Acyclic graphs with at least $2\ell + 1$ vertices are $\ell$-recognizable

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Joint work with
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The Classical Problem

**Def.** A card of a graph $G$ is an induced subgraph $G - v$. The deck of a graph is the multiset of its cards.
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**Def.** A *card* of a graph $G$ is an induced subgraph $G - \nu$. The *deck* of a graph is the multiset of its cards.

![Diagram of cards and their deck](image)

**Reconstruction Conj:** Kelly [1957], Ulam [1960]
Any graph with $\geq 3$ vertices is determined by its deck.
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**Ex.** $K_4^-$ is determined by three of its cards.
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**Ex.** $K_4^-$ is determined by three cards. Which three?
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**Ex.** $K_{4}^{-}$ is determined by three cards. Which three?

**Def.** Harary-Plantholt [1985]: The reconstruction number $\text{rn}(G)$ is the least number of cards that determine $G$. 
Another Direction

**Conj.** Kelly [1957], Manvel [1969]: For $\ell \in \mathbb{N}$, $\exists M_{\ell} \in \mathbb{N}$ s.t. $|V(G)| \geq M_{\ell} \implies G$ is reconstructible from the deck obtained by deleting $\ell$ vertices.
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**RC:** $M_1 = 3$. 
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**RC:** \( M_1 = 3 \). \( M_2 = 6 \)\? McMullen–Radziszowski [2007]
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**RC:** $M_1 = 3$. $M_2 = 6$? McMullen–Radziszowski [2007] ($C_4 + K_1$ and the tree $K'_{1,3}$ are not 2-reconstructible.)
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**Def.** $k$-deck $D_k(G) =$ set of $k$-vertex induced subgrs.
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![Graphs](image)

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**Obs.** $\mathcal{D}_k(G)$ determines $\mathcal{D}_{k-1}(G)$. 
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**Pf.** Each graph in \( D_{k-1} \) arises \( n - k + 1 \) times by deleting one vertex from a graph in \( D_k(G) \).
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- Another way to ask how hard it is to reconstruct \( G \).
What is known?

Spinoza–West [2019]: $D_\ell(P_{2\ell}) = D_\ell(C_{\ell+1} + P_{\ell-1})$, so $M_\ell > 2\ell$. 
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**Thm.** Nýdl [1992]: For $\epsilon > 0$, $\exists$ arb. large graphs not $\epsilon n$-reconstructible. $\therefore M_\ell$ grows superlinearly.
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**Thm.** Taylor [1990]: The degree list is $\ell$-reconstructible for $n \geq e\ell(1 + o(1))$. 
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Spinoza–West [2019]: $D_l(P_{2l}) = D_l(C_{l+1} + P_{l-1})$, so $M_l > 2l$.

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More Results

Connectedness is not $n/2$-reconstructible, but . . .
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Generalizes Chinn ’71, Müller ’76, Bollobás ’90 for \( \ell = 1 \).
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**Thm.** Nýdl’81: \( \exists \) trees with \( n = 2l \) and same \( l \)-deck.
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**Thm.** Kostochka-Nahvi-West-Zirlin [2021+]: For $n \geq 2\ell + 1$, $n$-vertex acyclic graphs are $\ell$-recognizable. (not (5,2))
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**Thm.** Müller [1976], S-W [2019]: For $\ell \leq (1 - o(1))n/2$, almost every graph is $\ell$-reconstr’bl. (From $\binom{\ell+2}{2}$ cards.) Generalizes Chinn ’71, Müller ’76, Bollobás ’90 for $\ell = 1$.

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**Conj.** Trees with $n \geq 2\ell+1$ are $\ell$-reconstr’ble.

**Thm.** KNWZ’21+: Trees with $n \geq 22$ are 3-reconstr’ble.
More Results

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**Thm.** Müller [1976], S-W [2019]: For $l \leq (1 - o(1))n/2$, almost every graph is $l$-reconstr’bl. (From $\binom{l+2}{2}$ cards.)

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Generalizes Chinn ’71, Müller ’76, Bollobás ’90 for $\ell = 1$.

