Acyclic graphs with at least $2\ell + 1$ vertices are $\ell$-recognizable

Douglas B. West

Departments of Mathematics  
Zhejiang Normal University and  
University of Illinois at Urbana-Champaign  
dwest@illinois.edu  
slides and papers on preprint page  
https://faculty.math.illinois.edu/~west/pubs/publink.html

Joint work with  
Alexandr V. Kostochka, Mina Nahvi, Dara Zirlin
The Classical Problem

**Def.** A card of a graph $G$ is an induced subgraph $G - \nu$. The deck of a graph is the multiset of its cards.

Equality for $K_{n/2,n/2}$ and $2K_{n/2}$.
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Any graph with $\geq 3$ vertices is determined by its deck.

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![Diagram of a graph and its cards]

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**Ex.** $K_4^-$ is determined by three of its cards.

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**Ex.** $K_4^-$ is determined by three cards. Which three?

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**Ex.** $K_4^-$ is determined by three cards. Which three?

**Def.** Harary-Plantholt [1985]: The reconstruction number $\text{rn}(G)$ is the least number of cards that determine $G$.

$\text{rn}(G)$ measures the difficulty of reconstructing $G$. Equality for $K_{n/2,n/2}$ and $2K_{n/2}$. 
Another Direction

**Conj.** Kelly [1957]: For \( \ell \in \mathbb{N} \), \( \exists M_{\ell} \in \mathbb{N} \) such that \( |V(G)| \geq M_{\ell} \implies G \) is reconstructible from the graphs obtained by deleting \( \ell \) vertices.
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(Similarly, \( l \)-reconstructible \( \Rightarrow (l-1) \)-reconstructible.)

- Another way to ask how hard it is to reconstruct \( G \).
What is known?

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**Thm.** Graphs w. same #verts and #edges have same $D_k$ if components are cycle w. $\geq k+1$ vts or path with $\geq k-1$ vts.
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**Cor.** For $n \geq 2l + 1$, all $n$-vertex graphs with maximum degree 2 are $l$-reconstructible (except $(n,l)=(5,2)$).
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**Thm.** Müller [1976], S-W [2019]: For $l \leq (1 - o(1))n/2$, almost every graph is $l$-reconstructible. (From $\binom{l+2}{2}$ cards.) Generalizes Chinn ’71, Müller ’76, Bollobás ’90 for $l = 1$. 
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**Thm.** Connectedness (Manvel [1974]) and the degree list (Chernyak [1982]) are 2-reconstructible for $n \geq 6$. 
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**Cor.** The degree list $\ell$-reconstruct’ble when $n \geq \ell + O \sqrt{\ell}$. 
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**Cor.** The degree list $l$-reconstr’ble when $n \geq l + O\sqrt{l}$.

**Thm.** SW’19: Connectedness is $l$-reconstr. for $n > \ell(\ell + 1)^2$. 
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Thm. SW’19: Connectedness is \( \ell \)-reconstr. for \( n > \ell^{(\ell+1)^2} \).

Thm. GJST’21: Connectedness is \( \ell \)-reconstr. for \( n \geq 10\ell \).
Regular Graphs and Components

**Thm.** Kostochka–West 2021: $r$-regular graphs that are not 2-connected are $(r+1)$-reconstr’ble.
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**Thm.** KW’21: If $n \geq 2l + 1$ and every component of $G$ has at most $n - l$ vertices, then $G$ is $l$-reconstructible.

**Thm.** KW’21: If graphs with at least $l + 2$ vertices having $l–1$ isolated vertices and one large component $(n–l+1)$ are $l$-reconstructible, then the original RC holds.
Thm. Kelly’57: Trees with at least 3 vertices are 1-reconstructible.
Trees - I

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**Thm.** Giles’76: Trees with at least 6 vertices are 2-reconstructible.
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Two steps to reconstruction of graphs in a family \( \mathcal{F} \).
(1) **Recognition**: Every graph with deck \( \mathcal{D} \) lies in \( \mathcal{F} \).
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![Diagram showing two trees with at least 6 vertices]

Two steps to reconstruction of graphs in a family $\mathcal{F}$.

1. **Recognition**: Every graph with deck $\mathcal{D}$ lies in $\mathcal{F}$.
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**Conj.** Nýdl’81: Trees with $n \geq 2\ell + 1$ are weakly $\ell$-reconstructible.
Trees - II

**Thm.** Kostochka-Nahvi-West-Zirlin [2021+]: For \( n \geq 2\ell + 1 \), \( n \)-vertex acyclic graphs are \( \ell \)-recognizable.
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GJST: 13 vertices, same 7-deck
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![Diagram of trees with 13 vertices and same 7-deck](image)

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**Cor.** Trees \( \ell \)-reconstr’ble for \( n \geq 9\ell + 24\sqrt{2\ell} + o(\sqrt{\ell}) \).
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<table>
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![Diagram showing two different trees with the same 7-deck.](image)

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![GJST: 13 vertices, same 7-deck](image)

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The case $\ell = 3$ of GJST covers $n \geq 194$.

