Generating Linear Extensions by Adjacent Transpositions

DOUGLAS B. WEST*

Department of Mathematics, University of Illinois,
Urbana, Illinois 61801

Received February 8, 1990

If \( P \) is a rooted forest in which no element has exactly one child, then the linear extensions of \( P \) can be successively generated by transpositions of consecutive elements; i.e., \( P \) is adjacent-traversable. More generally, if \( P \) is adjacent-traversable and \( P' \) is obtained from \( P \) by adding an antichain of size at least two covering (or covered by) a single element of \( P \), then \( P' \) is also adjacent-traversable. The proof is constructive. © 1993 Academic Press, Inc.

The linear extensions of a partial order \( P \) are the linear orderings of its elements that are consistent with all relations in \( P \). If \( P \) has \( n \) elements \( \{x_1, \ldots, x_n\} \), then the linear extensions correspond to a collection of permutations of \( \{x_1, \ldots, x_n\} \). For example, if \( P \) is a collection of disjoint chains of sizes \( n_1, \ldots, n_k \), then its linear extensions correspond to the permutations of \( k \) types of items with multiplicities \( n_1, \ldots, n_k \). For speed in applications where such collections must be searched, it helps to generate the linear extensions successively by making small changes in the corresponding permutation. Ideally, the extensions will be generated by successive transpositions or, better yet, by successive transpositions of consecutive elements. The poset is traversable or adjacent-traversable, respectively, if this can be done.

The transposition graph on the linear extensions of a poset joins two extensions by an edge if they differ by a transposition. The adjacent transposition graph, which we denote \( G(P) \), is the subgraph of the transposition graph in which we keep only the edges corresponding to transpositions of consecutive elements. A transposition changes the parity of a permutation, so the transposition graph is bipartite, and a necessary condition for traversability is that the number of odd extensions and number of even extensions must differ by at most one.

Ruskey [3] conjectured that any poset with the same number of even and odd extensions is traversable, and he proved this [4] when \( P \) is a disjoint union of chains and when \( P \) is a rooted forest in which no element

* Research supported in part by ONR Grant N00014-85K0570.
has exactly one child. Further references and discussion of the problem can be found in [1, 2, 4, 6]. Concerning the more restrictive property of adjacent-traversability, Stachowiak [7] has shown that if \( Q \) has an even number of linear extensions and is adjacent-traversable, then the disjoint union of \( Q \) with any poset is also adjacent-traversable. Applied to disjoint unions of chains, this improves Ruskey's result [4] for chains. In this note, we do the same for a rooted forest in which no element has exactly one child, showing it is adjacent-traversable. More generally, we prove the following:

**Theorem.** If \( P \) is an adjacent-traversable poset and \( P' \) is obtained from \( P \) by adding an antichain of at least two elements covered by (or covering) a single element, then \( P' \) is adjacent-traversable in the same way.

**Preliminary Remarks and Notation.** By "in the same way," we mean that if \( G = G(P) \) has a Hamiltonian cycle \( Q \), then we construct a Hamiltonian cycle in \( G' = G(P') \). If the traversal of \( G(P) \) is merely a Hamiltonian path \( Q \), then we construct a Hamiltonian path in \( G(P') \). By duality, we may assume that the new elements are maximal elements \( z_1, \ldots, z_m \) covering a single element \( x \) of \( P \). We write extensions as concatenations of elements of the poset, beginning with a minimal element and ending with a maximal element. If \( \sigma = xzx'z' \) is an extension of \( P \), let \( T(\sigma) \) denote the extensions of \( P' \) whose restriction to \( P \) is \( \sigma \). If \( \beta \) has size \( k \), the set \( T \) is given by \( \prod_{i=1}^{m} (k+i) \) ways to insert \( z_1, \ldots, z_m \) in any order anywhere following \( x \) in \( \sigma \); in particular, \( T(\sigma) \) contains \( \sigma z_1 \cdots z_m \).