**Thm.** Spinoza-West’19: When $\text{maxdeg}(G) \leq 2$, we know $\max\{\ell : G \text{ is } \ell\text{-reconstructible}\}$. Always $\geq (n - 1)/2$.

**Thm.** Graphs w. same #verts and #edges have same $D_k$ if each comp. is a cycle w. $\geq k+1$ vts or path with $\geq k-1$ vts.

**Thm.** Nýdl’81: $\exists$ trees with $n = 2\ell$ and same $\ell$-deck.

**Thm.** Kostochka-Nahvi-West-Zirlin [2021+]: For $n \geq 2\ell + 1$, $n$-vertex acyclic graphs are $\ell$-recognizable. (not $(5,2)$)

**Conj.** Trees with $n \geq 2\ell + 1$ are $\ell$-reconstr’ble.

**Thm.** KNWZ’21+: Trees with $n \geq 22$ are 3-reconstr’ble. 64

**Thm.** KNWZ’21: 3-regular graphs are 2-reconstr’ble.
Initial Thoughts

**Thm.** For $n \geq 2\ell + 2$, the $(n - \ell)$-deck $D$ of an $n$-vertex graph $G$ determines whether $G$ has a cycle.
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**Thm.** For $n \geq 2\ell + 2$, the $(n - \ell)$-deck $\mathcal{D}$ of an $n$-vertex graph $G$ determines whether $G$ has a cycle.

**Def.** $k$-vine = a tree with diameter $2k$.
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No $k$-vine with other center contains $B$. 
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∴ unique maximal $k$-vine having $B$ is the $k$-ball at $v$. ■
**Initial Thoughts**

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**Cor.** To count $k$-centers, count maximal $k$-vines.
The Counting Lemma

The Counting Lemma


**Def.** $\mathcal{F}$-subgraph = induced subgraph of $G$ in family $\mathcal{F}$.

$s(F,G) = \#$ induced copies of $F$ in $G$.

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**Lem.** If \( \mathcal{F} \) is absorbing for \( n \)-vertex \( G \) with \( (n - l) \)-deck \( \mathcal{D} \), and \( m(F, G) \) is known for each \( F \in \mathcal{F} \) with at least \( n - l \) vertices, then \( m(F, G) \) is determined for all \( F \in \mathcal{F} \).
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**Pf.** For each $F_0 \in \mathcal{F}$, use induction on max length of chain of induced $\mathcal{F}$-subgraphs $F_0, \ldots, F_r$ in $G$. 
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**Def.** $\mathcal{F}$-subgraph = induced subgraph of $G$ in family $\mathcal{F}$. 
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$r$ is computable from the decks of smaller subgraphs.
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\( r \) is computable from the decks of smaller subgraphs.

If \( r = 0 \), then \( m(F, G) = s(F, G) \).
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If \( r = 0 \), then \( m(F, G) = s(F, G) \).
For \( r > 0 \), gather copies of \( F \) by unique maximal \( H \).
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\( r \) is computable from the decks of smaller subgraphs.

If \( r = 0 \), then \( m(F, G) = s(F, G) \).
For \( r > 0 \), gather copies of \( F \) by unique maximal \( H \).
Now \( s(F, G) = \sum_{H \in \mathcal{F}} s(F, H)m(H, G) \), solve for \( m(F, G) \).  ■
Applications

**Cor.** For $n > 2\ell$, every $n$-vertex graph having no component with more than $n - \ell$ vertices is $\ell$-reconstr’bl.
Applications

Cor. For $n > 2\ell$, every $n$-vertex graph having no component w. more than $n - \ell$ vertices is $\ell$-reconstr’bl.

Pf. \{connected graphs\} is absorbing family for any $G$. 
Cor. For $n > 2l$, every $n$-vertex graph having no component w. more than $n - l$ vertices is $l$-reconstr’bl.

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\[\square\]
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**Cor.** If $D$ is an $(n - l)$-deck where every card is acyclic and has radius greater than $k$, then all $n$-vertex reconstructions have the same number of $k$-centers.
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**Cor.** For \( n > 2\ell \), every \( n \)-vertex graph having no component with more than \( n - \ell \) vertices is \( \ell \)-reconstr’bl.

