Acyclic graphs with at least $2\ell + 1$ vertices are
\ell-recognizable

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Abstract

The $(n - \ell)$-deck of an $n$-vertex graph is the multiset of subgraphs obtained from
it by deleting $\ell$ vertices. A family of $n$-vertex graphs is \ell-recognizable if every graph
having the same $(n - \ell)$-deck as a graph in the family is also in the family. We prove
that the family of $n$-vertex graphs having no cycles is \ell-recognizable when $n \geq 2\ell + 1$
(except for $(n, \ell) = (5, 2)$). It is known that this fails when $n = 2\ell$.

1 Introduction

The $k$-deck of a graph is the multiset of $k$-vertex induced subgraphs. We write this as the
$(n - \ell)$-deck when the graph has $n$ vertices and the focus is on deleting $\ell$ vertices. An $n$-
vertex graph is \ell-reconstructible if it is determined by its $(n - \ell)$-deck. It is an elementary
observation, via a counting argument, that the $k$-deck of a graph always determines its
$(k - 1)$-deck. Therefore, an enhancement of the Reconstruction Problem is to find for each
graph the maximum $\ell$ such that it is \ell-reconstructible. Kelly [5] extended the classical
Reconstruction Conjecture of Kelly [4] and Ulam [12] as follows:

Conjecture 1.1 ([5]). For $\ell \in \mathbb{N}$, there exists a threshold $M_\ell$ such that every graph with at
least $M_\ell$ vertices is \ell-reconstructible.

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Many reconstruction arguments have two parts. First, one proves that the deck determines that the graph is in a particular class or has a particular property. When the \((n-\ell)\)-deck determines this, the property is \(\ell\)-recognizable. Separately, using the knowledge that every reconstruction from the deck has that property, one determines that only one such graph has that deck. That is, one proves that the family is weakly \(\ell\)-reconstructible, meaning that no two graphs in the family have the same deck. Bondy and Hemminger [1] introduced this separation into two steps for the case \(\ell = 1\).

Here, toward \(\ell\)-reconstructibility of trees, we consider \(\ell\)-recognizability of acyclic graphs. We prove the following theorem.

**Theorem 1.2.** For \(n \geq 2\ell + 1\) (except \((n, \ell) = (5, 2)\)), the \((n-\ell)\)-deck of an \(n\)-vertex graph determines whether the graph contains a cycle.

Our proof is constructive; the information we need in order to confirm that every graph having \((n-\ell)\)-deck \(\mathcal{D}\) is acyclic is computed from the deck.

Since the \((n-\ell)\)-deck determines the 2-deck, in this setting we also know the number of edges. This yields the following corollary.

**Corollary 1.3.** For \(n \geq 2\ell + 1\) (except \((n, \ell) = (5, 2)\)), the \((n-\ell)\)-deck of an \(n\)-vertex graph determines whether the graph is a tree.

Spinoza and West [11] determined for every graph \(G\) with maximum degree at most 2 the maximum \(\ell\) such that \(G\) is \(\ell\)-reconstructible. Their full result is quite complicated to state, but a special case is that for \(n \geq 2\ell + 1\) (except \((n, \ell) = (5, 2)\)), every \(n\)-vertex graph with maximum degree at most 2 is \(\ell\)-reconstructible. A path with \(2\ell\) vertices has the same \(\ell\)-deck as the disjoint union of an \((\ell+1)\)-cycle and a path with \(\ell-1\) vertices, as shown in [11], so the result of [11] and the result in the present paper are both sharp.

Nýdl [10] conjectured that trees with at least \(2\ell + 1\) vertices are weakly \(\ell\)-reconstructible. This conjecture would be sharp, since Nýdl presented two trees with \(2\ell\) vertices having the same \(\ell\)-deck. The two trees are obtained from a path with \(2\ell-1\) vertices by adding one leaf, either to the central vertex of the path or to one of its neighbors. Kostochka and West [8] used the results of [11] to give a short proof that these two trees have the same \(\ell\)-deck. With our result, Nýdl’s conjecture can be strengthened as follows.

**Conjecture 1.4.** For \(\ell \neq 2\), every tree with at least \(2\ell + 1\) vertices is \(\ell\)-reconstructible.

When \(\ell = 2\), the correct threshold is 6 rather than 5, since the union of a 4-cycle and an isolated vertex has the same 3-deck as the tree obtained by subdividing one edge of the 4-vertex star. Giles [2] proved that trees with at least six vertices are 2-reconstructible. For \(\ell = 1\), the only non-acyclic \(n\)-vertex graph having no cycle with length at most \(n-1\) is the \(n\)-vertex cycle, distinguished by its number of edges, so \(n \geq 3\) suffices. In [6], the present
authors proved that \( n \geq 25 \) suffices when \( \ell = 3 \). For \( \ell \geq 4 \), it is not yet known whether there is a threshold \( T_\ell \) such that \( n \)-vertex trees are \( \ell \)-reconstructible when \( n \geq T_\ell \).

Besides acyclicity, another fundamental property of trees is connectedness. Spinoza and West [11] proved that connectedness is \( \ell \)-recognizable for \( n \)-vertex graphs when \( n > 2\ell(\ell+1)^2 \). This threshold is surely too large. Manvel [9] proved that connectedness is 2-recognizable for graphs with at least six vertices, and the present authors [7] proved that connectedness is 3-recognizable for graphs with at least seven vertices. Spinoza and West [11] suggested that (except for \( (n, \ell) = (5, 2) \)), connectedness is recognizable for \( n \)-vertex graphs when \( n \geq 2\ell+1 \).

As a first step toward the threshold on \( n \) for \( \ell \)-recognizability of connectedness, one can consider \( n \)-vertex graphs whose \((n-\ell)\)-deck has only acyclic cards. With \( n-\ell \geq 2 \), we know the number of edges in any reconstruction. Our result in this paper settles the question for graphs with \( n - 1 \) edges, where connectedness and acyclicity are equivalent. This motivates more detailed questions.