Our approach to proving the theorem is based on techniques used by Ruskey in [4]. We expand the traversal \( Q \) of \( G' \). We group the vertices of \( Q \) into consecutive pairs from the beginning. With each successive pair \((\sigma, \tau)\), we associate the subgraph \( G'(\sigma, \tau) \) of \( G' \) induced by \( T(\sigma) \cup T(\tau) \). We prove that \( G'(\sigma, \tau) \) has a Hamiltonian path from \( \tau z_1 \cdots z_m \) to \( \tau z_1 \cdots z_m \) (which themselves are adjacent). Since \( (\tau z_1 \cdots z_m, \rho z_1 \cdots z_m) \in E(G') \) whenever \((\pi, \rho) \in E(G)\), these paths string together to form a Hamiltonian path of \( G' \) (if \( Q \) has odd order, ending in \( \rho \), then the final path is a Hamiltonian path from \( \rho z_1 \cdots z_m \) in the subgraph induced by \( T(\rho) \)). If \( Q \) is a cycle, then \( Q \) has even order, and the final vertex of the expanded path is adjacent to the initial vertex, completing a Hamiltonian cycle for \( G' \).

It is convenient to prove this first for \( m = 2 \), letting \( y, z \) denote \( z_1, z_2 \). For this we study in more detail the subgraph induced by \( T(\sigma) \) when \( m = 2 \). For any \( \sigma \) in which \( x \) is followed by \( k \) elements, the subgraph \( H_k \) of \( G' \) induced by the \((k + 2)(k + 1) \) vertices of \( T(\sigma) \) will be the same. The graph \( H_k \) consists of two triangular grids, corresponding to \( y \) preceding \( z \) and \( z \) preceding \( y \), respectively. The grids are joined by edges corresponding to transposition of \( y \) and \( z \) when they are consecutive, so there are \( k + 1 \) of
these "diagonal" edges. In each grid, the vertical edges correspond to movement of \( y \), and the horizontal edges to movement of \( z \). \( H_0 \) is a single diagonal edge and \( H_1 \) is a 6-cycle; Figure 1 shows \( H_3 \).

Since \( \alpha \) appears at the beginning of all the elements of \( T(\sigma) \) when \( \sigma = \alpha x \beta \), we henceforth drop it from the designation of vertices in \( H_k \). Also, for paths in this subgraph the identity of elements of \( \beta \) is unimportant, since they are incomparable to \( y \) and \( z \), so we indicate them all by \( o^t \)'s, with the vertices in the lower grid of \( H_k \) being \( \{ x o^r y o^s z o^t : r, s, t \geq 0, r + s + t = k \} \). There are many ways to cover the vertices of \( H_k \) with a path (for example, if \( r \) is odd, with \( 1 \leq r \leq k \), then \( H_k \) has a Hamiltonian path from \( xo^r y z \) to \( xo^r y z o^{k-r} \)), but we will derive only the paths needed to give a short proof of the theorem.

**Lemma 1.** If \( k \) is even, then \( H_k \) has a Hamiltonian path from \( xo^k y z \) to \( x zo^k y \). If \( k \) is odd, then \( H_k \) has a Hamiltonian path from \( xo^k y z \) to \( x yo^k z \).

**Proof.** For \( H_0 \), the claimed path is the only edge of the graph. Figure 2 exhibits these paths in \( H_2 \) and \( H_3 \). Consider the subgraph in \( H_k \) induced
by the vertices beginning $xo$; this is a copy of $H_{k-1}$. Figure 2 illustrates the
fact that the path for $H_{k-1}$ on this copy can be extended to the desired
path in $H_k$ by adding $xzo^ky, ..., xzyo^k, xzyo^k, ..., xyo^kz$ if $k - 1$ is even and
by adding $xyo^kz, ..., xzyo^k, xzyo^k, ..., xzo^ky$ if $k - 1$ is odd.