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Recognition: components have \( \leq n - \ell \) verts \( \iff \) \( G \) has \( \leq 1 \) connected card. Then apply the Counting Lemma.

Sharpness by \( P_\ell + P_\ell \) vs. \( P_{\ell+1} + P_{\ell-1} \) when \( n = 2\ell \).

**Cor.** If \( \mathcal{D} \) is an \((n - \ell)\)-deck where every card is acyclic and has radius greater than \( k \), then all \( n \)-vertex reconstructions have the same number of \( k \)-centers.

**Pf.** \( n - \ell \geq 2k + 2 \) \( \Rightarrow \) reconstructions have girth \( \geq 2k + 3 \).
Applications

**Cor.** For $n > 2\ell$, every $n$-vertex graph having no component w. more than $n - \ell$ vertices is $\ell$-reconstr’bl.

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**Pf.** $n - \ell \geq 2k + 2 \implies$ reconstructions have girth $\geq 2k + 3$.  
\therefore Family of $k$-vines is absorbing.
Applications

**Cor.** For $n > 2l$, every $n$-vertex graph having no component w. more than $n - l$ vertices is $l$-reconstr’bl.

**Pf.** $\{\text{connected graphs}\}$ is absorbing family for any $G$.

**Recognition:** components have $\leq n - l$ verts $\iff G$ has $\leq 1$ connected card. Then apply the Counting Lemma.

**Sharpness** by $P_l + P_l$ vs. $P_{l+1} + P_{l-1}$ when $n = 2l$.

**Cor.** If $\mathcal{D}$ is an $(n - l)$-deck where every card is acyclic and has radius greater than $k$, then all $n$-vertex reconstructions have the same number of $k$-centers.

**Pf.** $n - l \geq 2k + 2 \Rightarrow$ reconstructions have girth $\geq 2k + 3$.

$\therefore$ Family of $k$-vines is absorbing.

All cards radius $> k \Rightarrow$ each $k$-vine has $< n - l$ vertices.
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**Pf.** \{connected graphs\} is absorbing family for any \( G \). Recognition: components have \( \leq n - \ell \) verts \( \iff \) \( G \) has \( \leq 1 \) connected card. Then apply the Counting Lemma. Sharpness by \( P_\ell + P_\ell \ vs. \ P_{\ell+1} + P_{\ell-1} \) when \( n = 2\ell \).

**Cor.** If \( \mathcal{D} \) is an \((n - \ell)\)-deck where every card is acyclic and has radius greater than \( k \), then all \( n \)-vertex reconstructions have the same number of \( k \)-centers.

**Pf.** \( n - \ell \geq 2k+2 \ \Rightarrow \) reconstructions have girth \( \geq 2k+3 \). \( \therefore \) Family of \( k \)-vines is absorbing. All cards radius \( > k \ \Rightarrow \) each \( k \)-vine has \( < n - \ell \) vertices. \( \therefore \) Counting Lemma yields \( \# \) maximal \( k \)-vines.
Applications

**Cor.** For \( n > 2l \), every \( n \)-vertex graph having no component w. more than \( n - l \) vertices is \( l \)-reconstr’bl.

**Pf.** \{connected graphs\} is absorbing family for any \( G \).

Recognition: components have \( \leq n - l \) verts \( \iff \) \( G \) has \( \leq 1 \) connected card. Then apply the Counting Lemma.

Sharpness by \( P_l + P_l \) vs. \( P_{l+1} + P_{l-1} \) when \( n = 2l \).  

**Cor.** If \( D \) is an \((n - l)\)-deck where every card is acyclic and has radius greater than \( k \), then all \( n \)-vertex reconstructions have the same number of \( k \)-centers.

**Pf.** \( n - l \geq 2k + 2 \) \( \Rightarrow \) reconstructions have girth \( \geq 2k + 3 \).

\( \therefore \) Family of \( k \)-vines is absorbing.

All cards radius \( > k \) \( \Rightarrow \) each \( k \)-vine has \( < n - l \) vertices.

\( \therefore \) Counting Lemma yields \#maximal \( k \)-vines.