**Problem 1.5.** For \( c, \ell \in \mathbb{N} \), determine the smallest thresholds \( N_{\ell,c} \) and \( N'_{\ell,c} \) such that for all \( n \)-vertex graphs with \( n + c \) edges whose \((n-\ell)\)-vertex induced subgraphs are all acyclic,

(a) if \( n \geq N_{\ell,c} \), then the \((n-\ell)\)-deck determines whether the graph is connected, and
(b) if \( n \geq N'_{\ell,c} \), then the graph is connected.

The thresholds when the cards are not required to be acyclic are also unknown.

When \( c = 1 \), the fact that a graph with \( p \) vertices and at least \( p + 2 \) edges has girth at most \( \lfloor (p + 2)/2 \rfloor \) (see Exercise 5.4.36 of [13], for example) can be used to prove \( N'_{\ell,1} \leq 2\ell \). That is, when the graph has \( n + 1 \) edges and the cards in the \((n-\ell)\)-deck are acyclic, every reconstruction is connected. The threshold of \( 2\ell \) vertices is sharp when \( \ell \) is even, by the graph with \( 2\ell - 1 \) vertices consisting of an isolated vertex plus four paths of length \( \ell/2 \) with common endpoints. However, it is possible that the threshold \( N_{\ell,1} \) for determining whether all reconstructions are connected may be smaller.

For \( c = 0 \), we believe \( N_{\ell,0} = 2\ell - 1 \); Zirlin [14] has proved this for sufficiently large \( \ell \). The threshold is at least this much because \( C_{2\ell-2} \) and the disjoint union \( C_{\ell-1} + C_{\ell-1} \) have the same deck. Zirlin [14] also proved \( N_{\ell,0} \leq 2\ell + 1 \) for \( \ell \geq 3 \).

## 2 General Tools

The proof for the special case \( n = 2\ell+1 \) requires additional care beyond the general argument. In this section we develop tools useful for all cases.

Let \( \mathcal{D} \) be the \((n-\ell)\)-deck of an \( n \)-vertex graph \( G \) (we henceforth just call it the “deck”). We will assume \( n > 2\ell \). The members of \( \mathcal{D} \) are the “cards” in the deck. We begin with a notion generalizing the degree list.
Definition 2.1. Given a vertex $v$ in a graph $G$, the $k$-ball at $v$, written $U_k(v)$, is the subgraph induced by all vertices within distance $k$ of $v$ in $G$. A $k$-vine is a tree with diameter $2k$. A $k$-center in a graph $G$ is a vertex $v$ that is the center of a $k$-vine.

The term “$k$-vine” continues the botanical theme of terminology about trees; a vine has a main path from which the rest grows. When $G$ is a forest, the maximal $k$-vine at a $k$-center $v$ is the $k$-ball at $v$. If the $k$-ball at $v$ does not contain a path of length $2k$, then $v$ is not a $k$-center. Our general approach is to consider an acyclic and a non-acyclic graph having the same $(n-\ell)$-deck, show that they have the same number of $k$-centers for an appropriate $k$, and obtain a contradiction by showing that they cannot have the same number of $k$-centers.

In order to count $k$-centers using the $(n-\ell)$-deck, we will count the $k$-centers whose $k$-balls have each size. The special case $k=1$ will yield the vertex degrees. The key point is uniqueness of the maximal $k$-vine containing a particular $k$-vine.

Lemma 2.2. In a graph $G$ with girth at least $2k+2$, every induced $k$-vine is contained in a unique maximal $k$-vine.

Proof. Since $G$ has girth at least $2k+2$, every $k$-vine in $G$ is an induced subgraph. Since a $k$-vine $B$ contains a path $P$ of length $2k$, its center $v$ is uniquely determined. No $k$-vine with a center $w$ other than $v$ contains $P$, because the distance from $w$ to one of the ends of $P$ would exceed $k$. Hence no $k$-vine with center $w$ contains $B$. Thus the maximal $k$-vine containing $B$ can only be $U_k(v)$. \hfill \Box

Lemma 2.2 will enable us to apply a general counting argument that has been used in other contexts. It combines ideas of Manvel [9] for $\ell$-reconstructibility and of Greenwell and Hemminger [3] for 1-reconstructibility. We follow the approach in a proof by Bondy and Hemminger [1] of the Greenwell–Hemminger result.

Definition 2.3. When $\mathcal{F}$ is a class of graphs, an $\mathcal{F}$-subgraph of a graph $G$ is an induced subgraph of $G$ in $\mathcal{F}$. Let $s(F,G)$ denote the number of induced subgraphs of $G$ isomorphic to $F$. Let $m(F,G)$ denote the number of occurrences of $F$ as a maximal $\mathcal{F}$-subgraph of $G$. An absorbing family for a graph $G$ is a family $\mathcal{F}$ of graphs such that every induced subgraph of $G$ belonging to $\mathcal{F}$ lies in a unique maximal induced subgraph of $G$ belonging to $\mathcal{F}$.

Lemma 2.4. Let $\mathcal{F}$ be an absorbing family for an $n$-vertex graph $G$. If $m(F,G)$ is known for each $F \in \mathcal{F}$ that has at least $n-\ell$ vertices, and the $(n-\ell)$-deck of $G$ is known, then $m(F,G)$ can be determined from the deck for all $F \in \mathcal{F}$.

Proof. For an $\mathcal{F}$-subgraph $F$ of $G$, let the depth of $F$ be the maximum length $k$ of a chain $F_0, \ldots, F_k$ of $\mathcal{F}$-subgraphs such that $F = F_0$ and each is an induced subgraph of the next.

By hypothesis, $m(F,G)$ is known for all $F \in \mathcal{F}$ with at least $n-\ell$ vertices. From the $(n-\ell)$-deck, we also know the $j$-deck of $G$ for $j < n-\ell$. Hence we know all the $\mathcal{F}$-subgraphs

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of $G$. From this we can determine the chains of $\mathcal{F}$-subgraphs in $G$, so we can compute the depth of $F$; let it be $k$. We prove the claim by induction on $k$.