Ours, only for $\ell = 3$, takes 48 pages (uses rooted trees).
Main Idea to Recognize Acyclic Graphs

**Thm.** For \( n \geq 2\ell + 2 \), the \((n - \ell)\)-deck \( D \) of an \( n \)-vertex graph \( G \) determines whether \( G \) has a cycle.
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**Cor.** To count \( k \)-centers, count maximal \( k \)-vines.
The Counting Lemma

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**Def.** $\mathcal{F}$-subgraph = induced subgraph of $G$ in family $\mathcal{F}$.

$s(F, G) = \#$induced copies of $F$ in $G$.

$m(F, G) = \#$copies of $F$ as a maximal $\mathcal{F}$-subgraph of $G$. 

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**Pf.** For each $F_0 \in \mathcal{F}$, use induction on max length of induced subgraph chain $F_0, \ldots, F_r$ in $G$. $r$ is known.
If $r = 0$, then $m(F, G) = s(F, G)$.
For $r > 0$, gather copies of $F$ by unique maximal $H$.
Now $s(F, G) = \sum_{H \in \mathcal{F}} s(F, H)m(H, G)$, solve for $m(F, G)$.
\[\blacksquare\]
Applications

**Cor.** For $n > 2\ell$, every $n$-vertex graph having no component with more than $n - \ell$ vertices is $\ell$-reconstr’bl.
Applications

**Cor.** For $n > 2l$, every $n$-vertex graph having no component with more than $n - l$ vertices is $l$-reconstructible.

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**Cor.** For $n > 2l$, every $n$-vertex graph having no component w. more than $n - l$ vertices is $l$-reconstr’bl.

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**Cor.** For $n > 2\ell$, every $n$-vertex graph having no component with more than $n - \ell$ vertices is $\ell$-reconstructible.

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Sharpness by $P_\ell + P_\ell$ vs. $P_{\ell+1} + P_{\ell-1}$ when $n = 2\ell$. 

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**Cor.** If $\mathcal{D}$ is an $(n - \ell)$-deck where every card is acyclic and has radius greater than $k$, then all $n$-vertex reconstructions have the same number of $k$-centers.
Cor. For \( n > 2l \), every \( n \)-vertex graph having no component w. more than \( n - l \) vertices is \( l \)-reconstr’bl.

Pf. \( \{ \text{connected graphs} \} \) is absorbing family for any \( G \).

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Cor. If \( \mathcal{D} \) is an \( (n - l) \)-deck where every card is acyclic and has radius greater than \( k \), then all \( n \)-vertex reconstructions have the same number of \( k \)-centers.

Pf. radius \( > k \) \( \Rightarrow \) \( n - l \geq 2k + 2 \) \( \Rightarrow \) girth \( \geq 2k+3 \).
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**Cor.** If \( \mathcal{D} \) is an \((n - \ell)\)-deck where every card is acyclic and has radius greater than \( k \), then all \( n \)-vertex reconstructions have the same number of \( k \)-centers.

**Pf.** radius \( > k \) \( \Rightarrow n - \ell \geq 2k + 2 \) \( \Rightarrow \) girth \( \geq 2k + 3 \).
\[ \therefore \text{Family of } k \text{-vines is absorbing.} \]
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**Cor.** For \( n > 2\ell \), every \( n \)-vertex graph having no component w. more than \( n - \ell \) vertices is \( \ell \)-reconstr’bl.

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No \( \#k \)-vines have \( \geq n - \ell \) vertices (too small radius).
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**Cor.** If $\mathcal{D}$ is an $(n - l)$-deck where every card is acyclic and has radius greater than $k$, then all $n$-vertex reconstructions have the same number of $k$-centers.

**Pf.** radius $> k \Rightarrow n - l \geq 2k + 2 \Rightarrow$ girth $\geq 2k + 3$.

$\therefore$ Family of $k$-vines is absorbing.

No $\#k$-vines have $\geq n - l$ vertices (too small radius).

$\therefore$ Counting Lemma yields $\#\text{maximal } k$-vines.

These correspond bijectively to $k$-centers.
The Marking Argument

**Def.** \( \hat{k} = \text{min radius connected card}: \text{short card} = \text{rad} \hat{k} \).

\( d = \#\text{disj. length-} \hat{k} \text{ paths from center.} \) Let \( k = \hat{k} - 1 \).

**Def.** **Marking argument:** For short card \( C \) from forest \( F \), each \( k \)-center \( x \) other than the center \( z \) marks a vertex \( x' \) at distance \( k \) from \( x \) (direction away from \( z \)).

\[ \begin{align*}
\hat{k} &= 3 \\
k &= 2 \\
d &= 3 \Rightarrow \#\text{paths}
\end{align*} \]
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**Def.** Marking argument: For short card $C$ from forest $F$, each $k$-center $x$ other than the center $z$ marks a vertex $x'$ at distance $k$ from $x$ (direction away from $z$).