These correspond bijectively to \( k \)-centers.
The Marking Argument

**Def.** $\hat{k} = \min$ radius of cards. short card $= \text{radius } \hat{k}$.

d $= \#\text{disj. length-}\hat{k} \text{ paths from center}$. Let $k = \hat{k} - 1$. 
The Marking Argument

**Def.** $\hat{k} = \text{min radius of cards.}$ short card = radius $\hat{k}$. $d = \#\text{disj. length-}\hat{k}\text{ paths from center.}$ Let $k = \hat{k} - 1$.

**Def.** Marking argument: Given short card $C$ with center $z$ from forest $F$, each $k$-center $x$ other than $z$ marks a vertex $x'$ at distance $k$ from $x$ away from $z$.

\[
\hat{k} = 3 \quad k = 2 \quad d = 3 = \#\text{paths}
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The Marking Argument

**Def.** \( \hat{k} = \min \) radius of cards. \( \text{short card} = \text{radius } \hat{k} \).
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\[ \hat{k} = 3 \quad k = 2 \quad d = 3 = \# \text{paths} \]

**Lem.** If \( C \) is a short card of \( F \), then \( \# k \)-centers in \( F \) is at most \( 1 + d + l \). Equality only if every vertex outside \( C \) is marked and \( F \) is a tree.
The Marking Argument

**Def.** $\hat{k} = \min$ radius of cards. short card = radius $\hat{k}$. 
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$\hat{k} = 3 \quad k = 2 \quad d = 3 = \# \text{paths}$

**Lem.** If $C$ is a short card of $F$, then $\# k$-centers in $F$ is at most $1 + d + l$. Equality only if every vertex outside $C$ is marked and $F$ is a tree.

**Pf.** $k$-centers not adj. to $z$ must mark verts. outside $C$. ■
Exclusions

**Def.** ambiguous deck $D = \text{the } (n - \ell)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}
Exclusions

**Def.** ambiguous deck $\mathcal{D}$ = the $(n - \ell)$-deck of both an acyclic $F$ and non-acyclic $H$ with $n$ vertices.

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \Rightarrow \hat{k} > 1.$
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**Pf.** If $\hat{k} = 1$, then short card is a star with $n - \ell$ vertices.
Exclusions

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Lem. ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \Rightarrow \hat{k} > 1.$

Pf. If $\hat{k} = 1,$ then short card is a star with $n - \ell$ vertices. All cards are acyclic $\Rightarrow H$ has girth $\geq n - \ell + 1,$
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Lem. ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \Rightarrow \hat{k} > 1.$

Pf. If $\hat{k} = 1,$ then short card is a star with $n - \ell$ vertices.
All cards are acyclic $\Rightarrow H$ has girth $\geq n - \ell + 1,$ $2n - 2\ell + 1 \geq n + 2 \Rightarrow$ cycle + star in same compon. of $H.$
**Def.** ambiguous deck $\mathcal{D} = \text{the } (n - l)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2l + 2 \Rightarrow \hat{k} > 1.$

**Pf.** If $\hat{k} = 1,$ then short card is a star with $n - l$ vertices. All cards are acyclic $\Rightarrow H$ has girth $\geq n - l + 1,$ $2n - 2l + 1 \geq n + 2 \Rightarrow \text{cycle+star in same compon. of } H.$ $2$-deck $\Rightarrow \leq n - 1 \text{ edges } \Rightarrow H \text{ is disconnected.}
Exclusions

**Def.** ambiguous deck $\mathcal{D} = \text{the } (n - l)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2l + 2 \implies \hat{k} > 1$.

**Pf.** If $\hat{k} = 1$, then short card is a star with $n - l$ vertices. All cards are acyclic $\implies H$ has girth $\geq n - l + 1$, \[ 2n - 2l + 1 \geq n + 2 \implies \text{cycle+star in same compon. of } H. \]

2-deck $\implies \leq n - 1 \text{ edges } \implies H \text{ is disconnected.}$

Girth $\geq 4 \implies \text{star & cycle share } \leq 3 \text{ verts.}$
Exclusions

**Def.** ambiguous deck $\mathcal{D}$ = the $(n - \ell)$-deck of both an acyclic $F$ and non-acyclic $H$ with $n$ vertices.