Let $j = |F|$, and suppose $j < n - \ell$. If $k = 0$, then $m(F, G)$ is the number of copies of $F$ in the $j$-deck of $G$. For $k > 0$, group the induced copies of $F$ in $G$ by which $\mathcal{F}$-subgraph $H$ of $G$ is the unique maximal $\mathcal{F}$-subgraph containing this copy of $F$. Now

$$s(F, G) = \sum_{H \in \mathcal{F}} s(F, H)m(H, G).$$

When $s(F, H) \neq 0$ and $F \neq H$, every chain of $\mathcal{F}$-subgraphs starting at $H$ can be augmented by adding $F$ at the beginning, so $H$ has smaller depth than $F$. By the induction hypothesis, we know every quantity in the displayed equation other than $m(F, G)$. The computation for $m(F, G)$ is the same for every graph having the same $(n - \ell)$-deck as $G$. \qed

For example, the family of connected graphs is absorbing for every graph. We formalize this application here because it was stated incorrectly in the paper by Kostochka and West [8] and because it illustrates the technique we use for $k$-vines. The special case for $\ell = 1$ was observed by Kelly [5] using different methods.

**Corollary 2.5.** If $n > 2\ell$, then $n$-vertex graphs having no component with more than $n - \ell$ vertices are $\ell$-reconstructible, and this threshold on $n$ is sharp. All $n$-vertex graphs having no component with at least $n - \ell$ vertices are $\ell$-reconstructible, with no restriction on $n$.

**Proof.** A graph with more than $2\ell$ vertices can only have one component with at least $n - \ell$ vertices, and it has no component with more vertices if and only if it has at most one connected $(n - \ell)$-card. Hence the condition is $\ell$-recognizable, and if there is a component with $n - \ell$ vertices it is seen as a card. Since the family of connected graphs is absorbing, by Lemma 2.4 graphs satisfying the condition are $\ell$-reconstructible.

The result is sharp, since $P_{\ell} + P_{\ell}$ and $P_{\ell+1} + P_{\ell-1}$ have the same $\ell$-deck. This follows from the result of Spinoza and West [11] that any two graphs with the same number of vertices and edges whose components are all cycles with at least $k + 1$ vertices or paths with at least $k - 1$ vertices have the same $k$-deck. \qed

By Lemma 2.2, the family of $k$-vines is absorbing for every graph with girth at least $2k + 2$ (the minimal $k$-vines in a graph all have $2k + 1$ vertices). This sometimes allows us to reconstruct the number of $k$-centers from the deck.

**Corollary 2.6.** Let $\mathcal{D}$ be the $(n - \ell)$-deck of an $n$-vertex graph. If every card in $\mathcal{D}$ is acyclic, and every card in $\mathcal{D}$ has radius greater than $k$, then all reconstructions from $\mathcal{D}$ have the same number of $k$-centers, which can be computed from $\mathcal{D}$.
Proof. A connected acyclic card with radius greater than $k$ has at least $2k + 2$ vertices. Hence $n - \ell \geq 2k + 2$. Since all cards are acyclic, every reconstruction has girth at least $2k + 3$.

Since every $(n - \ell)$-card has radius greater than $k$, every $k$-vine has fewer than $n - \ell$ vertices. Hence by examining the deck we see all the $k$-vines and determine the maximum number of vertices in a $k$-vine; call it $m$. We have $m < n - \ell$.

Thus any reconstruction has no $k$-vines with $i$ vertices whenever $i \geq n - \ell$. By Lemma 2.2, Lemma 2.4 implies that the $(n - \ell)$-deck determines the numbers of maximal $k$-vines with $i$ vertices for all $i$. Finally, since the $k$-centers correspond bijectively to maximal $k$-vines, the number of $k$-centers in any reconstruction is determined by the deck.

For the case $n = 2\ell + 1$ of our result, we will need the analogue of Corollary 2.6 for vertex degrees. Manvel [9] proved this without the restriction to triangle-free graphs, but without the triangles we obtain the result as a simple application of Lemma 2.4.

**Corollary 2.7 ([9]).** If the number of vertices of degree $i$ in an $n$-vertex triangle-free graph $G$ is known whenever $i \geq n - \ell$, then the degree list of $G$ is $\ell$-reconstructible.

**Proof.** The 1-vines in $G$ are the stars with at least two edges. With girth at least 4, Lemma 2.2 allows use of Lemma 2.4 to count the vertices with each degree at least 2. Since we also know the 2-deck and 1-deck, we know the numbers of edges and vertices, which also gives the numbers of vertices of degrees 1 and 0.

We will also need concepts for edges that are analogous to $k$-vines and $k$-centers.

**Definition 2.8.** Given an edge $e$ in a graph $G$, the $k$-eball at edge $e$ is the subgraph induced by all vertices within distance $k$ of either endpoint of $e$ in $G$. A $k$-evine is a tree with diameter $2k + 1$. A $k$-central edge in a graph $G$ is an edge $e$ whose $k$-eball contains a $k$-evine whose center is the vertex set of $e$.

Note that a $k$-evine has radius $k + 1$. Also, when the minimum radius among cards is $k + 1$, every card has diameter at least $2k + 1$.

**Lemma 2.9.** In a graph $G$ with girth at least $2k + 3$, every $k$-evine is contained in a unique maximal $k$-evine.

**Proof.** A $k$-evine $B$ in $G$ contains a path $P$ with $2k + 2$ vertices. Since $G$ has girth at least $2k + 3$, $B$ is an induced subgraph of $G$. The path $P$ determines a unique $k$-central edge $e$ in $B$, and no $k$-evine with a different central edge can contain $P$. Hence the unique maximal $k$-evine containing $B$ is the $k$-eball for $e$.

