Acyclic graphs with at least $2\ell + 1$ vertices are ℓ-recognizable

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Abstract

The $(n-\ell)$-*deck* of an n-vertex graph is the multiset of subgraphs obtained from it by deleting ℓ vertices. A family of n-vertex graphs is ℓ-*recognizable* if every graph having the same $(n-\ell)$-deck as a graph in the family is also in the family. We prove that the family of n-vertex graphs with no cycles is ℓ-recognizable when $n \ge 2\ell + 1$ (except for $(n, \ell) = (5, 2)$). It is known that this fails when $n = 2\ell$. As a consequence of the result, the family of trees is ℓ-recognizable.

1 Introduction

The j-*deck* of a graph is the multiset of its j-vertex induced subgraphs. We write this as the $(n-\ell)$-deck when the graph has n vertices and the focus is on deleting ℓ vertices. An n-vertex graph is ℓ-*reconstructible* if it is determined by its $(n-\ell)$-deck. Since every member of the $(j-1)$-deck arises $n-j+1$ times by deleting a vertex from a member of the j-deck, the j-deck of a graph determines its $(j-1)$-deck. Therefore, a natural reconstruction problem is to find for each graph the maximum ℓ such that it is ℓ-reconstructible. For this problem, Manvel [10, 11] extended the classical Reconstruction Conjecture of Kelly [5] and Ulam [16].

Conjecture 1.1 (Manvel [10, 11]). *For $\ell \in \mathbb{N}$, there exists a threshold M_ℓ such that every graph with at least M_ℓ vertices is ℓ-reconstructible.*

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Manvel named this "Kelly's Conjecture" in honor of the final sentence in Kelly [6], which suggested that one can study reconstruction from the $(n-2)$-deck. Manvel noted that Kelly may have expected the statement to be false.

Many reconstruction arguments have two parts. First, one proves that the deck determines that the graph is in a particular class or has a particular property. When the $(n-\ell)$-deck determines this, the property is *ℓ-recognizable*. Separately, using the knowledge that every reconstruction from the deck has that property, one proves that only one graph with that property has that deck. This makes the family *weakly ℓ-reconstructible*, meaning that no two graphs in the family have the same deck. Bondy and Hemminger [1] articulated the distinction between these two steps for the case $\ell = 1$.

Here, toward ℓ-reconstructibility of trees, we consider ℓ-recognizability of acyclic graphs. We prove the following theorem.

Theorem 1.2. *For $n \geq 2\ell + 1$, except when $(n, \ell) = (5, 2)$, the family of n-vertex acyclic graphs is ℓ-recognizable.*

We forbid $(n, \ell) = (5, 2)$ due to the two graphs in Figure 1, which have the same 3-deck. Indeed, this possibility must be excluded from many of the claims we prove.

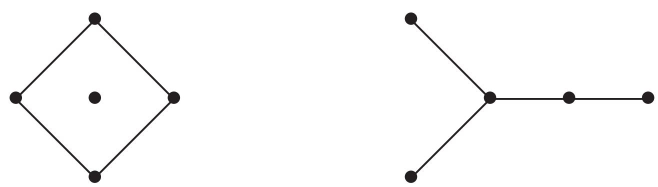

Figure 1: 5-vertex graphs with the same 3-deck

Since the $(n-\ell)$-deck determines the 2-deck when $n-\ell \geq 2$, in this setting we also know the number of edges. This yields the following corollary.

Corollary 1.3. *For $n \geq 2\ell + 1$, except when $(n, \ell) = (5, 2)$, the family of n-vertex trees is ℓ-recognizable.*

Spinoza and West [14] determined, for every graph G with maximum degree at most 2, the maximum ℓ such that G is ℓ-reconstructible. Their full result is complicated to state, but a special case is that for $n \geq 2\ell + 1$ (except $(n, \ell) = (5, 2)$), every n-vertex graph with maximum degree at most 2 is ℓ-reconstructible. They also show that a path with 2ℓ vertices has the same ℓ-deck as the disjoint union of an $(\ell + 1)$-cycle and a path with $\ell - 1$ vertices, so the threshold $n \geq 2\ell + 1$ in both [14] and Theorem 1.2 is sharp.

Nýdl [13] conjectured that trees with at least $2\ell+1$ vertices are weakly ℓ-reconstructible. This conjecture would be sharp, since Nýdl [12] found two trees with 2ℓ vertices having the same ℓ-deck. The two trees are obtained from a path with $2\ell-1$ vertices by adding one leaf, adjacent either to the central vertex of the path or to one of its neighbors. Kostochka and West [9] used the results of [14] to give a short proof of this result of Nýdl. However, one counterexample to Nýdl's conjecture is known: Groenland, Johnston, Scott, and Tan [4] obtained two 13-vertex trees having the same 7-deck. Excluding this example and incorporating the ℓ-recognizability of trees leads to a modification of Nýdl's conjecture.

Conjecture 1.4. *For $n \ge 2\ell+1$ (except $(n,\ell) \in \{(5,2),(13,6)\}$), every n-vertex tree is ℓ-reconstructible. The threshold on n is known to be sharp.*

For $\ell = 2$, the threshold on n must be at least 6 due to the graphs in Figure 1. Giles [2] proved that trees with at least six vertices are 2-reconstructible (using only the connected members of the deck). For general ℓ, Groenland et al. [4] proved that $n \ge 9\ell + 24\sqrt{2\ell} + o(\sqrt{\ell})$ suffices for ℓ-reconstructibility of n-vertex trees. For $\ell = 3$, their result applies when $n \ge 194$. In [7], the present authors proved that $n \ge 25$ suffices when $\ell = 3$.