**Lem.** If $C$ is a short card of $F$, then $\#k$-centers in $F$ is at most $1 + d + \ell$. Equality only if every vertex outside $C$ is marked and $F$ is a tree.
The Marking Argument

Def. \( \hat{k} = \min \text{ radius connected card: short card} = \text{rad} \hat{k} \).
\[ d = \# \text{disj. length-} \hat{k} \text{ paths from center.} \]
Let \( k = \hat{k} - 1 \).

Def. Marking argument: For short card \( C \) from forest \( F \), each \( k \)-center \( x \) other than the center \( z \) marks a vertex \( x' \) at distance \( k \) from \( x \) (direction away from \( z \)).

\[ \hat{k} = 3 \quad k = 2 \quad d = 3 = \# \text{paths} \]

Lem. If \( C \) is a short card of \( F \), then \#\( k \)-centers in \( F \) is at most \( 1 + d + \ell \). Equality only if every vertex outside \( C \) is marked and \( F \) is a tree.

Pf. \( k \)-centers not adj. to \( z \) must mark verts. outside \( C \).
Ambiguous Decks

**Def.** ambiguous deck $\mathcal{D} = \text{the (}n - \ell\text{)-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}$
Ambiguous Decks

**Def.** ambiguous deck \( D \) = the \((n - \ell)\)-deck of both an acyclic \( F \) and non-acyclic \( H \) with \( n \) vertices.

• all cards acyclic \( \Rightarrow \) girth\((H) \geq n - \ell + 1 \geq 2\hat{k} = 2k + 2 \)
  \( \Rightarrow \) \( F \) and \( H \) have the same number of \( k \)-centers.
Ambiguous Decks

**Def.** ambiguous deck $\mathcal{D} = \text{the } (n - \ell)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}

- all cards acyclic $\Rightarrow$ girth($H$) $\geq n - \ell + 1 \geq 2\hat{k} = 2k + 2$
  $\Rightarrow F \text{ and } H \text{ have the same number of } k\text{-centers.}$

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \Rightarrow \hat{k} > 1.$
Ambiguous Decks

**Def.** ambiguous deck $\mathcal{D} = \text{the } (n - \ell)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}

- all cards acyclic $\Rightarrow \text{girth}(H) \geq n - \ell + 1 \geq 2\hat{k} = 2k + 2 \\
  \Rightarrow F \text{ and } H \text{ have the same number of } k\text{-centers.}$

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \Rightarrow \hat{k} > 1.$

**Pf.** If $\hat{k} = 1,$ then short card is a star with $n - \ell$ vertices.
Ambiguous Decks

**Def.** ambiguous deck $\mathcal{D} = \text{the } (n - ℓ)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}

- all cards acyclic $\Rightarrow \text{girth}(H) \geq n - ℓ + 1 \geq 2\hat{k} = 2k + 2$
  $\Rightarrow F \text{ and } H \text{ have the same number of } k\text{-centers.}$

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2ℓ + 2 \Rightarrow \hat{k} > 1.$

**Pf.** If $\hat{k} = 1,$ then short card is a star with $n - ℓ$ vertices. $2n - 2ℓ + 1 \geq n + 2 \Rightarrow \text{cycle+star in same compon. of } H.$
**Ambiguous Decks**

**Def.** ambiguous deck $\mathcal{D} = $ the $(n - \ell)$-deck of both an acyclic $F$ and non-acyclic $H$ with $n$ vertices.

- all cards acyclic $\Rightarrow$ girth$(H) \geq n - \ell + 1 \geq 2\hat{k} = 2k + 2$  
  $\Rightarrow$ $F$ and $H$ have the same number of $k$-centers.

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \Rightarrow \hat{k} > 1$.

**Pf.** If $\hat{k} = 1$, then short card is a star with $n - \ell$ vertices.  
$2n - 2\ell + 1 \geq n + 2 \Rightarrow$ cycle+star in same compon. of $H$.  
2-deck $\Rightarrow \leq n - 1$ edges $\Rightarrow H$ is disconnected.
**Ambiguous Decks**

**Def.** ambiguous deck $\mathcal{D} = \text{the } (n - \ell)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}$

- all cards acyclic $\Rightarrow$ girth$(H) \geq n - \ell + 1 \geq 2\hat{k} = 2k + 2$
  $\Rightarrow$ $F$ and $H$ have the same number of $k$-centers.

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \Rightarrow \hat{k} > 1$.

**Pf.** If $\hat{k} = 1$, then short card is a star with $n - \ell$ vertices.

$2n - 2\ell + 1 \geq n + 2 \Rightarrow$ cycle+star in same compon. of $H$.

2-deck $\Rightarrow \leq n - 1$ edges $\Rightarrow H$ is disconnected.

Girth $\geq 4 \Rightarrow$ star & cycle share $\leq 3$ verts.
Ambiguous Decks

**Def.** ambiguous deck $\mathcal{D} = \text{the } (n - \ell)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}

- all cards acyclic $\implies$ girth$(H) \geq n - \ell + 1 \geq 2\hat{k} = 2k + 2$ $\implies F$ and $H$ have the same number of $k$-centers.