**Lemma 2.10.** Let $n, k, \ell$ be positive integers with $2k + 2 \leq n - \ell$. If all cards in an $(n - \ell)$-deck $\mathcal{D}$ are acyclic, with radius greater than $k$, and no card has diameter $2k + 1$, then the deck determines the number of $k$-central edges in any $n$-vertex reconstruction, and this number can be computed from the deck.
Proof. The acyclic cards with diameter $2k + 1$ are the $k$-evines with $n - \ell$ vertices. With such cards forbidden, no $k$-evine has more than $n - \ell$ vertices, because with $2k + 2 \leq n - \ell$ we could delete leaves outside a longest path to obtain a $k$-evine with $n - \ell$ vertices.

Since $n - \ell \geq 2k + 2$, any reconstruction from $D$ has girth at least $2k + 3$. By Lemma 2.9, the family of $k$-evines is an absorbing family for any reconstruction. Since no $k$-evines have at least $n - \ell$ vertices, Lemma 2.4 applies, and the deck determines the numbers of maximal $k$-evines with each number of vertices. In particular, it determines the total number of maximal $k$-evines, and this is the same as the number of $k$-central edges.

3 The Proof for $n \geq 2\ell + 2$

We begin by developing a tool for bounding the number of $k$-centers in a forest in terms of its deck. This tool does not depend on the relationship between $n$ and $\ell$.

**Definition 3.1.** Let $D$ be the $(n - \ell)$-deck $D$ of an $n$-vertex graph having a component with at least $n - \ell$ vertices. Let $\hat{k}$ be the minimum radius among cards in $D$, and let $k = \hat{k} - 1$. This henceforth fixes $k$.

A short card is a card with radius $\hat{k}$. When $C$ is a short card with center $z$ in the $(n - \ell)$-deck of an $n$-vertex forest $F$, the marking argument describes a relationship between $k$-central vertices of $F$ and vertices of $F$ outside $C$. Each $k$-center other than $z$ that is in the component of $F$ containing $C$ marks a vertex $x'$ at distance $k$ from $x$ along a path that extends the $z,x$-path in $F$. If $x$ is not adjacent to $z$, then $x'$ is outside $C$. Furthermore, since $F$ is a forest, every vertex outside $C$ is marked by at most one $k$-center.

For a short card $C$ with a center $z$, let $d_C$ denote the maximum number of edge-disjoint paths of length $\hat{k}$ in $C$ with common endpoint $z$. Note that short cards have diameter $2\hat{k}$ (with unique center) or $2\hat{k} - 1$. In the latter case, $d_C = 1$ when viewed from either center.

**Lemma 3.2.** If $C$ is a short card with center $z$ in the deck of a forest $F$, then the number of $k$-centers in $F$ is at most $1 + d_C + \ell$. If equality holds, then in the marking argument each vertex $x'$ of $F$ outside $C$ is marked by a $k$-center $x$ not adjacent to $z$, and $F$ is a tree.

**Proof.** Let $F'$ be the component of $F$ containing $C$, and let $\ell'$ be the number of vertices of $F'$ outside $C$. The neighbors of $z$ in $C$ along paths of length $\hat{k}$ in $C$ are $k$-centers, as is $z$. By the marking argument, $F'$ contains at most $\ell'$ additional centers. In any component of $F$ other than $F'$, the number of $k$-centers is strictly less than the number of vertices, since vertices with degree at most 1 cannot be $k$-centers. The total number of vertices of $F$ outside $C$ is $\ell$. Summing over all the components of $F$, the number of $k$-centers is at most $1 + d_C + \ell$, with equality only when $F$ is a tree.

Our task is to study when a deck determines whether all reconstructions have cycles.
Definition 3.3. We say that a deck $D$ is ambiguous if it is the $(n - \ell)$-deck of both an $n$-vertex acyclic graph and an $n$-vertex non-acyclic graph. Given an ambiguous deck, we typically let $F$ and $H$ be $n$-vertex acyclic and non-acyclic graphs having $(n - \ell)$-deck $D$, respectively. All cards of an ambiguous deck are acyclic, being induced subgraphs of a forest. Hence when $D$ is ambiguous the graph $H$ has girth at least $n - \ell + 1$, and thus $D$ has connected cards (in fact, paths).

Since all cards in $D$ are acyclic, and all cards have radius greater than $k$, it follows from Corollary 2.6 that the number of $k$-centers is the same for all reconstructions from $D$. We will study the number of $k$-central edges to eliminate the possibility of ambiguous decks when $n \geq 2\ell + 2$.

First we exclude the case $\hat{k} = 1$, after which $k$ will be positive.

Lemma 3.4. If $D$ is ambiguous and $n \geq 2\ell + 2$, then $\hat{k} > 1$.

Proof. A card with radius 1 is a star with $n - \ell$ vertices. A non-acyclic reconstruction $H$ has a cycle with at least $n - \ell + 1$ vertices. Since $2n - 2\ell + 1 \geq n + 2$, a star with $n - \ell$ vertices must lie in the same component of $H$ with any cycle. Since $D$ also gives us the 2-deck of a forest, any reconstruction has at most $n - 1$ edges. Since $H$ contains a cycle, $H$ must therefore be disconnected.

Since $H$ has girth at least $n - \ell + 1$, which is at least 4, a star shares at most three vertices with a cycle. Hence the number of vertices in their common component of $H$ is at least $(n - \ell + 1) + (n - \ell) - 3$. Since this is at most $n - 1$, we conclude $n \leq 2\ell + 1$. □

Lemma 3.5. If $D$ is ambiguous and $n \geq 2\ell + 2$, then $D$ has no card with diameter $2k + 1$.

Proof. Let $C$ be a card with diameter $2k + 1$. As remarked earlier, $d_C = 1$. By Lemma 3.2, at most $2 + \ell$ vertices of $F$ are $k$-centers. We have noted that all reconstructions from $D$ have the same number of $k$-centers. All vertices on a cycle in a reconstruction $H$ are $k$-centers in $H$, and the cycle has length at least $n - \ell + 1$. Hence $2 + \ell \geq n - \ell + 1$, which yields $n \leq 2\ell + 1$. □

Theorem 3.6. For $n \geq 2\ell + 2$, the family of $n$-vertex acyclic graphs is $\ell$-recognizable.