Besides acyclicity, another fundamental property of trees is connectedness. Spinoza and West [14] proved that connectedness is ℓ-recognizable for n-vertex graphs when $n > 2\ell^{(\ell+1)^2}$. Later, Groenland et al. [4] reduced the general threshold to $n \ge 10\ell$. Manvel [11] proved that connectedness is 2-recognizable for graphs with at least six vertices, and the present authors [8] proved that connectedness is 3-recognizable for graphs with at least seven vertices. Spinoza and West [14] suggested that (except for $(n,\ell) = (5,2)$), connectedness is recognizable for n-vertex graphs when $n \ge 2\ell+1$.

The $(n-\ell)$-deck of a graph is *acyclic* if each card in the deck is acyclic. As a step toward the threshold on n for ℓ-recognizability of connectedness, one can consider the special case of n-vertex graphs whose $(n-\ell)$-deck is acyclic. Our result in this paper settles the question for graphs with $n-1$ edges, where connectedness and acyclicity are equivalent (the number of edges is known from the 2-deck). This suggests other detailed questions.

Problem 1.5. *For $\ell \ge 1$ and $c \ge 0$, determine the smallest thresholds $N_{\ell,c}$ and $N'_{\ell,c}$ such that for all n-vertex graphs with $n+c$ edges whose $(n-\ell)$-deck $\mathcal{D}$ is acyclic,*

(a) if $n \ge N_{\ell,c}$, then $\mathcal{D}$ determines whether the graph is connected, and

(b) if $n \ge N'_{\ell,c}$, then the graph is connected.

The thresholds when the deck is not required to be acyclic are also unknown.

We note that $N'_{\ell,1} = 2\ell$. For the upper bound, consider a disconnected n-vertex graph with an acyclic $(n-\ell)$-deck, where $n \ge 2\ell$. A smallest component must be acyclic, since a cycle would have length at most $n/2$ and be seen in the deck. Hence some other component H must have at least $|V(H)|+2$ edges. However, a p-vertex graph with at least $p+2$ edges has girth at most $\lfloor (p+2)/2 \rfloor$ (see Exercise 5.4.36 of [17], for example), yielding a cycle in a card. For the lower bound, we seek a disconnected graph with $2\ell-1$ vertices whose $(\ell-1)$-deck is acyclic. When ℓ is even, the graph consists of an isolated vertex plus four paths of length $\ell/2$ with common endpoints. When n is odd, the nontrivial component consists of a cycle of length $2\ell-2$ plus two diametric chords creating cycles of length ℓ (this example was contributed by a referee). It is possible that the threshold $N_{\ell,1}$ for determining connectedness from the $(n-\ell)$-deck is smaller.

For $c=0$, we conjecture $N_{\ell,0}=2\ell-1$. The lower bound holds because a $(2\ell-2)$-cycle and the disjoint union of two $(\ell-1)$-cycles have the same $(\ell-2)$-deck. Zirlin [18] proved $N_{\ell,0}\le 2\ell+1$ for $\ell\ge 3$, and she proved $N_{\ell,0}=2\ell-1$ for $\ell\ge 45$.

In Section 2 we develop tools that are useful for reconstructing information from acyclic decks. In Section 3 we prove that $n\ge 2\ell+2$ suffices for ℓ-recognizability of n-vertex acyclic graphs. In Section 4 we obtain the sharp threshold, $2\ell+1$.

2 Vines, Diameter, and Marking

Let $\mathcal{D}$ be the $(n-\ell)$-deck of an n-vertex graph G (we henceforth just call it the "deck"). We will assume $n>2\ell$. The members of $\mathcal{D}$ are the "cards" in the deck.

Definition 2.1. The *eccentricity* of a vertex in a graph G is the maximum of the distances from it to other vertices. A *center* of G is a vertex of minimum eccentricity, and the minimum eccentricity is called the *radius* of G.

In G, the *j-ball* at a vertex v is the subgraph induced by all vertices within distance j of v in G. The *j-eball* at an edge e is the subgraph induced by all vertices within distance j of either endpoint of e. A *j-vine* or *j-evine* is a tree having diameter $2j$ or $2j+1$, respectively. A *j-center* is a vertex that is the center of a j-vine; a *j-central edge* is the central edge of a j-evine (joining the two centers).

The term "j-vine" follows the botanical theme in terminology about trees; a vine grows from its main path. Note that if the j-ball at a vertex v in a graph G is a tree but does not contain a path with $2j+1$ vertices, then v is not a j-center. When v *is* a j-center, the maximal j-vine with center v is just the j-ball at v.

We will be interested in j-vines and j-evines that are induced subgraphs of every reconstruction from the given deck. Our aim is to consider an acyclic and a non-acyclic graph having the same deck, show that they have the same number of j-centers for an appropriate value j, and obtain a contradiction by showing that they cannot have the same number of j-centers. The key property that will permit counting the j-centers in a reconstruction is in the next lemma; it implies (under the girth condition) that maximal j-vines correspond bijectively to centers of j-vines (that is, j-centers), and similarly for j-evines.

Lemma 2.2. *In a graph G with girth at least $2j+2$, every j-vine lies in a unique maximal j-vine. If G has girth at least $2j+3$, then every j-evine lies in a unique maximal j-evine.*

Proof. When H is a j-vine or a j-evine, respectively, let H' be the j-ball or j-eball in G at the center(s) of H. All vertices in any j-vine or j-evine containing H, respectively, lie in H'. Thus H' is the desired unique maximal object unless it contains a cycle.

Let Q be a shortest cycle in H'. Because the vertex or vertices on Q that are farthest from the center of H' have distance at most j from the center, Q has at most $2j+1$ vertices if H' has a unique center and at most $2j+2$ vertices if H' has a central edge, contradicting the hypothesis on the girth of G. □

Example 2.3. To see that the girth condition in Lemma 2.2 is sharp, let G be a graph consisting of a $(2j+1)$-cycle Q plus two paths of length j grown from a single vertex v on Q. Deleting from G the two vertices of Q that are farthest from v yields a j-vine H. Replacing either one of those vertices yields a maximal j-vine in G containing H, so the maximal j-vine containing H is not unique. An analogous example for j-evines consists of a $(2j+2)$-cycle plus paths of length j grown from two consecutive vertices.