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \Rightarrow \hat{k} > 1$.

**Pf.** If $\hat{k} = 1$, then short card is a star with $n - \ell$ vertices.
All cards are acyclic $\Rightarrow H$ has girth $\geq n - \ell + 1$,
$2n - 2\ell + 1 \geq n + 2 \Rightarrow$ cycle + star in same compon. of $H$.
2-deck $\Rightarrow \leq n - 1$ edges $\Rightarrow H$ is disconnected.
Girth $\geq 4 \Rightarrow$ star & cycle share $\leq 3$ verts.
Now $(n - \ell + 1) + (n - \ell) - 3 \leq n - 1 \Rightarrow n \leq 2\ell + 1$.  

**Exclusions**

**Def.** ambiguous deck $\mathcal{D} = \text{the (}n-\ell\text{)-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \Rightarrow \hat{k} > 1$.

**Pf.** If $\hat{k} = 1$, then short card is a star with $n-\ell$ vertices. All cards are acyclic $\Rightarrow H$ has girth $\geq n-\ell+1$, $2n-2\ell+1 \geq n+2 \Rightarrow$ cycle+star in same compon. of $H$. 2-deck $\Rightarrow \leq n-1$ edges $\Rightarrow H$ is disconnected. Girth $\geq 4 \Rightarrow$ star & cycle share $\leq 3$ verts. Now $(n-\ell+1)+(n-\ell)-3 \leq n-1 \Rightarrow n \leq 2\ell+1$.

**Lem.** ambig. $\mathcal{D}$ & $n \geq 2\ell+2 \Rightarrow$ no card of diam $2k+1$. 


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**Def.** ambiguous deck $\mathcal{D} = \text{the } (n - l)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}

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**Pf.** $\mathcal{D}$ has connected cards (paths) $\Rightarrow 2\hat{k} \leq n - l,$
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Def. ambiguous deck $D$ = the $(n - \ell)$-deck of both an acyclic $F$ and non-acyclic $H$ with $n$ vertices.

Lem. ambiguous $D$ and $n \geq 2\ell + 2$ $\Rightarrow$ $\hat{k} > 1$.

Pf. If $\hat{k} = 1$, then short card is a star with $n - \ell$ vertices. All cards are acyclic $\Rightarrow$ $H$ has girth $\geq n - \ell + 1$, $2n - 2\ell + 1 \geq n + 2$ $\Rightarrow$ cycle+star in same compon. of $H$. 2-deck $\Rightarrow$ $\leq n - 1$ edges $\Rightarrow$ $H$ is disconnected. Girth $\geq 4$ $\Rightarrow$ star & cycle share $\leq 3$ verts. Now $(n - \ell + 1) + (n - \ell) - 3 \leq n - 1$ $\Rightarrow$ $n \leq 2\ell + 1$. 

Lem. ambig. $D$ & $n \geq 2\ell + 2$ $\Rightarrow$ no card of diam $2k+1$.

Pf. $D$ has connected cards (paths) $\Rightarrow$ $2\hat{k} \leq n - \ell$, $\therefore$ Girth($H$) $\geq 2k + 3$, $\therefore$ $F$ and $H$ have same # $k$-centers. Diameter $2k + 1$ $\Rightarrow$ $d_C = 1$, $\therefore$ $F$ has $\leq 2 + \ell$ $k$-ctrs.
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All cards are acyclic \( \Rightarrow H \) has girth \( \geq n - \ell + 1 \),
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2-deck \( \Rightarrow \leq n - 1 \) edges \( \Rightarrow H \) is disconnected.
Girth \( \geq 4 \) \( \Rightarrow \) star & cycle share \( \leq 3 \) verts.
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Diameter \( 2k + 1 \) \( \Rightarrow d_C = 1, \therefore F \text{ has } \leq 2 + \ell \text{ } k\text{-ctrs.} \]
All verts. on cycle are \( k\)-ctrs, \( \therefore H \) has \( \geq n - \ell + 1 \) \( k\)-ctrs.
Exclusions

**Def.** ambiguous deck $\mathcal{D}$ = the $(n - \ell)$-deck of both an acyclic $F$ and non-acyclic $H$ with $n$ vertices.