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \implies \hat{k} > 1$.

**Pf.** If $\hat{k} = 1$, then short card is a star with $n - \ell$ vertices. $2n - 2\ell + 1 \geq n + 2 \implies$ cycle+star in same compon. of $H$. $2$-deck $\implies \leq n - 1$ edges $\implies H$ is disconnected. Girth $\geq 4 \implies$ star & cycle share $\leq 3$ verts. Now $(n - \ell + 1) + (n - \ell) - 3 \leq n - 1 \implies n \leq 2\ell + 1$. $\blacksquare$
Ambiguous Decks

**Def.** ambiguous deck $\mathcal{D} =$ the $(n - l)$-deck of both an acyclic $F$ and non-acyclic $H$ with $n$ vertices.

- all cards acyclic $\Rightarrow$ girth$(H) \geq n - l + 1 \geq 2\hat{k} = 2k + 2$ $\Rightarrow$ $F$ and $H$ have the same number of $k$-centers.

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2l + 2$ $\Rightarrow$ $\hat{k} > 1$.

**Pf.** If $\hat{k} = 1$, then short card is a star with $n - l$ vertices.

$2n - 2l + 1 \geq n + 2$ $\Rightarrow$ cycle + star in same compon. of $H$.

2-deck $\Rightarrow$ $\leq n - 1$ edges $\Rightarrow$ $H$ is disconnected.

Girth $\geq 4$ $\Rightarrow$ star & cycle share $\leq 3$ verts.

Now $(n - l + 1) + (n - l) - 3 \leq n - 1$ $\Rightarrow$ $n \leq 2l + 1$.

**Lem.** ambig. $\mathcal{D}$ & $n \geq 2l + 2$ $\Rightarrow$ no card of diam $2k + 1$.  

Def. ambiguous deck $D = \text{the } (n - l)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}$

- all cards acyclic $\Rightarrow$ girth$(H) \geq n - l + 1 \geq 2\hat{k} = 2k + 2$
  $\Rightarrow$ $F$ and $H$ have the same number of $k$-centers.

Lem. ambiguous $D$ and $n \geq 2l + 2 \Rightarrow \hat{k} > 1.$

Pf. If $\hat{k} = 1,$ then short card is a star with $n - l$ vertices.
$2n - 2l + 1 \geq n + 2 \Rightarrow$ cycle+star in same compon. of $H.$
2-deck $\Rightarrow \leq n - 1$ edges $\Rightarrow H$ is disconnected.
Girth $\geq 4 \Rightarrow$ star & cycle share $\leq 3$ verts.
Now $(n - l + 1) + (n - l) - 3 \leq n - 1 \Rightarrow n \leq 2l + 1.$

Lem. ambigu. $D$ & $n \geq 2l + 2 \Rightarrow$ no card of diam $2k+1.$

Pf. $F$ and $H$ have same number of $k$-centers.
**Def.** ambiguous deck $\mathcal{D}$ = the $(n - \ell)$-deck of both an acyclic $F$ and non-acyclic $H$ with $n$ vertices.

- all cards acyclic $\Rightarrow$ girth$(H) \geq n - \ell + 1 \geq 2\hat{k} = 2k + 2$
  $\Rightarrow$ $F$ and $H$ have the same number of $k$-centers.

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2$ $\Rightarrow$ $\hat{k} > 1$.

**Pf.** If $\hat{k} = 1$, then short card is a star with $n - \ell$ vertices.
$2n - 2\ell + 1 \geq n + 2$ $\Rightarrow$ cycle + star in same compon. of $H$.
2-deck $\Rightarrow$ $\leq n - 1$ edges $\Rightarrow$ $H$ is disconnected.
Girth $\geq 4$ $\Rightarrow$ star & cycle share $\leq 3$ verts.
Now $(n - \ell + 1) + (n - \ell) - 3 \leq n - 1$ $\Rightarrow$ $n \leq 2\ell + 1$.

**Lem.** ambig. $\mathcal{D}$ & $n \geq 2\ell + 2$ $\Rightarrow$ no card of diam $2k + 1$.

**Pf.** $F$ and $H$ have same number of $k$-centers.
Diameter $2k + 1$ $\Rightarrow$ $d_C = 1$, $\therefore F$ has $\leq 2 + \ell k$-ctrs.
Ambiguous Decks

**Def.** ambiguous deck $\mathcal{D} = \text{the } (n - \ell)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}

- all cards acyclic $\implies$ girth($H$) $\geq n - \ell + 1 \geq 2\hat{k} = 2k + 2$
  $\implies$ $F$ and $H$ have the same number of $k$-centers.

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \implies \hat{k} > 1$.