Definition 2.4. Given a family $\mathcal{F}$ of graphs, an *$\mathcal{F}$-subgraph* of a graph G is an induced subgraph of G belonging to $\mathcal{F}$. Let $s(F,G)$ denote the number of occurrences of F as an induced subgraph of G. Let $m(F,G)$ be the number of occurrences of F as a maximal $\mathcal{F}$-subgraph in G (with respect to induced subgraphs).

The special case $\ell = 1$ of the next lemma is due to Greenwell and Hemminger [3]. Similar statements for general ℓ appear for example in [4]. We include a proof for completeness; it is slightly simpler than proofs in the literature involving inclusion chains of subgraphs.

Lemma 2.5. *Fix an n-vertex graph G, and let $\mathcal{F}$ be a family of graphs such that every $\mathcal{F}$-subgraph of G lies in a unique maximal $\mathcal{F}$-subgraph of G. If for every $F \in \mathcal{F}$ with at least $n-\ell$ vertices the value of $m(F,G)$ is known, then for all $F \in \mathcal{F}$ the $(n-\ell)$-deck of G determines $m(F,G)$.*

Proof. Let $t = |V(G)| - |V(F)|$; we use induction on t. When $t \le \ell$, the value $m(F,G)$ is given. When $t > \ell$, group the induced subgraphs of G isomorphic to F according to the unique maximal $\mathcal{F}$-subgraph of G containing them (as an induced subgraph). Counting all copies of F then yields

$$s(F,G) = \sum_{H \in \mathcal{F}} s(F,H)m(H,G).$$

Since $|V(F)| < n-\ell$, we know $s(F,G)$ from the deck, and we know $s(F,H)$ when F and H are known. By the induction hypothesis, we know all values of the form $m(H,G)$ when F is an induced subgraph of H except $m(F,G)$. Therefore, we can solve for $m(F,G)$. □

Before continuing with preparation for ℓ-recognizability of acyclicity, we note one application of Lemma 2.5 that was stated incorrectly in the paper by Kostochka and West [9]; it also illustrates the technique we use with j-vines. The special case for $\ell = 1$ was observed by Kelly [6] using different methods. Let P_n and C_n denote a path and a cycle with n vertices, and let $G + H$ denote the disjoint union of graphs G and H.

Corollary 2.6. *If $n > 2\ell$, then n-vertex graphs having no component with more than $n - \ell$ vertices are ℓ-reconstructible, and this threshold on n is sharp. All n-vertex graphs having no component with at least $n - \ell$ vertices are ℓ-reconstructible, with no restriction on n.*

Proof. Let $\mathcal{F}$ be the family of connected graphs; $\mathcal{F}$ satisfies the property stated in the first sentence of Lemma 2.5 for any G.

Now consider $m(F, G)$ for $F \in \mathcal{F}$. An n-vertex graph with $n > 2\ell$ has at most one component with at least $n - \ell$ vertices, and it has no component with more vertices if and only if it has at most one connected $(n - \ell)$-card. Hence the hypothesized condition here is ℓ-recognizable, and if there is a component with $n - \ell$ vertices it is seen as a card. Hence $m(F, G)$ is known for $F \in \mathcal{F}$ with at least $n - \ell$ vertices, and by using Lemma 2.5 we obtain all the components of G.

The result is sharp, since $P_\ell + P_\ell$ and $P_{\ell+1} + P_{\ell-1}$ have the same ℓ-deck. This follows from the result of Spinoza and West [14] that any two graphs with the same number of vertices and edges whose components are all cycles with at least $j + 1$ vertices or paths with at least $j - 1$ vertices have the same j-deck. □

When we speak of j-vines and j-evines in an n-vertex graph G, we always consider only induced subgraphs. In addition, a particular value of j in terms of G will be of interest.

Definition 2.7. For a given graph G, let k denote the largest integer j such that G contains a j-evine and every j-evine in G has fewer than $n - \ell$ vertices. **This fixes k in terms of G for the remainder of the paper.**

Lemma 2.8. *The value k is determined by the deck of G.*

Proof. Since an edge forms a 0-evine and every j-evine with $j > 0$ contains a smaller $(j-1)$-evine, the value of k is well-defined. It remains to compute k from the deck.

All subgraphs with at most $n - \ell$ vertices are visible from the deck. The deck yields a largest value j' such that some induced subgraph of a card is a j'-evine and all j'-evines that appear in cards have fewer than $n - \ell$ vertices. Since k has these properties, we have $j' \geq k$.

We claim $j' = k$. Let T be a largest j'-evine in G among those having fewer than $n - \ell$ vertices. Note that T contains a path with $2j' + 2$ vertices, and hence $2j' + 2 < n - \ell$.

By the choice of j', no j'-evine in G has exactly $n - \ell$ vertices. If $j' > k$, then some j'-evine among the j'-evines that exist in G must have more than $n - \ell$ vertices. Let T' be a smallest j'-evine with more than $n - \ell$ vertices. Since G has no j'-evine with one less vertex, T' must be a path with $2j' + 2$ vertices. This requires $2j' + 2 > n - \ell$. However, we already observed $2j' + 2 < n - \ell$. Hence $j' = k$. □

Recall that we require $n \geq 2\ell + 1$ and consider n-vertex reconstructions from an acyclic $(n-\ell)$-deck $\mathcal{D}$.

Lemma 2.9. *Every reconstruction from $\mathcal{D}$ has girth at least $2k+4$.*

Proof. The claim holds for a reconstruction G having no cycle, so suppose that G has a cycle. Since $\mathcal{D}$ is acyclic, the girth of G is $n-\ell+c$ for some positive integer c. Let e be an edge in a shortest cycle Q in G. Define t by $2t+2 = n-\ell+c-\varepsilon$, where $\varepsilon \in \{1,2\}$. Deleting ε vertices from Q that are farthest from e yields a t-evine T in G with central edge e.