Proof. Suppose that there is an ambiguous deck $D$ with reconstructions $F$ and $H$ as we have been discussing. By Lemma 3.5, no card has diameter $2k + 1$. Hence by Lemma 2.10 the number $s'$ of $k$-central edges is the same in $F$ and $H$. Let $C$ be a short card, and let $d = d_C$.

Note that $C$ has $d$ $k$-central edges incident to its unique center $z$. An edge of $F$ in the component containing $z$ is a $k$-central edge if and only if its endpoint farther from $z$ is a $k$-center. In other components, the number of $k$-central edges is less than the number of $k$-centers. Hence Lemma 3.2 implies $s' \leq d + \ell$. □
Since $C$ is a card in $\mathcal{D}$, every reconstruction from $\mathcal{D}$ has $d$ $k$-central edges with a common endpoint. In $H$, only two of these can lie on a particular cycle. We have observed that $H$ has girth at least $2k+3$, and hence every edge on a cycle in $H$ is a $k$-central edge. Lemma 2.10 now yields $s' \geq n-\ell+1+d-2$. The upper and lower bounds on $s'$ now require $n \leq 2\ell+1$. □

The argument for $n \geq 2\ell+2$ in this section is valid for all $\ell$. In the next section we must restrict to $\ell \geq 3$.

4 The Case $n = 2\ell + 1$

We begin with a result of independent interest about special trees.

**Definition 4.1.** A *branch vertex* in a tree is a vertex with degree at least 3. A *spider* is a tree having at most one branch vertex. A *leg* of a tree is a path in the tree whose endpoints are a leaf and a branch vertex. When $d \geq 3$, a spider whose branch vertex has degree $d$ is the union of $d$ paths with a common endpoint. When those paths have lengths $m_1, \ldots, m_d$, we denote the spider by $S_{m_1, \ldots, m_d}$; note that $S_{m_1, \ldots, m_d}$ has $1 + \sum_{i=1}^{d} m_i$ vertices.

**Lemma 4.2.** When $n \geq 2\ell + 1 \geq 3$, an $n$-vertex spider contains at most $\ell + 3$ paths having exactly $n - \ell$ vertices, except for $S_{1,1,1,1}$ when $\ell = 2$.

*Proof.* Let $G$ be an $n$-vertex spider with maximum degree $d$. Let a *long path* be a path with $n - \ell$ vertices. We use induction on $\ell$. When $\ell = 1$, an $n$-vertex tree has at most two paths with $n - 1$ vertices, except that when $n = 4$ the spider $S_{1,1,1,1}$ has three such paths, still less than $\ell + 3$.

For $\ell \geq 2$, consider first the case that some leaf $x$ lies in at most one long path. Let $G' = G - x$. Let $n' = |V(G')| = n - 1$ and $\ell' = \ell - 1$. Since $n \geq 2\ell + 1$, we have $n' > 2\ell' + 1$. In particular, $(n', \ell') \neq (5, 2)$, so we can apply the induction hypothesis without considering the exception. Thus $G'$ contains at most $\ell' + 3$ paths with $n' - \ell'$ vertices. That is, $G$ has at most $\ell + 2$ paths with $n - \ell$ vertices avoiding $x$. Adding (at most) one long path containing $x$ yields the desired bound for $G$.

In the remaining case, every leaf appears in at least two long paths. Here we argue directly, without needing the induction hypothesis. Let $a$ be the length of a shortest leg of $G$, and let $x$ be the leaf in a leg of length $a$. Since $x$ lies in two path-cards, $G$ must have at least two other legs of length at least $n - \ell - 1 - a$, so $d \geq 3$.

If $d \geq 4$, then some fourth leg (with leaf $y$) also has length at least $a$. Summing the lengths of these four legs yields $2n - 2\ell - 2 \leq n - 1$, or $n \leq 2\ell + 1$. Since we consider only $n \geq 2\ell + 1$, equality holds, requiring $G = S_{a,a,\ell-a,\ell-a}$ and $n - \ell = \ell + 1$. If $a < \ell - a$, then exactly four long paths use $x$ or $y$ and a leg of length $\ell - a$, and $\ell - 2a + 1$ long paths use the two long legs. The total is $\ell - 2a + 5$, which is at most $\ell + 3$ since $a \geq 1$. If $a = \ell - a$,
then there is also one long path from \( x \) to \( y \), but now exceeding \( \ell + 3 \) requires \( a = 1 \) and \( \ell = 2 \), which occurs precisely for the exceptional case \( S_{1,1,1,1} \).

Hence we may assume \( d = 3 \). The graph is \( S_{a,b,c} \). To have each leaf in two path-cards, the lengths of any two legs must sum to at least \( n - \ell - 1 \). The number of vertices in the union of two legs is the sum of their lengths plus 1, and the last \( n - \ell - 1 \) vertices cannot start a long path. Hence the number of long paths is \( 2(a + b + c) + 3 - 3(n - \ell - 1) \), which equals \( 3\ell - n + 4 \). Since \( n \geq 2\ell + 1 \), the number of long paths is at most \( \ell + 3 \).

For \( \ell \geq 4 \), equality in Lemma 4.2 requires \( n = 2\ell + 1 \) and occurs for \( S_{a,b,c} \) and for \( S_{1,1,\ell-1,\ell-1} \). When these are excluded, the bound can be improved to \( \ell + 1 \) except for the special case \( S_{2,2,2,2} \) when \( \ell = 4 \), but we will not need this stronger bound.

In the remainder of the paper we restrict to the setting \( n = 2\ell + 1 \geq 7 \). We maintain the notation and definitions for \( F, H, \mathcal{D}, \hat{k}, k \) as in the previous section. In particular, an ambiguous deck \( \mathcal{D} \) is the \((n-\ell)\)-deck of both an acyclic \( n \)-vertex graph \( F \) and a non-acyclic \( n \)-vertex graph \( H \). Also \( \hat{k} \) is the minimum radius among cards in \( \mathcal{D} \), and \( k = \hat{k} - 1 \). We again begin by excluding the case \( \hat{k} = 1 \).