**Pf.** If $\hat{k} = 1$, then short card is a star with $n - \ell$ vertices.
2$n$−2$\ell$+1 $\geq$ n+2 $\implies$ cycle+star in same compon. of $H$.
2-deck $\implies$ $\leq$ n−1 edges $\implies$ $H$ is disconnected.
Girth $\geq 4 \implies$ star & cycle share $\leq$ 3 verts.
Now $(n - \ell + 1) + (n - \ell) - 3 \leq n - 1 \implies n \leq 2\ell + 1$. ■

**Lem.** ambig. $\mathcal{D}$ & $n \geq 2\ell+2 \implies$ no card of diam $2k+1$.

**Pf.** $F$ and $H$ have same number of $k$-centers.
Diameter $2k + 1 \implies d_C = 1, \quad \therefore F \text{ has } \leq 2 + \ell \text{ } k\text{-ctrs.}$
All verts. on cycle are $k$-ctrs, $\therefore H \text{ has } \geq n - \ell + 1 \text{ } k\text{-ctrs.}$
Ambiguous Decks

**Def.** ambiguous deck $\mathcal{D} =$ the $(n - l)$-deck of both an acyclic $F$ and non-acyclic $H$ with $n$ vertices.

- all cards acyclic $\Rightarrow$ girth$(H) \geq n - l + 1 \geq 2\hat{k} = 2k + 2$
  $\Rightarrow$ $F$ and $H$ have the same number of $k$-centers.

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2l + 2 \Rightarrow \hat{k} > 1$.

**Pf.** If $\hat{k} = 1$, then short card is a star with $n - l$ vertices.
$2n - 2l + 1 \geq n + 2 \Rightarrow$ cycle + star in same compon. of $H$.
$2$-deck $\Rightarrow \leq n - 1$ edges $\Rightarrow$ $H$ is disconnected.
Girth $\geq 4 \Rightarrow$ star & cycle share $\leq 3$ verts.
Now $(n - l + 1) + (n - l) - 3 \leq n - 1 \Rightarrow n \leq 2l + 1$.

**Lem.** ambig. $\mathcal{D}$ & $n \geq 2l + 2 \Rightarrow$ no card of diam $2k + 1$.

**Pf.** $F$ and $H$ have same number of $k$-centers.
Diameter $2k + 1 \Rightarrow d_C = 1$, $\therefore F$ has $\leq 2 + l$ k-ctrs.
All verts. on cycle are $k$-ctrs, $\therefore H$ has $\geq n - l + 1$ k-ctrs.
Now $n - l + 1 \leq 2 + l \Rightarrow n \leq 2l + 1$.  ■
**Def.** $k$-evine = a tree with diameter $2k + 1$.

$k$-central edge = the central edge of a $k$-evine.
**k-Central Edges and End of Proof**

**Def.**  $k$-evine = a tree with diameter $2k + 1$.

$k$-central edge = the central edge of a $k$-evine.

**Lem.** All cards acyclic w. radius $> k$, none w. diam $2k + 1$, and $2k + 2 \leq n - \ell \Rightarrow \mathcal{D}$ fixes $\#k$-central edges.
**k-Central Edges and End of Proof**

**Def.**  
$k$-evine = a tree with diameter $2k + 1$.  
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All cards acyclic w. radius $> k$, none w. diam $2k+1$, and $2k + 2 \leq n - \ell \Rightarrow \mathcal{D}$ fixes $\#k$-central edges.

**Pf.**  
No card w. diam $2k+1 \Rightarrow$ no $k$-evine has $> n - \ell$ vrts.
**k-Central Edges and End of Proof**

**Def.**  
$k$-evine = a tree with diameter $2k+1$.  
$k$-central edge = the central edge of a $k$-evine.

**Lem.**  
All cards acyclic w. radius $> k$, none w. diam $2k+1$, and $2k + 2 \leq n – l \Rightarrow \mathcal{D}$ fixes $\#k$-central edges.

**Pf.**  
No card w. diam $2k+1 \Rightarrow$ no $k$-evine has $>n–l$ vrts.  
$2k+2 \leq n–l \Rightarrow$ any reconstruction has girth $\geq 2k+3$.  

**k-Central Edges and End of Proof**

**Def.** *k-evine* = a tree with diameter $2k + 1$.

*k-central edge* = the central edge of a *k-evine*.

**Lem.** All cards acyclic w. radius $> k$, none w. diam $2k+1$, and $2k + 2 \leq n - \ell \Rightarrow \mathcal{D}$ fixes #*k*-central edges.

**Pf.** No card w. diam $2k+1 \Rightarrow$ no *k*-evine has $> n - \ell$ vrts.

$2k+2 \leq n - \ell \Rightarrow$ any reconstruction has girth $\geq 2k+3$.

∴ {*k*-evines} is absorbing; Counting Lemma applies. ■
**k-Central Edges and End of Proof**

**Def.** *k-evine* = a tree with diameter $2k + 1$.

*k-central edge* = the central edge of a *k-evine*.

**Lem.** All cards acyclic w. radius > $k$, none w. diam $2k + 1$, and $2k + 2 \leq n - l$ ⇒ $D$ fixes #*k*-central edges.

**Pf.** No card w. diam $2k + 1$ ⇒ no *k-evine* has $>n-l$ vrts.

$2k + 2 \leq n - l$ ⇒ any reconstruction has girth $\geq 2k + 3$.