If $c \geq \varepsilon$, then $|V(T)| \geq n-\ell$. Since no k-evine has at least $n-\ell$ vertices, $2k+2 < n-\ell$, so $k < t$ and hence $2k+4 \leq n-\ell+c-\varepsilon$. This yields $n-\ell+c \geq 2k+5$.

The remaining case is $c=1$ and $\varepsilon=2$, giving Q even length $n-\ell+1$. Deleting adjacent vertices from Q yields a t-evine with $n-\ell-1$ vertices. If G has an edge xy with $x \in V(Q)$ and $y \notin V(Q)$, then deleting a neighbor of x on Q and adding y and the edge xy yields an induced path with $n-\ell+1$ vertices (since Q is a shortest cycle). This is a $(t+1)$-evine with more than $n-\ell$ vertices, so $k \leq t$. Now $2k+2 \leq n-\ell-1$, so G has girth at least $2k+4$.

If G has no such edge xy, then Q is a component of G with $n-\ell+1$ vertices, and outside Q there are fewer vertices. Now G has no $(t+1)$-evine. Deleting two adjacent vertices from Q yields a t-evine with fewer than $n-\ell$ vertices, and no t-evine is larger than this. Hence $k=t$ and G has girth $2k+4$. □

Lemma 2.10. *Every reconstruction from $\mathcal{D}$ contains k-vines. If some reconstruction contains a cycle, then every k-vine in any reconstruction has fewer than $n-\ell$ vertices.*

Proof. A k-evine contains two k-vines whose centers lie in the central edge of the k-evine.

Let G be a reconstruction from $\mathcal{D}$; k is determined. Suppose that some k-vine in G has at least $n-\ell$ vertices. We recognize this by the existence of a card that is a k-vine unless a smallest k-vine with more than $n-\ell$ vertices is a path with $2k+1$ vertices. Since a k-evine (which exists) has a path with $2k+2$ vertices, we would then have $2k+2 > n-\ell$, contradicting the definition of k.

Let z be the center of a largest k-vine R. The distance between any two vertices in R is at most $2k$. By Lemma 2.9, G has girth at least $2k+4$, so no vertex outside R has two neighbors in R. Thus any single outside vertex having a neighbor in R can be added to the induced subgraph without introducing a cycle. Since R is a largest k-vine, we conclude that all vertices of G within distance k of z lie in R. If the component of G containing R has any vertex outside C, then it has one at distance $k+1$ from z. Adding this vertex to R creates a k-evine with more than $n-\ell$ vertices, contradicting the definition of k.

Hence R must be a component of G. Now, since $n-\ell > \ell$ and the deck is acyclic, not enough vertices remain for a cycle in G. Hence no k-vine in G has at least $n-\ell$ vertices, so no card is a k-vine, and no k-vine in G has at least $n-\ell$ vertices. □

Corollary 2.11. *Suppose that some reconstruction from $\mathcal{D}$ contains a cycle. For $j \le k$, the deck $\mathcal{D}$ determines the maximal j-evines and maximal j-vines, with multiplicity. Also, all reconstructions from $\mathcal{D}$ have the same numbers of j-centers and j-central edges.*

Proof. Fix j, and let $\mathcal{F}$ be the family of j-vines or the family of j-evines. Using the definition of k, Lemma 2.8, and Lemma 2.10 for various cases, we obtain $m(T, G) = 0$ whenever G is a reconstruction from $\mathcal{D}$ and T is a member of $\mathcal{F}$ having at least $n - \ell$ vertices. Hence Lemma 2.5 applies to compute $m(T, G)$ for all $T \in \mathcal{F}$.

Since G has girth at least $2k+4$ (Lemma 2.9), the k-balls and k-eballs around vertices and edges are acyclic. Hence the maximal k-vines and k-evines are simply the (e)balls around their centers (Lemma 2.2). Thus there is a one-to-one correspondence between the maximal j-vines and the j-centers, and similarly for the maximal j-evines and j-central edges. Thus we obtain the total number of j-centers and the total number of j-central edges. □

Setting $j = 1$ in Corollary 2.11 almost provides the degree list. Groenland et al. [4] proved the nearly sharp result that the degree list is ℓ-reconstructible for all n-vertex graphs whenever $n - \ell > \sqrt{2n \log(2n)}$. Taylor [15] had shown that asymptotically $n > \ell e$ is enough. For the context of acyclic decks we obtain a simpler intermediate threshold.

Corollary 2.12. *For $n \ge 2\ell + 1$ with $(n, \ell) \ne (5, 2)$, the degree list of any n-vertex graph with an acyclic $(n - \ell)$-deck is determined by its deck.*

Proof. All 1-vines are stars. Each vertex with degree at least 2 is the center of exactly one maximal 1-vine. Suppose first that no star has at least $n - \ell$ vertices. For $t \ge 3$, by the counting argument (Lemma 2.5) the deck determines the number of maximal 1-vines having t vertices. This is the number of vertices with degree $t - 1$ in any reconstruction.

Now suppose that some star has at least $n - \ell$ vertices. Since the deck is acyclic and $n - \ell \ge 4$, two stars share at most an edge joining the centers or a common leaf. Having two stars with at least $n - \ell$ vertices thus requires $n \le 2\ell + 1$, with equality only when two stars with $n - \ell$ vertices share a leaf.

When only one vertex has degree at least $n - \ell - 1$, its degree is d if and only if exactly $\binom{d}{n-\ell-1}$ cards are stars. If there are exactly two such cards, then we have the special case described above with $n = 2\ell + 1$. This determines $m(T, G)$ for any reconstruction G and every star T with at least $n - \ell$ vertices. Since every star with at least three vertices lies in a unique maximal star in G, again Lemma 2.5 applies. Again we obtain the number of vertices with degree $t - 1$ whenever $t \ge 3$.