**Lemma 4.3.** In an ambiguous deck \( \mathcal{D} \), no card is a star. Hence \( \hat{k} > 1 \) and all reconstructions have the same degree list.

*Proof.* Suppose that some card is a star, which is equivalent to \( \hat{k} = 1 \). Since cards have \( \ell + 1 \) vertices, there is no room for a star and a cycle in separate components of a reconstruction \( H \). Since \( H \) has girth at least \( n - \ell + 1 \), which is at least 5, a star shares at most three vertices with a cycle. When they are in the same component \( H' \) of \( H \), the cycle must have exactly \( n - \ell + 1 \) vertices and share exactly three with the star, since \( n - \ell + 1 + \ell + 1 = n + 2 \) and \( H \) is disconnected. With \( H' \) having \( m \) vertices, we have \( n - \ell + 1 + \ell - 2 \leq m \leq n - 1 \), so equality holds, and we know \( H \) exactly.

In particular, no other star has at least \( n - \ell \) vertices. Thus Manvel’s result (Corollary 2.7) applies, and the deck determines the degree list of every reconstruction. However, the 2-deck guarantees that every reconstruction has the same number of edges. We have found \( H \) to be unicyclic, with \( n - 1 \) edges and an isolated vertex. An acyclic reconstruction \( F \) is a tree, with no isolated vertex, so the degree lists are different.

This contradiction implies that no card is a star. Hence \( \hat{k} > 1 \), and therefore Corollary 2.7 again implies that the deck determines the degree list. \( \Box \)

**Lemma 4.4.** Graphs with an ambiguous deck \( \mathcal{D} \) have maximum degree at least 3.

*Proof.* Since \( n - \ell \geq 4 \), we can see in the deck whether there is a vertex of degree at least 3 in the reconstructions. If not, then an acyclic reconstruction is a subgraph of the path \( P_{2\ell+1} \) and hence has at most \( \ell + 1 \) cards that are paths. (We have \( n - \ell = \ell + 1 \), and the last \( \ell \) vertices cannot start paths with \( \ell + 1 \) vertices.) On the other hand, a cycle with at least
\(n - \ell + 1\) vertices contains at least \(\ell + 2\) cards that are paths. Hence for an ambiguous deck maximum degree at least 3 is required. \( \blacksquare \)

**Lemma 4.5.** When \(D\) is an ambiguous deck, \(2k \leq \ell\).

*Proof.* By Corollary 2.6, \(D\) determines the number of \(k\)-centers in any reconstruction. When a reconstruction contains a cycle with length at least \(n - \ell + 1\), every vertex on it is a \(k\)-center, so at most \(\ell - 1\) vertices are not \(k\)-centers. By Lemma 4.4, an acyclic reconstruction has a branch vertex, and we know that it also has a path with at least \(n - \ell\) vertices. Hence at least \(2k + 1\) vertices are not \(k\)-centers, consisting of \(k\) vertices at each end of a longest path plus one additional leaf. We conclude \(2k + 1 \leq \ell - 1\), which yields \(2k \leq \ell\). \( \blacksquare \)

Recall that by Corollary 2.6, \(D\) determines the number of \(k\)-centers in \(F\) and \(H\). As in Section 3, we will want to determine the number of \(k\)-central edges and even the number of \(k\)-centers, but this is more difficult when \(n = 2\ell + 1\). Again let \(d_C\) be the maximum number of edge-disjoint paths of length \(\hat{k}\) with common endpoint at a center \(z\) of \(C\) when \(C\) is a short card. The next two lemmas consider only one type of reconstruction from \(D\).

**Lemma 4.6.** Let \(C\) be a short card in an \((n - \ell)\)-deck \(D\) having an acyclic reconstruction \(F\). If \(F\) has at least \(1 + d_C + \ell\) \(k\)-centers, then for any two vertices \(v_1, v_2 \in V(F) - V(C)\) at the same distance from a center \(z\) of \(C\), the \(z, v_1\)-path and the \(z, v_2\)-path in \(F\) share at most one edge. As a consequence, all vertices having distance at least \(\hat{k}\) from \(z\) have degree at most 2.

*Proof.* By Lemma 3.2, the number of \(k\)-centers in \(F\) is at most \(1 + d_C + \ell\), with equality only when \(F\) is a tree. Let \(Y\) be a set of \(d_C\) neighbors of \(z\) along which edge-disjoint paths of length \(\hat{k}\) in \(C\) depart from \(z\). By Lemma 3.2, the marking argument of Definition 3.1 marks all vertices of \(V(F) - V(C)\) using \(k\)-centers not in \(Y \cup \{z\}\).

If the claim fails, then let \(v_1\) and \(v_2\) be two vertices of \(F - V(C)\) closest to \(z\) whose paths from \(z\) share at least two edges. For \(i \in \{1, 2\}\), let \(P_i\) be the \(z, v_i\)-path in \(F\), and let \(x_i\) be the vertex at distance \(\hat{k}\) from \(v_i\) along \(P_i\). Since \(v_1\) and \(v_2\) must be marked by distinct \(k\)-centers outside \(Y \cup \{z\}\), the vertices \(x_1\) and \(x_2\) are distinct. Since \(P_1\) and \(P_2\) share at least two edges, \(x_1\) and \(x_2\) have distance at least 3 from \(z\). Now \(z\) has distance at least \(\hat{k} + 1\) to the neighbors of \(v_1\) and \(v_2\) on the paths, so they are not in \(C\), contradicting the choice of \(v_1\) and \(v_2\). \( \blacksquare \)

Let \(d\) denote the maximum of \(d_C\) over short cards \(C\).

**Lemma 4.7.** When cards in \(D\) are acyclic and there is a non-acyclic reconstruction \(H\), the number of \(k\)-centers and the number of \(k\)-central edges in \(H\) are both at least \(n - \ell + d - 1\), which is \(d + \ell\) when \(n = 2\ell + 1\). If also \(D\) is ambiguous, then \(H\) is unicyclic.