∴ \{*k-evines*\} is absorbing; Counting Lemma applies. □

**Thm.** For $n \geq 2l + 2$, acyclicity is $l$-recognizable.
**$k$-Central Edges and End of Proof**

**Def.** $k$-evine = a tree with diameter $2k+1$.
$k$-central edge = the central edge of a $k$-evine.

**Lem.** All cards acyclic w. radius > $k$, none w. diam $2k+1$, and $2k+2 \leq n - \ell \Rightarrow \mathcal{D}$ fixes $\#k$-central edges.

**Pf.** No card w. diam $2k+1 \Rightarrow$ no $k$-evine has $>n-\ell$ vrtxs.
$2k+2 \leq n - \ell \Rightarrow$ any reconstruction has girth $\geq 2k+3$.
$\therefore \{k$-evines$\}$ is absorbing; Counting Lemma applies.  ■

**Thm.** For $n \geq 2\ell + 2$, acyclicity is $\ell$-recognizable.

**Pf.** No cards of diam $2k+1 \Rightarrow \#k$-central edges fixed.
Def. $k$-evine = a tree with diameter $2k+1$.

$k$-central edge = the central edge of a $k$-evine.

Lem. All cards acyclic w. radius $> k$, none w. diam $2k+1$, and $2k+2 \leq n - \ell \Rightarrow \mathcal{D}$ fixes #k-central edges.

Pf. No card w. diam $2k+1 \Rightarrow$ no $k$-evine has $> n - \ell$ vrts.

$2k+2 \leq n - \ell \Rightarrow$ any reconstruction has girth $\geq 2k+3$.

∴ \{$k$-evines} is absorbing; Counting Lemma applies. ■

Thm. For $n \geq 2\ell + 2$, acyclicity is $\ell$-recognizable.

Pf. No cards of diam $2k+1 \Rightarrow$ #k-central edges fixed.

Card $C$: edge is $k$-central $\iff$ end away from $z$ is $k$-cntr.
**k-Central Edges and End of Proof**

**Def.**  
$k$-evine = a tree with diameter $2k + 1$.  
$k$-central edge = the central edge of a $k$-evine.

**Lem.** All cards acyclic w. radius $> k$, none w. diam $2k+1$, and $2k + 2 \leq n - l \Rightarrow \mathcal{D}$ fixes $\#k$-central edges.

**Pf.** No card w. diam $2k+1 \Rightarrow$ no $k$-evine has $>n-l$ vrts.  
$2k+2 \leq n-l \Rightarrow$ any reconstruction has girth $\geq 2k+3$.  
∴ $\{k$-evines$\}$ is absorbing; Counting Lemma applies.  

**Thm.** For $n \geq 2l + 2$, acyclicity is $l$-recognizable.

**Pf.** No cards of diam $2k + 1 \Rightarrow \#k$-central edges fixed.  
Card $C$: edge is $k$-central $\iff$ end away from $z$ is $k$-cntr.  
∴ $\#k$-centers $\leq 1 + d + l \Rightarrow \#k$-central edges $\leq d + l$.  


**k-Central Edges and End of Proof**

**Def.** $k$-evine = a tree with diameter $2k + 1$.

$k$-central edge = the central edge of a $k$-evine.

**Lem.** All cards acyclic w. radius $> k$, none w. diam $2k + 1$, and $2k + 2 \leq n - l \Rightarrow D$ fixes $\#k$-central edges.

**Pf.** No card w. diam $2k + 1 \Rightarrow$ no $k$-evine has $> n - l$ vrts.

$2k + 2 \leq n - l \Rightarrow$ any reconstruction has girth $\geq 2k + 3$.

$\therefore \{k$-evines\} is absorbing; Counting Lemma applies. □

**Thm.** For $n \geq 2l + 2$, acyclicity is $l$-recognizable.

**Pf.** No cards of diam $2k + 1 \Rightarrow \#k$-central edges fixed.

Card $C$: edge is $k$-central $\iff$ end away from $z$ is $k$-cntr.

$\therefore \#k$-centers $\leq 1 + d + l \Rightarrow \#k$-central edges $\leq d + l$.

$C \Rightarrow H$ has $d$ $k$-central edges w. common endpoint.
**Def.** $k$-evine $=$ a tree with diameter $2k+1$. $k$-central edge $=$ the central edge of a $k$-evine.

**Lem.** All cards acyclic w. radius $> k$, none w. diam $2k+1$, and $2k+2 \leq n - \ell \Rightarrow \mathcal{D}$ fixes $\#k$-central edges.

