JOURNAL OF COMBINATORIAL THEORY, Series B 58, 58–64 (1993)

Generating Linear Extensions by Adjacent Transpositions

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Received February 8, 1990

If P is a rooted forest in which no element has exactly one child, then the linear extensions of P can be successively generated by transpositions of consecutive elements; i.e., P is *adjacent-traversable*. More generally, if P is adjacent-traversable and P' is obtained from P by adding an antichain of size at least two covering (or covered by) a single element of P, then P' is also adjacent-traversable. The proof is constructive.

The linear extensions of a partial order P are the linear orderings of its elements that are consistent with all relations in P. If P has n elements $\{x_1, ..., x_n\}$, then the linear extensions correspond to a collection of permutations of $\{x_1, ..., x_n\}$. For example, if P is a collection of disjoint chains of sizes $n_1, ..., n_k$, then its linear extensions correspond to the permutations of k types of items with multiplicities $n_1, ..., n_k$. For speed in applications where such collections must be searched, it helps to generate the linear extensions successively by making small changes in the corresponding permutation. Ideally, the extensions will be generated by successive transpositions or, better yet, by successive transpositions of consecutive elements. The poset is *traversable* or *adjacent-traversable*, respectively, if this can be done.

The *transposition graph* on the linear extensions of a poset joins two extensions by an edge if they differ by a transposition. The *adjacent transposition graph*, which we denote $G(P)$, is the subgraph of the transposition graph in which we keep only the edges corresponding to transpositions of consecutive elements. A transposition changes the parity of a permutation, so the transposition graph is bipartite, and a necessary condition for traversability is that the number of odd extensions and number of even extensions must differ by at most one.

Ruskey [3] conjectured that any poset with the same number of even and odd extensions is traversable, and he proved this [4] when P is a disjoint union of chains and when P is a rooted forest in which no element

* Research supported in part by ONR Grant N00014-85K0570.

0095-8956/93 $5.00

has exactly one child. Further references and discussion of the problem can be found in [1, 2, 4–6]. Concerning the more restrictive property of adjacent-traversability, Stachowiak [7] has shown that if Q has an even number of linear extensions and is adjacent-traversable, then the disjoint union of Q with any poset is also adjacent-traversable. Applied to disjoint unions of chains, this improves Ruskey's result [4] for chains. In this note, we do the same for a rooted forest in which no element has exactly one child, showing it is adjacent-traversable. More generally, we prove the following:

THEOREM. *If P is an adjacent-traversable poset and P' is obtained from P by adding an antichain of at least two elements covered by (or covering) a single element, then P' is adjacent-traversable in the same way.*

Preliminary Remarks and Notation. By "in the same way," we mean that if $G = G(P)$ has a Hamiltonian cycle Q, then we construct a Hamiltonian cycle in $G' = G(P')$. If the traversal of $G(P)$ is merely a Hamiltonian path Q, then we construct a Hamiltonian path in $G(P')$. By duality, we may assume that the new elements are maximal elements $z_1, ..., z_m$ covering a single element x of P. We write extensions as concatenations of elements of the poset, beginning with a minimal element and ending with a maximal element. If $\sigma = \alpha x \beta$ is an extension of P, let $T(\sigma)$ denote the extensions of P' whose restriction to P is σ. If β has size k, the set T is given by the $\prod_{i=1}^{m} (k+i)$ ways to insert $z_1, ..., z_m$ in any order anywhere following x in σ; in particular, $T(\sigma)$ contains $\sigma z_1 \cdots z_m$.

Our approach to proving the theorem is based on techniques used by Ruskey in [4]. We expand the traversal Q of G'. We group the vertices of Q into consecutive pairs from the beginning. With each successive pair (σ, τ), we associate the subgraph $G'(\sigma, \tau)$ of G' induced by $T(\sigma) \cup T(\tau)$. We prove that $G'(\sigma, \tau)$ has a Hamiltonian path from $\sigma z_1 \cdots z_m$ to $\tau z_1 \cdots z_m$ (which themselves are adjacent). Since $(\pi z_1 \cdots z_m, \rho z_1 \cdots z_m) \in E(G')$ whenever $(\pi, \rho) \in E(G)$, these paths string together to form a Hamiltonian path of G' (if Q has odd order, ending in ρ, then the final path is a Hamiltonian path from $\rho z_1 \cdots z_m$ in the subgraph induced by $T(\rho)$). If Q is a cycle, then Q has even order, and the final vertex of the expanded path is adjacent to the initial vertex, completing a Hamiltonian cycle for G'.