*Proof.* Let \(C\) be a short card with \(d_C = d\). Since \(n - \ell > n/2\), the card \(C\) lies in the component of \(H\) containing any cycle. The induced subgraph \(C\) has \(d\) \(k\)-centers with a
common neighbor and $d$ $k$-central edges with a common endpoint (in any reconstruction). At most two of these (in either case) lie on a cycle in $H$, but with $2\hat{k} \leq n - \ell$, all the vertices on a cycle in $H$ are $k$-centers (and its edges are $k$-central edges), and there are at least $n - \ell + 1$ of each. Hence at least $n - \ell + 1 + d - 2$ vertices in $H$ are $k$-centers, and at least the same number of edges are $k$-central edges.

If $H$ has more than one cycle, then since each has at least $\ell + 2$ vertices and $H$ has only $2\ell + 1$ vertices, any two cycles share at least three vertices. When two cycles share at least two vertices, their union contains two vertices joined by three edge-disjoint paths. Let $R$ be the union of three such paths with least total length. Note that $R$ contains three cycles, with each edge of $R$ appearing in two of the cycles. Let the lengths of the three paths be $a$, $b$, and $c$. Summing the girth requirement over the three cycles yields $2(a + b + c) \geq 3(n - \ell + 1)$.

There are $a + b + c - 1$ vertices in $R$, and every vertex on a cycle is a $k$-center, since the girth is at least $\ell + 2$. Also, the short card $C$ provides $d$ $k$-centers with a common neighbor, of which at least $d - 3$ are not in $R$, since $R$ has maximum degree 3 and is chosen with minimum number of edges. Hence $H$ has at least $(a + b + c - 1) + (d - 3)$ $k$-centers.

By Corollary 2.6, all reconstructions have the same number of $k$-centers. By Lemma 3.2, any acyclic reconstruction has at most $1 + d + \ell$ $k$-centers. Since $D$ is ambiguous, there is such a reconstruction. With $s$ denoting the number of $k$-centers, we obtain

$$\frac{3}{2}(n - \ell + 1) + d - 4 \leq a + b + c + d - 4 \leq s \leq 1 + d + \ell.$$  

With $n - \ell + 1 = \ell + 2$, this simplifies to $3(\ell + 2) - 8 \leq 2 + 2\ell$ and then $\ell \leq 4$.

To eliminate the remaining cases of small $\ell$, note that $a + b + c \leq n - 1$, since there is an acyclic reconstruction. Hence $2n - 2 \geq 2(a + b + c) \geq 3(n - \ell + 1)$. With $n = 2\ell + 1$, this simplifies to $4\ell \geq 3\ell + 6$, which requires $\ell \geq 6$.

Hence there is no value of $\ell$ that allows $H$ to have more than once cycle. \qed

We can now prove the analogue of Lemma 3.5 for the case $n = 2\ell + 1$.

**Lemma 4.8.** If $D$ is ambiguous and $n = 2\ell + 1$, then $D$ has no card with diameter $2k + 1$.

**Proof.** Let $C$ be a connected card with diameter $2k + 1$; note that $d_C = 1$. Since $d_C = 1$, by Lemma 3.2 $F$ has at most $\ell + 2$ $k$-centers. By Corollary 2.6, $F$ and $H$ have the same number of $k$-centers, $s$. Let $C'$ be a short card chosen to maximize $d_{C'}$; let $d = d_{C'}$. By Lemma 4.7, $H$ has at least $d + \ell$ $k$-centers, but also at least $\ell + 2$ along its unique cycle $Q$. Hence $\ell + 2 \geq s \geq \ell + \max\{2, d\}$. If $d \geq 3$, then we have a contradiction.

Hence we may assume $d \leq 2$ and $s = \ell + 2$. Therefore Lemma 4.6 applies to $C$. First, by Lemma 3.2, in the marking argument every vertex of $F$ outside $C$ is marked by a vertex outside the central edge $e$ of $C$, and $F$ is a tree. Vertices within distance $k$ of $e$ cannot be marked, and therefore $C'$ contains all such vertices.
Thus vertices of $F$ outside $C$ have distance at least $\hat{k}$ from each center of $C$. Since we can apply Lemma 4.6 with either vertex of the central edge $e$ being the center $z$, the paths in $F$ from any two vertices outside $C$ to $e$ cannot meet before reaching $e$.

**Case 1:** Two paths from vertices outside $C$ reach $e$ at the same endpoint $z$. Let $v$ and $v'$ be the vertices at distance $\hat{k}$ from $z$ on these two paths. Since $C$ has diameter $2k + 1$, a third path of length $\hat{k}$ reaches $z$ through $e$. If $C$ has vertices $w$ and $w'$ outside these three paths, then deleting $\{w, w'\}$ and adding $\{v, v'\}$ produces a short card exhibiting $d \geq 3$, a contradiction. Hence $C$ is either the spider $S_{k,k,k+1}$ with three legs and $3k + 2$ vertices or is obtained from $S_{k,k,k+1}$ by adding one vertex.

By Lemma 4.6, $F$ has no branch vertices outside $C$, and no vertex of $C$ has two neighbors outside $C$. Thus $F$ is obtained from $C$ only by extending paths from leaves of $C$ at distance $k$ from $e$. Hence $F$ also is a spider or a spider plus one extra vertex.

Again call a path with $n - \ell$ vertices a long path. If $F$ is a spider, then by Lemma 4.2 $F$ has at most $\ell + 3$ long paths, since $n = 2\ell + 1$ with $(n, \ell) \neq (5, 2)$. If $F$ consists of a spider plus one leaf $v$, then by Lemma 4.2 $F - v$ has at most $\ell + 2$ paths with $n - \ell$ vertices, since $n - \ell = (n - 1) - (\ell - 1)$ and $n - 1 > 2(\ell - 1) + 1$ with $(n - 1, \ell - 1) \neq (5, 2)$. Since $F - v$ is a spider with branch vertex of degree 3, adding $v$ adds at most three long paths, so $F$ has at most $\ell + 5$ long paths.