Now consider vertices with degree at most 1. Since we know the number of edges from the 2-deck, the number of vertices with degree 1 is given by the degree-sum being twice the number of edges, and then the remaining vertices have degree 0. □

Lemma 2.13. *Every connected card in $\mathcal{D}$ has diameter at least $2k + 2$, and some connected card has diameter at most $2k + 3$.*

Proof. Since the deck is acyclic, every connected card is a tree. A connected card with diameter at most $2k+1$ would be a j-vine or j-evine with $j \le k$ having $n-\ell$ vertices, contradicting Lemma 2.10 or the definition of k.

For the second claim, let C be a connected card. If C has diameter at least $2k+3$, then C contains a path with $2k+4$ vertices. Hence $n-\ell \ge 2k+4$ and any reconstruction contains a $(k+1)$-evine. By the definition of k, some $(k+1)$-evine has at least $n-\ell$ vertices. Since $n-\ell \ge 2k+4$, we can iteratively delete leaves of such a $(k+1)$-evine outside a fixed longest path in it to trim it to $n-\ell$ vertices. We thus obtain a card that is a $(k+1)$-evine, which has diameter $2k+3$. Hence some card has diameter at most $2k+3$. □

For the case $n \ge 2\ell+2$, we will show that no deck can have both an acyclic reconstruction F and a non-acyclic reconstruction H by showing that H would have more k-centers or $(k+1)$-centers than F. We next introduce our tool for bounding the number of j-centers in a forest F; we call it the **marking process**.

Definition 2.14. Let z be a central vertex of a connected $(n-\ell)$-card C with radius $j+1$ in a forest F. Every j-center x other than z in the component of F containing C **marks** one vertex x' at distance j from x along a path that extends the z,x-path in F. Also, let d_C denote the maximum number of edge-disjoint paths of length $j+1$ in C with endpoint z. Note that $d_C = 1$ when C has diameter $2j+1$ and $d_C \ge 2$ when C has diameter $2j+2$.

Figure 2 illustrates the marking process for a card C (in bold) within a tree F. Here C has radius 3 with center z, also $d_C = 3$, and the vertex x_i marks the vertex x_i'.

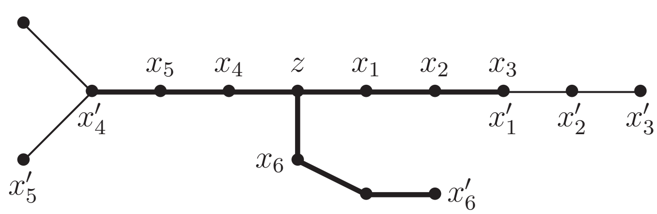

Figure 2: The Marking Process

Lemma 2.15. *If $j \ge 1$ and C is a connected card with radius $j+1$ in the $(n-\ell)$-deck $\mathcal{D}$ of an n-vertex forest F, then the number of j-centers in F is at most $1 + d_C + \ell$. If equality holds, then in the marking process each vertex of F outside C is marked and F is a tree.*

Proof. Let F' be the component of F containing C, and let ℓ' be the number of vertices of F' outside C. Since F' has no cycles, every vertex of F' is marked by at most one j-center. Each j-center in F' that is not adjacent to z marks a vertex that has distance at least $j+2$ from z and hence lies outside C. Leaves of C at distance $j+1$ from z can only be marked by j-centers in F' adjacent to z, and only if they reach z via edge-disjoint paths. Including also the j-center z, we conclude that F has at most $\ell' + d_C + 1$ j-centers in F'.

There are $\ell - \ell'$ vertices of F outside F', and any that have degree at most 1 cannot be j-centers (since $j \geq 1$). Hence F has at most $1 + d_C + \ell$ j-centers, with equality only if F is a tree and all vertices outside C are marked. □

3 The Case $n \geq 2\ell + 2$

Given an acyclic $(n-\ell)$-deck $\mathcal{D}$ for $n \geq 2\ell+1$, with k defined as in Section 2, we have proved that all n-vertex reconstructions from $\mathcal{D}$ have the same number of k-centers and have the same number of k-central edges. If $\mathcal{D}$ has no card with diameter $2k+2$, then all $(k+1)$-vines in any reconstruction have fewer than $n-\ell$ vertices, and hence by Lemmas 2.5 and 2.11) all reconstructions have the same number of $(k+1)$-centers. Our task is now to show that such equalities cannot hold for an acyclic reconstruction and a non-acyclic reconstruction when $n \geq 2\ell + 2$. This will prevent having reconstructions of both types.

Definition 3.1. We say that a deck $\mathcal{D}$ is *ambiguous* if it is the $(n - \ell)$-deck of both an n-vertex acyclic graph F and an n-vertex non-acyclic graph H.

Remark 3.2. An ambiguous deck is acyclic, since F contains no cycle. Hence when $\mathcal{D}$ is ambiguous the graph H has girth at least $n - \ell + 1$, and thus $\mathcal{D}$ has connected cards (in particular, paths).

Lemma 3.3. *Let $\mathcal{D}$ be ambiguous, with $n \geq 2\ell + 1$ and $(n, \ell) \neq (5, 2)$. If C is a connected card in $\mathcal{D}$ and Q is a shortest cycle in H, then C and Q share at least four vertices and lie in a component with at most $n - 2$ vertices. (In particular, H is disconnected.)*

Proof. Since $\mathcal{D}$ is acyclic, Q has at least $n - \ell + 1$ vertices. Since C has $n - \ell$ vertices and $2n-2\ell+1 > n$, subgraphs C and Q of H intersect. Since F has at most $n-1$ edges and the component H' of H containing C and Q has as many edges as vertices, H' cannot be all of H. If H' has $n - 1$ vertices, then F has $n - 1$ edges and is a tree, with no isolated vertices. Since F and H have the same degree list (Corollary 2.12), H' therefore has at most $n - 2$ vertices. With $t = |V(C) \cap V(Q)|$, we have $n - \ell + (n - \ell + 1) - t \leq n - 2$, so $t \geq 4$. □

Henceforth let $\mathcal{D}$ be the ambiguous $(n - \ell)$-deck of reconstructions F and H as in Definition 3.1, with k as in Definition 2.7. When we want to use the marking process to compare the numbers of k-centers, we need to exclude the possibility $k = 0$. This is easy when $n \geq 2\ell + 2$; we leave the boundary case $n = 2\ell + 1$ to the next section.