**Lemma 2.** If $k \geq 1$, then the vertices of $H_k$ can be covered by two disjoint
paths with endpoints at $A = xo^kyz, B = xo^kzy, C = xo^{k-1}yoz, D = xo^{k-1}yzo$.
If $k$ is even, then we construct these as the edge $(A, B)$ and a $C, D$-path, and
if $k$ is odd, then we construct these as the edge $(A, C)$ and a $B, D$-path.

**Proof.** As in Lemma 1, there are many such pairs of paths, but again
it is easy to construct the specified pairs inductively. For $H_1$, they are the
edge $(xoyz, xyo)$ and the path $(xoyz, xyzo, xzyo, xyo)$. For $H_2$, they are
the edge $(xoyo_z, xoyo_y)$ and the path $(xoyo_z, xyo_o, xzyo, xyo, xzyo, xzyo, xzyo, xzyo, xzyo, xzyo, xzyo)$. Figure 3 exhibits such pairs of paths
for $H_3$ and $H_4$. For $k \geq 3$, consider the subgraph in $H_k$ induced by the
vertices beginning $xoo$; this is a copy of $H_{k-2}$. Figure 3 illustrates the fact
that the pair of paths for $H_{k-2}$ on this copy can be extended to the desired
pair in $H_k$ be deleting the edge $(xoyo^{k-2}yo, xoyo^{k-2}y)$ (shown as a dashed line in the figure) and adding the edges of the path $(xoyo^{k-3}yo,
xoyo^{k-2}yo, xoyo^{k-2}o, xoyo^{k-2}o, xoyo^{k-1}, ..., xoyo^{k-1}z, xyo^kz, ..., xzo^k, xzo^k, ..., xzo^k, xzo^{k-1}y, xoyo^{k-2}y)$ to include the remaining vertices. The arrow
indicates the special edge for the next step.

Armed with these paths, we can complete the proof of the Theorem.

**Theorem.** If $P$ is an adjacent-traversable poset and $P'$ is obtained from
$P$ by adding an antichain of $m \geq 2$ elements covered by (or covering) a single
element, then $P'$ is also adjacent-traversable.

![Fig. 3. Pairs of paths covering $H_1$ and $H_4$.](image-url)
Proof. As suggested in the earlier discussion, we start with the case of \( m = 2 \), where it suffices to show that if \( \sigma, \tau \) are consecutive in the traversal \( Q \) of \( G \), then \( G'(\sigma, \tau) \) has a Hamiltonian path from \( \sigma yz \) to \( \tau yz \) (the final path for a Hamiltonian path \( Q \) of odd order is provided by Lemma 1). Suppose \( \sigma = zx\beta \) and \( \tau = \gamma x\delta \).

If the transposition between \( \sigma \) and \( \tau \) does not involve \( x \), then \( \beta \) and \( \delta \) have the same length \( k \), and the graph \( G'(\sigma, \tau) \) contains the edges \((xxy\beta z, \gamma xy\delta z)\) and \((xxz\beta y, \gamma xz\delta y)\) that are copies of the edge \((\sigma, \tau)\) in \( G \). Now Lemma 1 applies to the two copies of \( H_k \) in \( G'(\sigma, \tau) \). Together with one of the edges just described, this completes the desired path.

