Acyclic graphs with at least $2\ell+1$ vertices are ℓ-recognizable

Douglas B. West

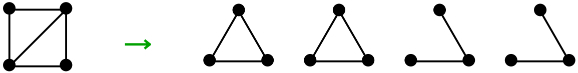
Departments of Mathematics
Zhejiang Normal University and
University of Illinois at Urbana-Champaign
dwest@illinois.edu

slides and papers on preprint page
https://faculty.math.illinois.edu/~west/pubs/publink.html

Joint work with
Alexandr V. Kostochka, Mina Nahvi, Dara Zirlin

The Classical Problem

Def. A card of a graph G is an induced subgraph $G - v$. The deck of a graph is the multiset of its cards.



Equality for $K_{n/2,n/2}$ and $2K_{n/2}$.

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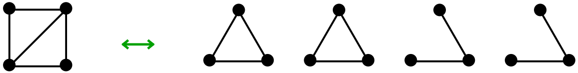

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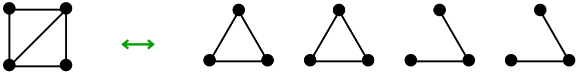

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Any graph with ≥ 3 vertices is determined by its deck.

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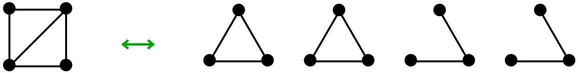

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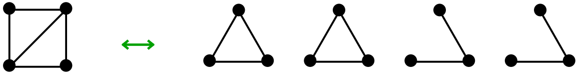

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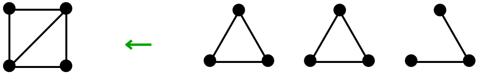

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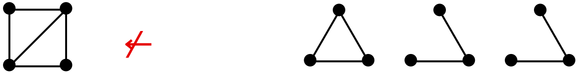

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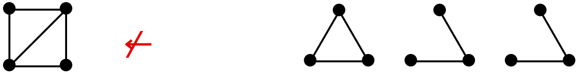

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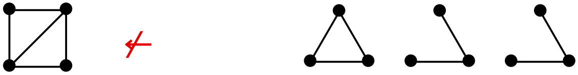

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Def. Harary-Plantholt [1985]: The reconstruction number $\text{rn}(G)$ is the least number of cards that determine G.

$\text{rn}(G)$ measures the difficulty of reconstructing G.
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Conj. Kelly [1957]: For $\ell \in \mathbf{N}$, $\exists\, M_\ell \in \mathbf{N}$ such that $|V(G)| \geq M_\ell \;\Rightarrow\; G$ is reconstructible from the graphs obtained by deleting ℓ vertices.

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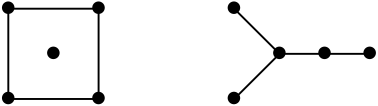

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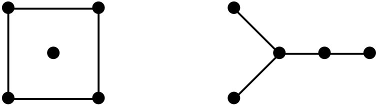

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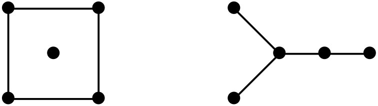

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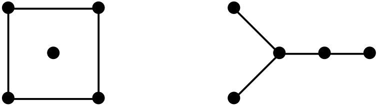

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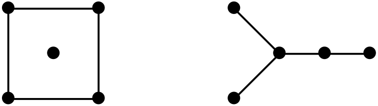

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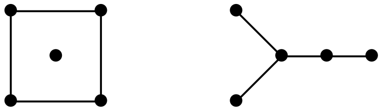

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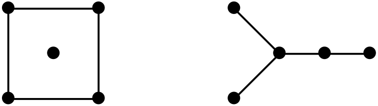

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- Another way to ask how hard it is to reconstruct G.

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Generalizes Chinn '71, Müller '76, Bollobás '90 for $\ell = 1$.

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Cor. The degree list ℓ-reconstr'ble when $n \geq \ell + O\sqrt{\ell}$.