It is convenient to prove this first for $m = 2$, letting y, z denote z_1, z_2. For this we study in more detail the subgraph induced by $T(\sigma)$ when $m = 2$. For any σ in which x is followed by k elements, the subgraph H_k of G' induced by the $(k+2)(k+1)$ vertices of $T(\sigma)$ will be the same. The graph H_k consists of two triangular grids, corresponding to y preceding z and z preceding y, respectively. The grids are joined by edges corresponding to transposition of y and z when they are consecutive, so there are $k+1$ of

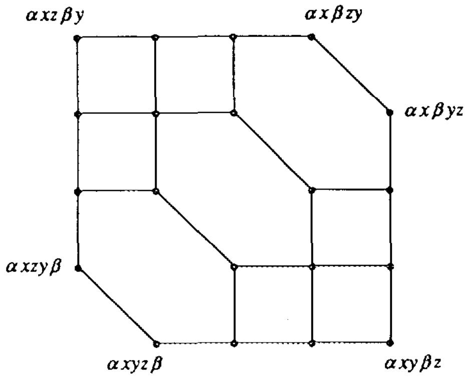

FIG. 1. The subgraph H_3.

these "diagonal" edges. In each grid, the vertical edges correspond to movement of y, and the horizontal edges to movement of z. H_0 is a single diagonal edge and H_1 is a 6-cycle; Figure 1 shows H_3.

Since α appears at the beginning of all the elements of $T(\sigma)$ when $\sigma = \alpha x \beta$, we henceforth drop it from the designation of vertices in H_k. Also, for paths in this subgraph the identity of elements of β is unimportant, since they are incomparable to y and z, so we indicate them all by o's, with the vertices in the lower grid of H_k being $\{xo^r yo^s zo^t \colon r, s, t \geqslant 0, r+s+t=k\}$. There are many ways to cover the vertices of H_k with a path (for example, if r is odd, with $1 \leqslant r \leqslant k$, then H_k has a Hamiltonian path from $xo^k yz$ to $xo^r zyo^{k-r}$), but we will derive only the paths needed to give a short proof of the theorem.

LEMMA 1. *If k is even, then H_k has a Hamiltonian path from $xo^k yz$ to $xzo^k y$. If k is odd, then H_k has a Hamiltonian path from $xo^k yz$ to $xyo^k z$.*

Proof. For H_0, the claimed path is the only edge of the graph. Figure 2 exhibits these paths in H_2 and H_3. Consider the subgraph in H_k induced

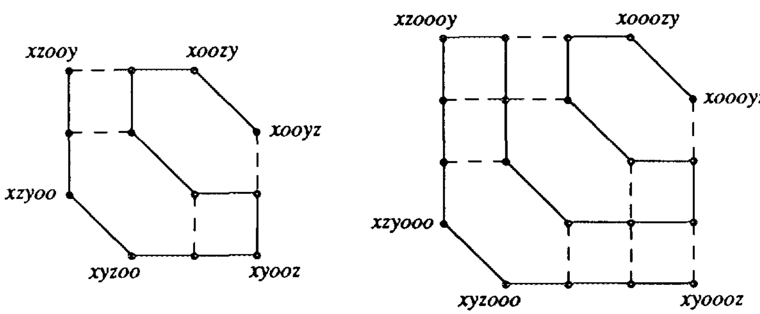

FIG. 2. Hamiltonian paths in H_2 and H_3.

by the vertices beginning xo; this is a copy of H_{k-1}. Figure 2 illustrates the fact that the path for H_{k-1} on this copy can be extended to the desired path in H_k by adding $xzo^ky, ..., xzyo^k, xyzo^k, ..., xyo^kz$ if $k-1$ is even and by adding $xyo^kz, ..., xyzo^k, xzyo^k, ..., xzo^ky$ if $k-1$ is odd. ▮