Lemma 3.4. *If $\mathcal{D}$ is ambiguous and $n \geq 2\ell + 2$ with $\ell \geq 2$, then $k \geq 1$.*

Proof. All 0-evines are single edges. If $k = 0$, then in any reconstruction either there is no 1-evine or some 1-evine has at least $n-\ell$ vertices. If there is no 1-evine, then every connected card is a star. However, since H has girth at least $n-\ell+1$, it has a path with $n-\ell$ vertices as a card. Since $n - \ell \geq 4$, this card is not a star.

Hence some card C is a 1-evine, with exactly two non-leaf vertices. By Lemma 3.3, C and a shortest cycle Q share at least four vertices in H. Since Q has at least five vertices and is a shortest cycle in H, in H there is no chord of Q and no vertex outside Q with two neighbors in Q. Hence C and Q share exactly four vertices, consecutive along Q.

Now the component of H containing C and Q has at least $2n - 2\ell + 1 - 4$ vertices. By Lemma 3.3, we have $2n - 2\ell - 3 \leq n - 2$, which simplifies to $n \leq 2\ell + 1$. □

Theorem 3.5. *For $n \geq 2\ell + 2$, the family of n-vertex acyclic graphs is ℓ-recognizable.*

Proof. Suppose that some ambiguous deck $\mathcal{D}$ has reconstructions F and H as we have been discussing. By Lemma 2.13, no card has diameter less than $2k + 2$. By Corollary 2.11, F and H have the same number of k-centers and the same number s of k-central edges.

Suppose that $\mathcal{D}$ has a card C with diameter $2k + 2$. Note that C has d_C k-central edges incident to its unique center z; these edges are also k-central in F. An edge of F in the component containing z is a k-central edge if and only if its endpoint farther from z is a k-center. In other components, the number of k-central edges is less than the number of k-centers. Hence Lemma 2.15 implies $s \leq d_C + \ell$.

Since C is a card, every reconstruction has d_C k-central edges with a common endpoint. In H, a cycle can contain only two of these. Also, since H has girth at least $2k + 4$, every edge on a cycle in H is a k-central edge. Since F and H have the same number of k-central edges, $s \geq n - \ell + 1 + d_C - 2$. The upper and lower bounds on s now require $n \leq 2\ell + 1$.

If $\mathcal{D}$ has no card with diameter $2k + 2$, then every $(k + 1)$-vine has fewer than $n - \ell$ vertices. As noted earlier, F and H thus have the same number s' of $(k + 1)$-centers, since reconstructions have girth at least $2k + 4$ (Lemma 2.9). Also, Lemma 2.13 guarantees in $\mathcal{D}$ a card C' with diameter $2k + 3$, which has radius $k + 2$. Let z be a center in C'. By the marking process with $j = k + 1$, we have $s' \leq 2 + \ell$, since $d_{C'} = 1$. On the other hand, every vertex of a cycle in H is a $(k + 1)$-center, since we only need it to be the center of a path with $2k + 3$ vertices and H has girth at least $2k + 4$. Hence $n - \ell + 1 \leq s' \leq 2 + \ell$, which again implies $n \leq 2\ell + 1$. □

4 The Extreme Case $n = 2\ell + 1$

The arguments of the previous section leave open the possibility of an ambiguous deck when $n = 2\ell + 1$, and the graphs in Figure 1 yield an ambiguous deck when $(n, \ell) = (5, 2)$. In this section we will prohibit ambiguous decks when $n = 2\ell + 1$ and $\ell \geq 3$, yielding the sharp threshold on n for ℓ-recognizability of acyclicity.

To apply the marking process with $j = k$, again we must exclude the possibility $k = 0$. This will require counting special paths in special trees, a result of independent interest.

Definition 4.1. A *branch vertex* in a tree is a vertex with degree at least 3. A *spider* is a tree with at most one branch vertex. A *near-spider* is a tree with two branch vertices, which are adjacent (contracting the edge joining them yields a spider). A *leg* of a non-path tree is a path in the tree whose endpoints are a leaf and the branch vertex closest to it.

A spider whose branch vertex has degree d is the union of d legs with a common endpoint. When the legs have lengths $m_1, \ldots, m_d$, we denote the spider by $S_{m_1,\ldots,m_d}$; it has $1+\sum_{i=1}^d m_i$ vertices. (An n-vertex path can be described as $S_{m,n-1-m}$ for any m with $1 \le m \le n-2$.)

A *full path* in a graph with n vertices is a path that is a card in the $(n-\ell)$-deck.

Lemma 4.2. *When $n \ge 2\ell+1 \ge 3$, an n-vertex spider or near-spider contains at most $\ell+3$ full paths, except for $S_{1,1,1,1}$ when $\ell = 2$.*

Proof. We use induction on ℓ. When $\ell = 1$, an n-vertex tree has at most two full paths, except that $S_{1,1,1}$ has three full paths when $n = 4$. Still, $3 < 4 = \ell + 3$.

