} - e, d(e))$ determines $K_{\overline{n}}$.

Case 2: $K_{\overline{n}}$ has a 3-part and a 2-part but no 1-part. Let S be the union of a 3-part and a 2-part. Let F be a 4-cycle plus a pendant edge, and let F' be a 5-cycle plus a chord. Both $K_{2,3}$ and F' have two copies of $(F,3)$ as decards. Hence substituting F' for the subgraph of $K_{\overline{n}}$ induced by S yields a graph sharing two decards with $K_{\overline{n}}$, so $\mathrm{adern}(K_{\overline{n}}) \geq 3$.

For the upper bound, since $K_{\overline{n}}$ has no 1-part, Lemma 4.2 implies that we only need to prove that $K_{\overline{n}}$ is determined by three copies of the decard $(K_{\overline{n}} - e, d(e))$, where e joins two vertices of S. Note that $d(e) = 2m - 7$, where $m = |V(K_{\overline{n}})|$. Construct H from $K_{\overline{n}} - e$ by adding an edge e' of degree $2m-7$. If $H \ncong K_{\overline{n}}$, then e' joins vertices of degrees $m-4$ and $m-3$ in the 3-part. The subgraph of H induced by S is F'. Each vertex of S has $m-5$ neighbors outside S. Hence to obtain $(K_{\overline{n}} - e, 2m-7)$ as a decard of H, we must delete an edge with degree 3 in F'. There are four such edges, but only two yield F when deleted. Hence H has only two copies of $(K_{\overline{n}} - e, 2m-7)$ in its dedeck, so $\mathrm{adern}(K_{\overline{n}}) \leq 3$.

Case 3: $K_{\overline{n}}$ has a 3-part and a 1-part. Let S be the union of a 3-part and a 1-part. Substituting $K_1 + K_3$ for the subgraph of $K_{\overline{n}}$ induced by S (which is $K_{1,3}$) yields a graph sharing three decards with $K_{\overline{n}}$, so $\mathrm{adern}(K_{\overline{n}}) \geq 4$.

For the upper bound, by Lemma 4.2 we only need to prove that four identical decards obtained by deleting an edge joining a 3-part to a smaller part determine $K_{\overline{n}}$. If the smaller part is a 2-part, then the argument for Case 2 applies. Hence we only need to prove that four copies of $(K_{\overline{n}} - e, 2m-6)$ determine $K_{\overline{n}}$, where e joins a 3-part and a 1-part.

Construct H from $K_{\overline{n}} - e$ by adding an edge of degree $2m-6$. If $H \ncong K_{\overline{n}}$, then H is formed by adding an edge joining the two vertices of degree $m-3$ in a 3-part. The graph H has at most three copies of $(K_{\overline{n}} - e, 2m-6)$ in its dedeck. Hence, $\mathrm{adern}(K_{\overline{n}}) \leq 4$.

Case 4: Some edge fails the condition of Lemma 4.1, and $K_{\overline{n}}$ does not have both a 3*-part and a smaller part.* Since some edge fails the condition of Lemma 4.1, adern$(K_{\overline{n}}) > 1$. Every edge of $K_{\overline{n}}$ joins i-part and j-part where $(i, j) \notin \{(1, 2), (2, 3)\}$. By Lemma 4.2, any two decards determine $K_{\overline{n}}$. □

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