LEMMA 2. *If $k \geqslant 1$, then the vertices of H_k can be covered by two disjoint paths with endpoints at $A = xo^kyz$, $B = xo^kzy$, $C = xo^{k-1}yoz$, $D = xo^{k-1}yzo$. If k is even, then we construct these as the edge (A, B) and a C, D-path, and if k is odd, then we construct these as the edge (A, C) and a B, D-path.*

Proof. As in Lemma 1, there are many such pairs of paths, but again it is easy to construct the specified pairs inductively. For H_1, they are the edge $(xoyz, xyoz)$ and the path $(xozy, xzoy, xzyo, xyzo)$. For H_2, they are the edge $(xooyz, xoozy)$ and the path $(xoyoz, xyooz, xyozo, xyzoo, xzyoo, xzoyo, xzooy, xozoy, xozyo, xoyzo)$. Figure 3 exhibits such pairs of paths for H_3 and H_4. For $k \geqslant 3$, consider the subgraph in H_k induced by the vertices beginning xoo; this is a copy of H_{k-2}. Figure 3 illustrates the fact that the pair of paths for H_{k-2} on this copy can be extended to the desired pair in H_k be deleting the edge $(xoozo^{k-3}yo, xoozo^{k-2}y)$ (shown as a dashed line in the figure) and adding the edges of the path $(xoozo^{k-3}yo, xozo^{k-2}yo, ..., xozyo^{k-2}o, xoyzo^{k-1}, ..., xoyo^{k-1}z, xyo^kz, ..., xyzo^k, xzyo^k, ..., xzo^ky, xozo^{k-1}y, xoozo^{k-2}y)$ to include the remaining vertices. The arrow indicates the special edge for the next step. ▮

Armed with these paths, we can complete the proof of the Theorem.

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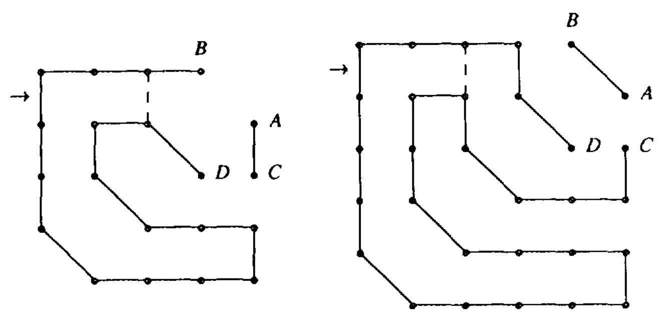

FIG. 3. Pairs of paths covering H_3 and H_4.

Proof. As suggested in the earlier discussion, we start with the case of $m=2$, where it suffices to show that if σ, τ are consecutive in the traversal Q of G, then $G'(\sigma, \tau)$ has a Hamiltonian path from σyz to τyz (the final path for a Hamiltonian path Q of odd order is provided by Lemma 1). Suppose $\sigma = \alpha x\beta$ and $\tau = \gamma x\delta$.

If the transposition between σ and τ does not involve x, then β and δ have the same length k, and the graph $G'(\sigma, \tau)$ contains the edges $(\alpha xy\beta z, \gamma xy\delta z)$ and $(\alpha xz\beta y, \gamma xz\delta y)$ that are copies of the edge (σ, τ) in G. Now Lemma 1 applies to the two copies of H_k in $G'(\sigma, \tau)$. Together with one of the edges just described, this completes the desired path.