For $\ell \ge 2$, suppose first that some leaf x lies in at most one full path. Let $G' = G - x$. Let $n' = |V(G')| = n-1$ and $\ell' = \ell - 1$. Since $n \ge 2\ell+1$, we have $n' > 2\ell'+1$. In particular, $(n', \ell') \ne (5, 2)$, so G' is not the exception $S_{1,1,1,1}$. Thus G' contains at most $\ell' + 3$ paths with $n' - \ell'$ vertices. Since $n' - \ell' = n - \ell$, now G has at most $\ell + 2$ full paths that avoid x. Adding (at most) one full path containing x yields the desired bound for G.

Hence we may assume that every leaf appears in at least two full paths. We argue directly, without using the induction hypothesis.

First let G be an n-vertex spider with maximum degree d. Let x be the leaf in a shortest leg of G, with length a. Since x lies in two full paths, $d \ge 3$. If $d \ge 4$, then two legs not containing x must each have length at least $n - \ell - 1 - a$, and some fourth leg with leaf y has length at least a. Summing the lengths of these four legs yields $2n - 2\ell - 2 \le n - 1$, or $n \le 2\ell + 1$. By the restriction to $n \ge 2\ell + 1$, equality holds, requiring $G = S_{a,a,\ell-a,\ell-a}$ and $n - \ell = \ell + 1$. If $a < \ell - a$, then exactly four full paths use x or y and a leg of length $\ell - a$, and $\ell - 2a + 1$ full paths use the two long legs. The total is $\ell - 2a + 5$, which is at most $\ell + 3$ since $a \ge 1$. If $a = \ell - a$, then there is also one full path from x to y, but now exceeding $\ell + 3$ requires $a = 1$ and $\ell = 2$, which occurs precisely for the exceptional case $S_{1,1,1,1}$.

If G is a spider with $d = 3$, then $G = S_{a,b,c}$. To have each leaf in two full paths, the lengths of any two legs sum to at least $n - \ell - 1$. The union of two legs together having t vertices contains $t-(n-\ell-1)$ full paths. Hence the number of full paths is $2(a+b+c)+3-3(n-\ell-1)$, which simplifies to $3\ell - n + 4$. Since $n \ge 2\ell + 1$, the value is at most $\ell + 3$.

Now suppose that G is a near-spider; the argument is similar. Deleting the edge joining the branch vertices leaves two spiders, with vertex sets X and Y. We may assume $|X| \le n/2$. Since $n \ge 2\ell + 1$, there is no full path within X. Let x be the leaf in a shortest leg in X, say with length a. Since x lies in at least two full paths, Y must have at least two legs with length at least $n - \ell - a - 2$, and X has another leg with length at least a. Summing the lengths yields $2(n - \ell - a - 2) + 2a \le n - 2$, which simplifies to $n \le 2\ell + 2$.

If $n = 2\ell + 2$, then by the equality there are exactly four legs. The number of full paths in Y is $2(n-\ell-a-2)+1-(n-\ell-1)$, which simplifies to $n-\ell-2a-2$. Since $n = 2\ell+2$, adding the four full paths that use both branch vertices yields $\ell + 4 - 2a$, less than $\ell + 3$.

If $n = 2\ell + 1$, then $n - \ell = \ell + 1$ and G has a subgraph G' with 2ℓ vertices that has two legs of length a in X and two of length $\ell - a - 1$ in Y. Full paths have $\ell+1$ vertices; G' has $\ell-2a-2$ such paths in Y plus four containing the two branch vertices. Now G is completed by adding one vertex y, which may extend a leg of G' or start a new leg at a branch vertex. This adds at most three full paths, with equality only when y extends a leg in Y. Hence G has at most $\ell - 2a + 5$ full paths. Since $a \ge 1$, this yields the desired bound. □

In the final case, when $a = 1$ the vertex y adds only two paths, since a full path from it to X ends at the branch vertex in X. Thus a near-spider in fact has at most $\ell+2$ full paths. Similarly, when the spiders $S_{a,b,c}$ and $S_{1,1,\ell-1,\ell-1}$ with $n = 2\ell+1$ (and the special case $S_{2,2,2,2}$ with $\ell = 4$) are excluded, the bound for spiders can be improved to $\ell + 1$. We will not need these stronger bounds.

As in the case $n \ge 2\ell+2$, we exclude $k = 0$ so that the marking process will be valid for cards with radius k. We maintain the notation and definitions as in Definitions 3.1 and 2.7.

Lemma 4.3. *If $\mathcal{D}$ is ambiguous and $n = 2\ell + 1$ with $\ell \ge 3$, then $k \ge 1$.*

Proof. Under the assumption $k = 0$, the first part of the argument in Lemma 3.4 remains valid. That is, since $n - \ell \ge 4$, a full path is not a star, so $\mathcal{D}$ contains a 1-evine C as a card. By Lemma 3.3, C and a shortest cycle Q in H share at least four vertices and lie in a component with at most $n - 2$ vertices. As in Lemma 3.4, they cannot share more, so with $n = 2\ell+1$ their union in H is a component having exactly $n-2$ vertices, and the four shared vertices form a path in both C and Q.

This component is an $(\ell+2)$-cycle plus a total of $\ell-3$ pendant edges at two consecutive vertices. The cards of H that are paths (with $\ell+1$ vertices) lie along Q or start at a leaf of C. There are $\ell + 2$ along Q and $2(\ell - 3)$ that start at leaves of C, for a total of $3\ell - 4$.

Already $C \cup Q$ has $n-2$ edges. An n-vertex forest with at least $n-2$ edges cannot have two isolated vertices. Since H and F have the same degree list (Corollary 2.12), without isolated vertices H must have $n - 1$ edges, so also F has $n - 1$ edges and is a tree.

Since F and H have the same degree list, the tree F grows from C only by appending edges to leaves, extending paths. Since C is a near-spider or is $S_{2,1,\ldots,1}$, extending at leaves makes F also a near-spider or a spider. By Lemma 4.2, F has at most $\ell + 3$ full paths.