If the transposition between \( \sigma \) and \( \tau \) does involve \( x \), then by symmetry in the endpoints of the desired path we may assume that \( \sigma = zx\rho \alpha \rho \) and \( \tau = ax\rho \alpha \rho \), where \( \rho \) has length \( k \). Since every vertex in \( G'( \sigma, \tau ) \) begins with \( \alpha \), again we ignore \( \alpha \) in the vertex labels. If \( k = 0 \), then the desired path is \( xayz, wyaz, xya, axya, axay, axzy, xazy, ayzy, axyz \). If \( k \geq 1 \), suppose that \( \rho \) ends in \( a \) and \( \rho = \pi \alpha \rho \). Note that \( G'(\sigma, \tau) \) consists of the copy of \( H_{k+1} \) on \( T(\sigma) \), the copy of \( H_k \) on \( T(\tau) \), and an edge from each element of \( T(\tau) \) to the corresponding element in \( T(\sigma) \). These are copies of the edge \((\sigma, \tau)\) in \( G \); in particular, we have the three edges \((xapzy, axpzy), (xap\rho z, axp\rho z), \) and \((xap\rho z, axp\rho z)\). The endpoints of these edges are copies of the vertices \( B, C, D \) described in Lemma 2. We use these three edges and the pairs of paths constructed in Lemma 2 in the copies of \( H_k \) and \( H_{k+1} \) in \( G'(\sigma, \tau) \). Instead of writing out the extensions to describe vertices of the resulting Hamiltonian paths, we simply write \( A^+, A^- \) for the copies of \( A \) in \( T(\sigma) \) and \( T(\tau) \), and similarly for \( B, C, D \). If \( k \) is odd, then the resulting path is \((A^+, B^+, B^-, ..., D^-, D^+, ..., C^+, C^-)\). If \( k \) is even, then the resulting path is \((A^+, C^+, C^-, ..., D^+, D^-, ..., B^+, B^-, A^-)\). Figure 4 illustrates the paths for the cases \( k = 1 \) and \( k = 2 \), where we have put \( H_{k+1} \)

![Fig. 4. Hamiltonian paths in \( G'(\sigma, \tau) \).](image)
in the lower left and $H_k$ in the upper right, with the latter reflected around a line with slope $-1$.

If $m > 2$, we complete the argument by induction on $m$, with $m = 2$ as the basis. If $\sigma, \tau$ are consecutive in the traversal $Q$ of $G$, then the induction hypothesis is the existence of a path $Q''$ from $\sigma z_1 \cdots z_{m - 1}$ to $\tau z_1 \cdots z_{m - 1}$ in the graph $G'(\sigma, \tau)$ whose vertices are all ways to add $\{z_{m - 1}, \ldots, z_m\}$, and that furthermore this path $Q''$ consists of successive odd-even pairs between which the transposition does not move $x$. If $x$ does not move in the transposition between $\sigma$ and $\tau$ in $Q$, this additional statement is trivial. When $x$ does move, we must verify it for the basis case $m = 2$. The graph $H_k$ has an even number of vertices. In Lemma 2, the pairs of paths always consist of a single edge and another path that therefore also has odd length. In the construction above, we combine four paths of odd length by adding three edges. Each of the four subpaths thus consists of odd-even pairs along the resulting path $Q' = Q''$. The three added edges are the only edges on which $x$ moves; hence $x$ does not move between any odd-even pair.

Now, assume the full induction hypothesis for $m - 1 \geq 2$. Let $(\sigma', \tau')$ be an odd-even pair in $Q''$. Because $x$ does not move, we may assume $\sigma' = x_x' \beta$ and $\tau' = x_x' \delta$, with $\beta, \delta$ having the same length. We expand this pair into a path from $\sigma' z_m$ to $\tau' z_m$ by sweeping $z_m$ in from the right through $\beta$, performing the transposition from $\sigma'$ to $\tau'$ while $z_m$ sits next to $x$, and then sweeping $z_m$ back to the right through $\delta$. Since $\beta$ and $\delta$ have the same length, $z_m$ has the same number of insertions into $\sigma'$ and $\tau'$, so this path itself consists of odd-even pairs within which $x$ does not move. Because $z_m$ is rightmost at this ends of these paths, they link together via the even-odd transpositions in $Q''$ to form the desired path from $\sigma z_1 \cdots z_m$ to $\tau z_1 \cdots z_m$.

We remark that the argument works equally well if the added antichain covers more than one element, as long as they all cover the same set. In this case, the element playing the role of $x$ in a particular $G'(\sigma, \tau)$ is the rightmost of the elements they cover. If the transposition $\sigma, \tau$ interchanges two elements they cover, then the behavior is as in the proof above when $x$ does not move, because here again the positions available for the new elements are the same in both $\sigma$ and $\tau$.

References