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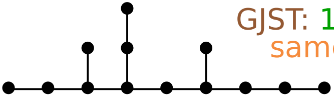

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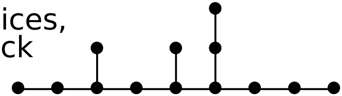

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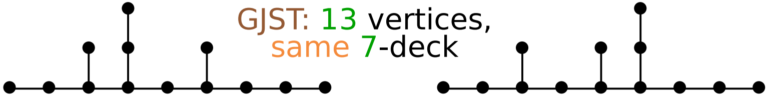

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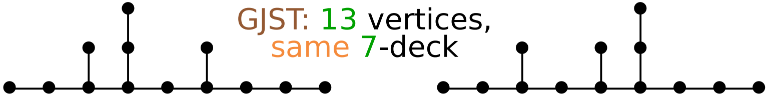

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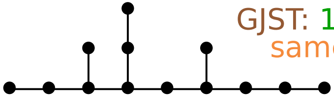

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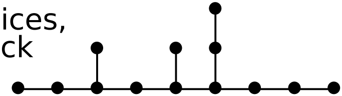

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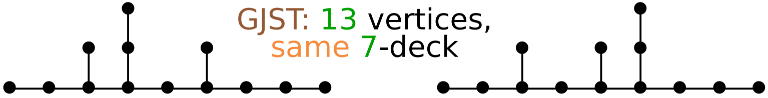

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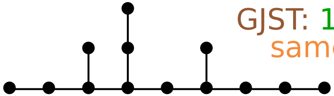

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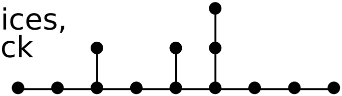

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Ours, only for $\ell = 3$, takes 48 pages (uses rooted trees).

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Cor. To count k-centers, count maximal k-vines.

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Now $s(F,G) = \sum_{H \in \mathcal{F}} s(F,H) m(H,G)$, solve for $m(F,G)$. ■

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Cor. If $\mathcal{D}$ is an $(n-\ell)$-deck where every card is acyclic and has radius greater than k, then all n-vertex reconstructions have the same number of k-centers.

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Cor. For $n > 2\ell$, every n-vertex graph having no component w. more than $n - \ell$ vertices is ℓ-reconstr'bl.

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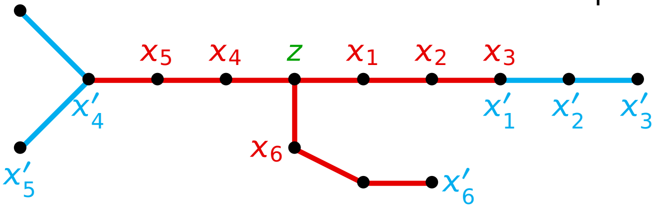
Def. $\hat{k}$ = min radius connected card: short card = rad $\hat{k}$.
d = #disj. length-$\hat{k}$ paths from center. Let $k = \hat{k} - 1$.

Def. Marking argument: For short card C from forest F, each k-center x other than the center z marks a vertex x' at distance k from x (direction away from z).

$\hat{k} = 3$ $\quad k = 2$ $\quad d = 3 =$ #paths



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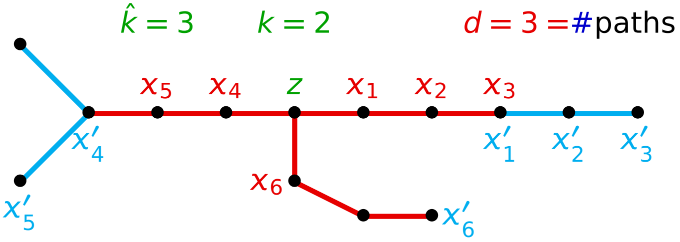

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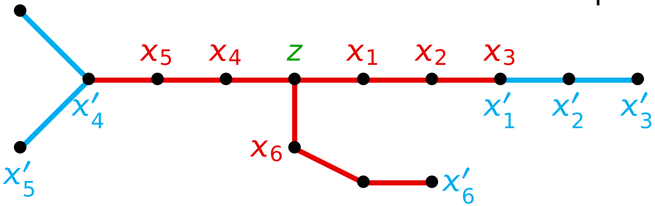

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Lem. When $n \ge 2\ell+1$ (except $(5, 2)$), an n-vertex spider has at most $\ell+3$ paths w. exactly $n-\ell$ vertices (cards).

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Use of the cycle in H yields at least $\ell+2$ $\hat{k}$-centers.

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Prob. Find thresholds on n for ℓ-reconstructibility of connectivity, matching number, $\chi(G)$, planarity, etc.