**Pf.** No card w. diam $2k+1 \Rightarrow$ no $k$-evine has $> n - \ell$ vrts. $2k+2 \leq n - \ell \Rightarrow$ any reconstruction has girth $\geq 2k+3$. \[\therefore \{k$-evines$\}$ is absorbing; Counting Lemma applies. \[\blacksquare\]

**Thm.** For $n \geq 2\ell + 2$, acyclicity is $\ell$-recognizable.

**Pf.** No cards of diam $2k+1 \Rightarrow$ $\#k$-central edges fixed. Card $C$: edge is $k$-central $\iff$ end away from $z$ is $k$-cntr. \[\therefore \#k$-centers $\leq 1 + d + \ell \Rightarrow \#k$-central edges $\leq d + \ell$. $C \Rightarrow H$ has $d$ $k$-central edges w. common endpoint. Only two can be on a cycle.
**k-Central Edges and End of Proof**

**Def.**  
A **k-evine** is a tree with diameter $2k + 1$.  
A **k-central edge** is the central edge of a k-evine.

**Lem.**  
All cards acyclic w. radius $> k$, none w. diam $2k + 1$, and $2k + 2 \leq n - l \implies \mathcal{D}$ fixes $\#k$-central edges.

**Pf.**  
No card w. diam $2k + 1 \implies$ no k-evine has $> n - l$ vrts.  
$2k + 2 \leq n - l \implies$ any reconstruction has girth $\geq 2k + 3$.  
$\therefore \{k$-evines$\}$ is absorbing; Counting Lemma applies.  

**Thm.**  
For $n \geq 2l + 2$, acyclicity is $l$-recognizable.

**Pf.**  
No cards of diam $2k + 1 \implies \#k$-central edges fixed.  
Card $C$: edge is $k$-central $\iff$ end away from $z$ is $k$-cntr.  
$\therefore \#k$-centers $\leq 1 + d + l \implies \#k$-central edges $\leq d + l$.  
$C \implies H$ has $d$ $k$-central edges w. common endpoint.  
Only two can be on a cycle.  
Girth $\geq 2k + 3 \implies$ every edge on a cycle is $k$-central.
**k-Central Edges and End of Proof**

**Def.** \(k\)-evine = a tree with diameter \(2k + 1\).

\(k\)-central edge = the central edge of a \(k\)-evine.

**Lem.** All cards acyclic w. radius > \(k\), none w. diam \(2k + 1\), and \(2k + 2 \leq n - l \Rightarrow \mathcal{D}\) fixes \#\(k\)-central edges.

**Pf.** No card w. diam \(2k + 1 \Rightarrow \) no \(k\)-evine has > \(n - l\) vrts.

\(2k + 2 \leq n - l \Rightarrow \) any reconstruction has girth \(\geq 2k + 3\).

\(\therefore\) \(\{k\text{-evines}\}\) is absorbing; Counting Lemma applies.

**Thm.** For \(n \geq 2l + 2\), acyclicity is \(l\)-recognizable.

**Pf.** No cards of diam \(2k + 1 \Rightarrow \) \#\(k\)-central edges fixed.

Card \(C\): edge is \(k\)-central \(\iff\) end away from \(z\) is \(k\)-cntr.

\(\therefore\) \#\(k\)-centers \(\leq 1 + d + l \Rightarrow \) \#\(k\)-central edges \(\leq d + l\).

\(C \Rightarrow \) \(H\) has \(d\) \(k\)-central edges w. common endpoint.

Only two can be on a cycle.

Girth \(\geq 2k + 3 \Rightarrow\) every edge on a cycle is \(k\)-central.

\(\therefore\) \(n - l + 1 + d - 2 \leq d + l\). Hence \(n \leq 2l + 1\).
Ideas for $n = 2\ell + 1$

Lemmas again show (1) $\hat{k} > 1$, (2) $2\hat{k} \leq \ell = n - \ell - 1$, (3) Same $#k$-centers and same $#k$-central edges.
Ideas for $n = 2\ell + 1$

Lemmas again show (1) $\hat{k} > 1$, (2) $2\hat{k} \leq \ell = n - \ell - 1$, (3) Same $\#k$-centers and same $\#k$-central edges.

**Def.** spider = a tree with at most one branch vertex.
Ideas for $n = 2\ell + 1$

Lemmas again show (1) $\hat{k} > 1$, (2) $2\hat{k} \leq \ell = n - \ell - 1$, (3) Same $\#k$-centers and same $\#k$-central edges.

**Def.** spider = a tree with at most one branch vertex.

**Lem.** When $n \geq 2\ell + 1$ (except $(5, 2)$), an $n$-vertex spider has at most $\ell + 3$ paths w. exactly $n - \ell$ vertices (cards).
Ideas for $n = 2\ell + 1$

Lemmas again show (1) $\hat{k} > 1$, (2) $2\hat{k} \leq \ell = n - \ell - 1$, (3) Same $\#k$-centers and same $\#k$-central edges.

Def. spider = a tree with at most one branch vertex.

Lem. When $n \geq 2\ell + 1$ (except $(5, 2)$), an $n$-vertex spider has at most $\ell + 3$ paths w. exactly $n - \ell$ vertices (cards).

Lem. Ambiguous $D \Rightarrow$ no card with diameter $2k + 1$. 
Ideas for $n = 2\ell + 1$

Lemmas again show (1) $\hat{k} > 1$, (2) $2\hat{k} \leq \ell = n - \ell - 1$, (3) Same $\#k$-centers and same $\#k$-central edges.