If the transposition between σ and τ does involve x, then by symmetry in the endpoints of the desired path we may assume that $\sigma = \alpha xa\rho$ and $\tau = \alpha ax\rho$, where ρ has length k. Since every vertex in $G'(\sigma, \tau)$ begins with α, again we ignore α in the vertex labels. If $k=0$, then the desired path is $xayz$, $xyaz$, $xyza$, $xzya$, $xzay$, $xazy$, $axzy$, $axyz$. If $k \geqslant 1$, suppose that ρ ends in o and $\rho = \pi o$. Note that $G'(\sigma, \tau)$ consists of the copy of H_{k+1} on $T(\sigma)$, the copy of H_k on $T(\tau)$, and an edge from each element of $T(\tau)$ to the corresponding element in $T(\sigma)$. These are copies of the edge (σ, τ) in G; in particular, we have the three edges $(xa\rho zy, ax\rho zy)$, $(xa\pi yoz, ax\pi yoz)$, and $(xa\pi yzo, ax\pi yzo)$. The endpoints of these edges are copies of the vertices B, C, D described in Lemma 2. We use these three edges and the pairs of paths constructed in Lemma 2 in the copies of H_k and H_{k+1} in $G'(\sigma, \tau)$. Instead of writing out the extensions to describe vertices of the resulting Hamiltonian paths, we simply write A^+, A for the copies of A in $T(\sigma)$ and $T(\tau)$, and similarly for B, C, D. If k is odd, then the resulting path is $(A^+, B^+, B, \ldots, D, D^+, \ldots, C^+, C, A)$. If k is even, then the resulting path is $(A^+, C^+, C, \ldots, D, D^+, \ldots, B^+, B, A)$. Figure 4 illustrates the paths for the cases $k=1$ and $k=2$, where we have put H_{k+1}

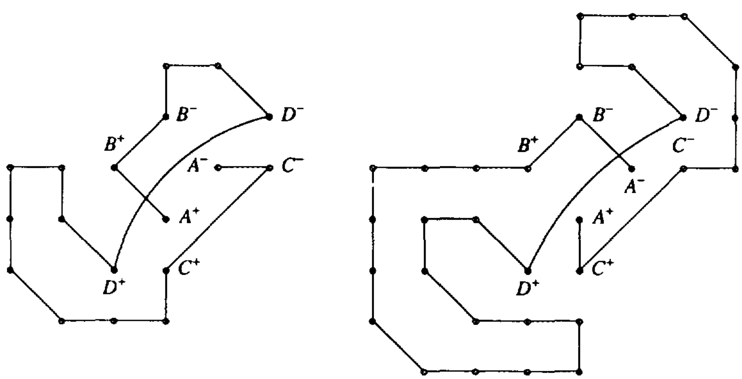

FIG. 4. Hamiltonian paths in $G'(\sigma, \tau)$.

in the lower left and H_k in the upper right, with the latter reflected around a line with slope -1.

If $m > 2$, we complete the argument by induction on m, with $m = 2$ as the basis. If σ, τ are consecutive in the traversal Q of G, then the induction hypothesis is the existence of a path Q'' from $\sigma z_1 \cdots z_{m-1}$ to $\tau z_1 \cdots z_{m-1}$ in the graph $G''(\sigma, \tau)$ whose vertices are all ways to add $\{z_1, ..., z_{m-1}\}$, and that furthermore this path Q'' consists of successive odd-even pairs between which the transposition does not move x. If x does not move in the transposition between σ and τ in Q, this additional statement is trivial. When x does move, we must verify it for the basis case $m = 2$. The graph H_k has an even number of vertices. In Lemma 2, the pairs of paths always consist of a single edge and another path that therefore also has odd length. In the construction above, we combine four paths of odd length by adding three edges. Each of the four subpaths thus consists of odd-even pairs along the resulting path $Q' = Q''$. The three added edges are the only edges on which x moves; hence x does not move between any odd-even pair.

Now, assume the full induction hypothesis for $m - 1 \geqslant 2$. Let (σ', τ') be an odd-even pair in Q''. Because x does not move, we may assume $\sigma' = \alpha x \beta$ and $\tau' = \gamma x \delta$, with β, δ having the same length. We expand this pair into a path from $\sigma' z_m$ to $\tau' z_m$ by sweeping z_m in from the right through β, performing the transposition from σ' to τ' while z_m sits next to x, and then sweeping z_m back to the right through δ. Since β and δ have the same length, z_m has the same number of insertions into σ' and τ', so this path itself consists of odd-even pairs within which x does not move. Because z_m is rightmost at this ends of these paths, they link together via the even-odd transpositions in Q'' to form the desired path from $\sigma z_1 \cdots z_m$ to $\tau z_1 \cdots z_m$. ∎

We remark that the argument works equally well if the added antichain covers more than one element, as long as they all cover the same set. In this case, the element playing the role of x in a particular $G'(\sigma, \tau)$ is the rightmost of the elements they cover. If the transposition σ, τ interchanges two elements they cover, then the behavior is as in the proof above when x does not move, because here again the positions available for the new elements are the same in both σ and τ.

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