Pairs of Adjacent Hamiltonian Circuits with Small Intersection

By Douglas B. West

Consider the question: When can the edges in a pair of Hamiltonian circuits be redistributed to form another pair of circuits with the same union and intersection? A class of pairs is exhibited which intersect in two edges and cannot be rearranged in this way. A connection to algorithms for the traveling salesman problem is explained using the convex polytope of Hamiltonian circuits in $\binom{n}{2}$-space. The exhibited pair is shown to be an edge of that polytope.

1. Introduction

The *traveling salesman problem* is the problem of finding the least cost Hamiltonian circuit in a complete graph on n vertices, where the cost of a circuit is the sum of the costs of its edges. Some algorithms begin with a candidate solution and look for small changes to improve it. Given n vertices, any Hamiltonian circuit corresponds to a corner with n ones of the 0-1 cube in $\binom{n}{2}$-dimensional space, where each dimension corresponds to an edge of the complete graph. Consider the polytope which is the convex hull of such points.

"Local search" algorithms start from some feasible circuit and examine some set of "nearby" circuits for improvement. These have been characterized by Savage, Weiner, and Krone [7]. Papadimitriou and Steiglitz [5] show that if such an algorithm guarantees proceeding to the optimum, then when looking for improvements it must examine all circuits adjacent to the current circuit on the skeleton of the polytope described. Determining the adjacency structure of this polytope is extremely difficult. Papadimitriou [4] shows that in general the problem of deciding whether two circuits are adjacent is NP-complete.

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It is possible to give necessary or sufficient conditions for non-adjacency and thus reduce the number of circuits to be examined. Lebensold [1], Murty [3], and Rao [6] have made progress in this direction. By definition, two vertices are adjacent on the skeleton of a convex polytope if no point on the segment joining them is expressible as a convex combination of other vertices. (A convex combination is a linear combination where the vertices are non-negative and sum to 1.) Suppose two pairs (P,Q) and (R,S) of Hamiltonian circuits exist having the same union and intersection. Then the segments PQ and RS between vertices of the polytope intersect at their midpoints. So, a sufficient condition for a pair of Hamiltonian circuits to be non-adjacent is if their edges can be redistributed to form another pair of circuits with the same union and intersection. Rao [6] gives an example to show the condition is not necessary.

The problem of whether an arbitrary pair of Hamiltonian circuits can be rearranged in this way is quite difficult. Papadimitriou [4] mentions some complexity results. The conjecture that all *disjoint* pairs are rearrangeable was raised by Lin [2] and has been outstanding for a number of years. We extend this conjecture to include all pairs whose intersection contains exactly one edge. In the next section we show that this would be the best possible result by exhibiting non-rearrangeable pairs which intersect in two edges. Failing this test means the pairs may be adjacent; we show in addition that they do form edges in the polytope of Hamiltonian circuits.

2. A Non-rearrangeable pair of circuits

THEOREM 1. *Suppose P and Q are two undirected Hamiltonian circuits on $v_1, v_2, \ldots, v_n$ defined as follows:*

(1) *Q consists of the edges (v_n, v_1) and $\{(v_i, v_{i+1})\colon i = 1,2,\ldots,n-1\}$.*
(2) *P consists of the edges (v_1, v_2), (v_{n-1}, v_n), and $(v_i, v_{i+2})\colon i = 1,2,\ldots,n-2$.*

Then there is no other pair of Hamiltonian circuits with the same union and intersection as P and Q.

Proof: (See Fig. 1.) Suppose R and S exist rearranging P and Q. Let us try to construct them with (v_n, v_1) in S. We show by induction that (v_i, v_{i+1}) must also

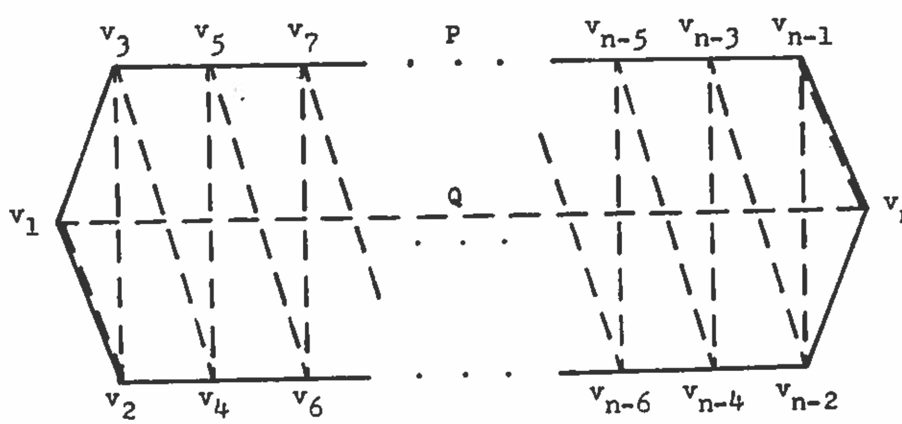

Figure 1. A non-rearrangeable pair.