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**Lem.** Ambiguous $D \Rightarrow$ no card with diameter $2k + 1$.

Two pages, using reduction to $F$ being spider and then $H$ having at least $\ell + 4$ paths with exactly $n - \ell$ vertices.
Ideas for $n = 2\ell + 1$

Lemmas again show (1) $\hat{k} > 1$, (2) $2\hat{k} \leq \ell = n - \ell - 1$, (3) Same $\#k$-centers and same $\#k$-central edges.

**Def.** spider = a tree with at most one branch vertex.

**Lem.** When $n \geq 2\ell + 1$ (except $(5, 2)$), an $n$-vertex spider has at most $\ell + 3$ paths w. exactly $n - \ell$ vertices (cards).

**Lem.** Ambiguous $D \Rightarrow$ no card with diameter $2k + 1$.

Two pages, using reduction to $F$ being spider and then $H$ having at least $\ell + 4$ paths with exactly $n - \ell$ vertices.

Using diameter $2k + 2$ for short cards, in both $F$ and $H$ no $\hat{k}$-vine has more than $n - \ell$ vertices, so Counting Lemma applies to get $\#\hat{k}$-centers (determined by $D$).
Ideas for \( n = 2\ell + 1 \)

Lemmas again show (1) \( \hat{k} > 1 \), (2) \( 2\hat{k} \leq \ell = n - \ell - 1 \), (3) Same \#k-centers and same \#k-central edges.

Def. spider = a tree with at most one branch vertex.

Lem. When \( n \geq 2\ell+1 \) (except \((5, 2)\)), an \( n \)-vertex spider has at most \( \ell + 3 \) paths w. exactly \( n - \ell \) vertices (cards).

Lem. Ambiguous \( D \Rightarrow \) no card with diameter \( 2k+1 \).
  
Two pages, using reduction to \( F \) being spider and then \( H \) having at least \( \ell + 4 \) paths with exactly \( n - \ell \) vertices.

Using diameter \( 2k+2 \) for short cards, in both \( F \) and \( H \) no \( \hat{k} \)-vine has more than \( n - \ell \) vertices, so Counting Lemma applies to get \#\( \hat{k} \)-centers (determined by \( D \)).

By marking argument, \( F \) has at most \( \ell + 1 \) \( \hat{k} \)-centers.
Ideas for $n = 2\ell + 1$

Lemmas again show (1) $\hat{k} > 1$, (2) $2\hat{k} \leq \ell = n - \ell - 1$, (3) Same $\#k$-centers and same $\#k$-central edges.

**Def.** spider = a tree with at most one branch vertex.

**Lem.** When $n \geq 2\ell+1$ (except $(5, 2)$), an $n$-vertex spider has at most $\ell+3$ paths w. exactly $n-\ell$ vertices (cards).

**Lem.** Ambiguous $\mathcal{D} \Rightarrow$ no card with diameter $2k+1$.

Two pages, using reduction to $F$ being spider and then $H$ having at least $\ell+4$ paths with exactly $n-\ell$ vertices.

Using diameter $2k+2$ for short cards, in both $F$ and $H$ no $\hat{k}$-vine has more than $n-\ell$ vertices, so Counting Lemma applies to get $\#\hat{k}$-centers (determined by $\mathcal{D}$).

By marking argument, $F$ has at most $\ell+1 \hat{k}$-centers. Use of the cycle in $H$ yields at least $\ell+2 \hat{k}$-centers.
Open Questions

**Conj.** Trees with \( n \geq 2\ell + 1 \) are \( \ell \)-reconstructible.
(Not for \((5, 2)\) or \((13, 6)\); known for \( n \geq 9\ell + O(\sqrt{\ell}) \).)
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(True for $\ell = 3$; known threshold is $n \geq 10\ell$.)
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**Ques.** S-W’19 Any complete \(r\)-partite \(G\) is determined by \(D_{r+1}(G)\). Is this sharp? \((D_3(K_{7,4,3}) = D_3(K_{6,6,1,1})\).)
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**Ques.** What is the max \( n \) such that every \( n \)-vertex complete multipartite \( G \) is determined by its \( k \)-deck? (Nýdl [1985]: it is between \( k \ln(k/2) \) and \( (k + 1)2^{k-1} \).)
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**Ques.** S-W’19 Any complete \( r \)-partite \( G \) is determined by \( \mathcal{D}_{r+1}(G) \). Is this sharp? \( (\mathcal{D}_3(K_{7,4,3}) = \mathcal{D}_3(K_{6,6,1,1})) \).

**Ques.** What is the max \( n \) such that every \( n \)-vertex complete multipartite \( G \) is determined by its \( k \)-deck? (Nýdl [1985]: it is between \( k \ln(k/2) \) and \((k + 1)2^{k-1}\).

**Prob.** Find thresholds on \( n \) for \( \ell \)-reconstructibility of connectivity, matching number, \( \chi(G) \), planarity, etc.