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Def. \textit{k-evine} = a tree with diameter $2k + 1$.
\textit{k-central edge} = the central edge of a \textit{k-evine}.
**k-Central Edges and End of Proof**

**Def.** *k*-evine = a tree with diameter $2k + 1$.

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**Lem.** All cards acyclic w. radius $> k$, none w. diam $2k + 1$, and $2k + 2 \leq n - \ell \Rightarrow \mathcal{D}$ fixes $\#k$-central edges.
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$\therefore \{k\text{-evines}\}$ is absorbing; Counting Lemma applies. \(\blacksquare\)
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**Thm.** For $n \geq 2\ell + 2$, acyclicity is $\ell$-recognizable.
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**Def.** $k$-evine = a tree with diameter $2k+1$. $k$-central edge = the central edge of a $k$-evine.

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**Pf.** No card w. diam $2k+1 \Rightarrow$ no $k$-evine has $> n - l$ vrts. $2k+2 \leq n - l \Rightarrow$ any reconstruction has girth $\geq 2k+3$. $\therefore \{k$-evines\} is absorbing; Counting Lemma applies.

**Thm.** For $n \geq 2l + 2$, acyclicity is $l$-recognizable.

**Pf.** No cards of diam $2k+1 \Rightarrow$ #k-central edges fixed. Card C: edge is $k$-central $\iff$ end away from $z$ is $k$-cntr.
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Card \( C \): edge is \( k \)-central \( \iff \) end away from \( z \) is \( k \)-cntr.

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\( C \) \( \Rightarrow \) \( H \) has \( d \) \( k \)-central edges w. common endpoint.
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**Pf.** No card w. diam $2k + 1 \Rightarrow$ no *k-evine* has $> n - l$ vrts. $2k + 2 \leq n - l \Rightarrow$ any reconstruction has girth $\geq 2k + 3$.

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**Thm.** For $n \geq 2l + 2$, acyclicity is $l$-recognizable.

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$2k+2 \leq n-\ell \Rightarrow$ any reconstruction has girth $\geq 2k+3$.
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**Pf.** No card w. diam $2k + 1 \Rightarrow$ no *k*-evine has $> n - l$ vrts.

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∴ $n - l + 1 + d - 2 \leq d + l$.

Hence $n \leq 2l + 1$.  

\[\blacksquare\]
Ideas for $n = 2\ell + 1$

Lemmas again show (1) $\hat{k} > 1$, (2) $2\hat{k} \leq \ell = n - \ell - 1$, (3) Same $#k$-centers and same $#k$-central edges.
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Def. spider = a tree with at most one branch vertex.
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Using diameter $2k + 2$ for short cards, in both $F$ and $H$ no $\hat{k}$-vine has more than $n - \ell$ vertices, so Counting Lemma applies to get $\#\hat{k}$-centers (determined by $D$).
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By marking argument, $F$ has at most $\ell + 1 \hat{k}$-centers.
Ideas for $n = 2\ell + 1$

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By marking argument, $F$ has at most $\ell + 1 \hat{k}$-centers. Use of the cycle in $H$ yields at least $\ell + 2 \hat{k}$-centers.
Further Questions

**Conj.** Trees with $n \geq 2\ell + 1$ are $\ell$-reconstructible. (We have shown that the family is $\ell$-recognizable.)
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The 2-deck gives $|E(G)|$, so we can also ask:

**Prob.** Given $c$ and $\ell$, find $N$ such that when $n \geq N$, all acyclic $(n - \ell)$-decks of $n$-vertex graphs with $n + c$ edges determine whether the reconstructions are connected.
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$c = -1 \Rightarrow N = 2\ell + 1$, done here.

$c = 0 \Rightarrow N = 2\ell - 1$ for $\ell \geq 42$ (Zirlin [2021+]).

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Further Questions

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The thresholds in the more general setting where the cards need not all be acyclic are also unknown.