Since $3\ell - 4 > \ell + 3$, there is no ambiguous $\mathcal{D}$ with $k = 0$ unless $\ell = 3$. In that case $\ell - 3 = 0$, and C is just a 4-vertex path. We find that F is P_7 and contains four copies of P_4, while H is the disjoint union of a 5-cycle and an edge, containing five copies of P_4. □

In Section 3, F and H could have the same number of k-central edges or $(k+1)$-centers when $n = 2\ell + 1$. In this special case, we prohibit an ambiguous deck $\mathcal{D}$ by showing that F and H cannot have the same number of full paths. Again we use the marking process.

Lemma 4.4. *Given $n = 2\ell+1$, let F be an n-vertex forest whose $(n-\ell)$-deck has a connected card C with radius $j+1$, where $j \geq 1$. If F has $1 + d_C + \ell$ j-centers, then F is a tree with at most $\ell + 3$ full paths.*

Proof. Again, *full paths* are those with $n - \ell$ (that is, $\ell + 1$) vertices.

Let z be a center of C (there is one choice for z when C has even diameter, two when C has odd diameter). By the marking process (Lemma 2.15), F has at most $1 + d_C + \ell$ j-centers, and equality requires that F is a tree and that every vertex outside C is marked. This further requires that every vertex outside C has distance at least $j + 1$ from z.

Let v and w be two vertices at distance $j + 1$ from z, with $v \notin V(C)$. If the path joining v and w in F does not go through z, then the paths from v and w to z share a vertex x at distance j from both v and w. In using the marking process to count $1 + d_C + \ell$ j-centers, we cannot count x in both d_C and ℓ, so when $w \in V(C)$ vertex x cannot mark both v and w. Similarly, when $w \notin V(C)$ vertices v and w cannot both be marked.

We conclude that in growing C to form F, we can only extend edge-disjoint paths from z beyond distance j or $j + 1$ from z, without introducing branch vertices. The paths that get extended in forming F form a spider with branch vertex z. Obtain C' by adding to this spider any shorter legs of C having an end at branch vertex z.

All vertices of C are within distance $j + 1$ of z, and C has $n - \ell$ vertices. There is no path with $n - \ell$ vertices contained in C unless C is a path, in which case F is a path with $2\ell + 1$ vertices and contains only $\ell + 1$ full paths.

Hence we may assume that every full path starting from a leaf v of C that is not in C' ends at a vertex of $F - V(C)$ along a leg of C'. Since the existence of v requires C to have at least three vertices not in C', at most one leg of C' has length at least $\ell - 1$. Obtain F' from $F - v$ by adding a leaf v' to extend a shortest leg of C'. Any full path starting at v in F is replaced by a new full path in F', since v' has distance at most $\ell - 1$ from z.

Thus F' has at least as many full paths as F. Removing v may allow another leg to end at branch vertex v. Repeating the transformation as long as a vertex such as v exists eventually converts F to a spider without reducing the number of full paths. Thus F has at most $\ell + 3$ full paths, by Lemma 4.2. □

Theorem 4.5. *For $n = 2\ell + 1$, the family of n-vertex acyclic graphs is ℓ-recognizable.*

Proof. Suppose that there is an ambiguous deck $\mathcal{D}$ with reconstructions F and H as we have been discussing. By Lemma 2.13, no card has diameter less than $2k + 2$. By Corollary 2.11, F and H have the same number of k-centers and the same number s of k-central edges. Let

C be a card of least diameter in $\mathcal{D}$, and let Q be a shortest cycle in H. By Lemma 3.3, C and Q lie in the same component of H and share at least four vertices.

Suppose that C has diameter $2k+2$ and center z. As in Theorem 3.5, using the marking process, F has at most $d_C+\ell$ k-central edges, and H has at least $|V(Q)|+d_C-2$ k-central edges. Recall that Q has length at least $n-\ell+1$. Thus $n-\ell+1+d_C-2 \le d_C+\ell$. Since $n=2\ell+1$, both F and H have exactly $d_C+\ell$ k-central edges, and Q has length exactly $n-\ell+1$, which equals $\ell+2$. With $d_C+\ell$ k-central edges, F has $1+d_C+\ell$ k-centers, and hence by Lemma 2.15 F is a tree with $n-1$ edges.

On the other hand, if $\mathcal{D}$ has no card with diameter $2k+2$, then C has diameter $2k+3$. Now every $(k+1)$-vine has fewer than $n-\ell$ vertices. Since also H has girth at least $2k+4$ (Lemma 2.9), F and H have the same number of $(k+1)$-centers, by Lemmas 2.2 and 2.5. Since C has radius $k+2$ and $d_C=1$, the marking process with $j=k+1$ implies that the number of $(k+1)$-centers in F is at most $2+\ell$. Since H has girth at least $2k+4$, each vertex of Q is a $(k+1)$-center in H, so the number of $(k+1)$-centers in H is at least $\ell+2$. Thus the number of $(k+1)$-centers in both F and H is exactly $\ell+2$, and H has girth exactly $\ell+2$. The equality again requires that F is a tree (by the marking process).

Hence in either case F is a tree and Q has length $\ell+2$. Now consider paths with $n-\ell$ vertices in H. There are $\ell+2$ such paths in Q. Let $t=|V(C)\cap V(Q)|$. From each vertex of C outside $V(Q)$, one can travel to $V(Q)$ and then complete a path with $n-\ell$ vertices in either direction along Q. Hence H contains at least $\ell+2+2(n-\ell-t)$ full paths. If $t<n-\ell$, then this contradicts the ambiguity of the deck, since by Lemma 4.4 with $j=k$ or $j=k+1$ there are at most $\ell+3$ full paths in F.

Since Q has no chords, we can only have $t=n-\ell$ if C is a path contained in Q. Now the argument in Lemma 4.4 implies that F is a path with $2\ell+1$ vertices and contains only $\ell+1$ paths with $\ell+1$ vertices, again yielding a contradiction. $\square$

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