be in S and (v_i, v_{i+2}) in R. If this is true for all $i<n-1$, then $R=P$, $S=Q$, and there is no suitable rearrangement. For the basis step, note that (v_1, v_2) and (v_{n-1}, v_n) must lie in both R and S, being edges of intersection. (v_1, v_3), the remaining edge incident to v_1, must lie in R, since (v_n, v_1) is in S. Now suppose the assertion is true for $i<j$, where $j<n-1$, and consider the edges incident to v_j. (v_{j-2}, v_j) is in R, and (v_{j-1}, v_j) is in S. We have two possible ways to distribute the remaining edges. If (v_j, v_{j+1}) is in R and (v_j, v_{j+2}) in S, then we create a circuit in R. For example, if j is even, we have $(v_1, v_2, v_4, v_6, \ldots, v_j, v_{j+1}, v_{j-1}, \ldots, v_3, v_1)$. R could not then be a Hamiltonian circuit, so we must place (v_j, v_{j+1}) in S and (v_j, v_{j+2}) in R.

3. Adjacency on the polytope

THEOREM 2. *Let P and Q be as described in Theorem 1. Then P and Q are adjacent on the polytope of Hamiltonian circuits.*

Proof: First we show that in any traveling salesman problem, the cheapest circuit R and the next-cheapest circuit S must be adjacent on the polytope of circuits. If c_i is the cost of the ith edge, then to any point $(x_1, \ldots, x_{\binom{n}{2}})$ in $\binom{n}{2}$-space we may assign $\Sigma c_i x_i$ as its cost. Then the cost of a convex combination of circuits is the same convex combination of the circuit costs. The cost of the combination (a weighted average) is no less than the cost of the cheapest circuit used to form it. Now, if R and S are not adjacent, then some point on the segment joining them is a convex combination of other circuits, the cheapest of which we call T. That point is also a convex combination of R and S, and has a cost between those of R and S. Since its cost is no less than that of T, T must be cheaper than S. So if R and S are the two cheapest circuits, they must be adjacent.

To establish the theorem we need only exhibit an assignment of edge costs for which P and Q are the cheapest circuits. To any edge not in P or Q, assign extremely high cost, say 10^{2n}, so none will appear in any cheap circuit. To the edges (v_{n-1}, v_n) and (v_1, v_2), assign cost 0. Let the other edges of P cost 10^n. Let (v_n, v_1) cost 10^{n+1}, and let the remaining edges of Q cost 10^n-1. Clearly the cheap circuits will contain the edges that are both in P and in Q, but no edge that is in neither. By the argument used in Theorem 1, the only such circuit containing (v_1, v_3) is P, which costs $(n-2)10^n$. Any such circuit not containing (v_1, v_3) must use (v_n, v_1). Among these, Q is the cheapest, costing $10^{n+1}+(n-3)(10^n-1)$, which is greater than the cost of P. Another possibility is $(v_n, v_1, v_2, v_4, v_3, v_5, v_6, \ldots, v_{n-1}, v_n)$ (see Fig. 1), but whenever an edge of P replaces an edge of Q other than (v_n, v_1), the cost increases. Thus, P and Q are the cheapest Hamiltonian circuits.

From Theorem 2 and the results mentioned in the introduction, we conclude that local search algorithms which guarantee optimality must at each stage examine circuits with as few as two edges in common with the current circuit. This contributes to the extensive evidence that local search algorithms for optimality cannot be efficient. (See Weiner, Savage, and Bagchi [8].)

Of course, since rearrangeability implies non-adjacency, adjacency implies non-rearrangeability, and Theorem 2 implies Theorem 1. However, finding cost assignments to test adjacency has proven to be inefficient. Rearrangeability might be more useful, and is of independent interest.

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