By Lemma 4.7, $H$ has a unique cycle $Q$. With $n = 2\ell + 1$, there are at least $\ell + 2$ vertices in $Q$. Since $H$ is disconnected, the component $H'$ of $H$ containing $Q$ has at most $\ell - 2$ vertices outside $Q$. Hence every vertex of $H'$ has distance at most $\ell - 2$ from $Q$. From each vertex of $H'$ outside $Q$, we can follow a shortest path to $Q$ and then turn either direction along $Q$ to complete a long path. With at least $\ell + 2$ long paths in $Q$, the number of long paths in $H$ is at least $\ell + 2 + 2t$, where $t$ is the number of vertices of $H'$ outside $Q$.

Since $H'$ is unicyclic, $C \cap Q$ is a single path along $Q$. Since $C$ has diameter $2k + 1$, it shares at most $2k + 2$ vertices with $Q$. With $|V(C)| \in \{3k + 2, 3k + 3\}$, we obtain $t \geq k$ if $F$ is a spider, and $t \geq k + 1$ if $F$ is a spider plus one vertex. Thus $H$ has at least $\ell + 2 + 2k$ long paths in the former case and at least $\ell + 4 + 2k$ in the latter. With $k \geq 1$, in each case $H$ has more long paths than $F$, which is a contradiction since the long paths are cards in $D$.

**Case 2:** Each endpoint of $e$ is reached by at most one path from outside $C$. In this case, $F$ extends $C$ by at most two paths, one grown from each end of $e$. The path joining them in $C$ has $2k + 2$ vertices. Together, these paths form a path $P$ with $\ell + 2k + 2$ vertices. It contains only $2k + 2$ long paths, since the last $\ell$ vertices cannot start a path with $\ell + 1$ vertices. The vertices of $F$ outside $P$ are $\ell + 1 - (2k + 2)$ vertices of $C$. Each such vertex can start a long path only by traveling to $P$ and turning one way or the other along $P$. Hence the total number of long paths in $F$ is at most $2(\ell + 1) - (2k + 2)$, which equals $2\ell - 2k$.

In $H$, again $C$ lies in the component $H'$ containing $Q$. Any vertex $v \in V(H') - V(Q)$ is within distance $\ell - 1$ from $Q$. Hence from $v$ a long path can be followed to $Q$ and then along $Q$ in either direction. Also at least $n - \ell + 1$ long paths lie in $Q$. As before, $C$ shares
at most $2k + 2$ of its $\ell + 1$ vertices with $Q$, since $H$ is unicyclic. Hence the total number of long paths in $H$ is at least $(n - \ell + 1) + 2(\ell + 1 - 2k - 2)$, which equals $3\ell - 4k$.

Since $2k + 2 \leq n - \ell = \ell + 1$, we have $2k < \ell$, which is equivalent to $2\ell - 2k < 3\ell - 4k$. \qed

We may henceforth assume that all short cards have diameter $2k + 2$, so they are $k$-vines.

**Lemma 4.9.** When $D$ is ambiguous and $n = 2\ell + 1$, every reconstruction has $d + \ell$ $k$-central edges, every acyclic reconstruction $F$ has at least $1 + d + \ell$ $k$-centers, and every $\hat{k}$-vine in $F$ has at most $n - \ell$ vertices.

**Proof.** By Lemma 4.8, no card has diameter $2k + 1$. Hence every $k$-evine has at most $n - \ell - 1$ vertices, and by Lemma 2.10 the number $s'$ of $k$-central edges is the same in $F$ and $H$. By Lemma 4.7, $H$ has at least $d + \ell$ $k$-central edges. Hence $F$ also has $d + \ell$ $k$-central edges.

Since any short card has diameter $2k + 2$, it has a unique central vertex, and this vertex is a $k$-center. In every $k$-central edge in the card, the endpoint farther from the center of the card is also a $k$-center. Hence $F$ has at least $1 + d + \ell$ $k$-centers.

If $F$ contains a $\hat{k}$-vine with more than $n - \ell$ vertices, then we can delete vertices to obtain a $\hat{k}$-vine $B$ with $n - \ell + 1$ vertices (since $2\hat{k} \leq n - \ell$). A longest path in $B$ has $2k + 3$ vertices. Deleting a leaf of $B$ yields a card of $F$. Since no card has diameter $2k + 1$, $B$ has a leaf $v$ such that $B - v$ is a card $C$ that is a $\hat{k}$-vine. Let $z$ be the common center of $B$ and $C$.

Under the marking argument, vertices at distance $\hat{k}$ from $z$ are marked only by $k$-centers adjacent to $z$. Since all vertices of $C$ are within distance $\hat{k}$ of $z$, at most $d$ vertices of $C$ and the $\ell$ vertices of $V(F) - V(C)$ can be marked. Hence $F$ has at most $1 + d_C + \ell$ $k$-centers.

We conclude that equality holds, so $d_C = d$ and every vertex outside $C$ is marked. Now we ask where is $v$? Since $v \in V(B)$, it is within distance $\hat{k}$ of $z$; to be marked it must have distance exactly $\hat{k}$ from $z$, marked by a neighbor of $z$. Thus if $C$ has any leaf vertex $w$ that is not on one of the $d$ edge-disjoint paths from $z$ or the $z, v$-path, then replacing $w$ with $v$ in $C$ yields a card $C'$ with $d_{C'} > d$. By the choice of $d$, we conclude that $C$ is a spider with $d$ legs of length $\hat{k}$ and one leg of length $k$, and $v$ extends that leg to length $\hat{k}$.

All vertices outside $B$ have distance greater than $\hat{k}$ from $z$. By Lemma 4.6, $F$ is a spider and has at most $\ell + 3$ cards that are paths. The unique cycle $Q$ in $H$ contains at least $\ell + 2$ cards that are paths. With one more vertex in the component $H'$ of $H$ containing $Q$, we obtain $\ell + 4$ such cards and a contradiction. Since no short card has diameter $2k + 1$, we have $d \geq 2$, and the card $C$ has at least two paths of length $\hat{k}$ and one path of length $k$ from the center, edge-disjoint. Also $C$ and $Q$ each contain more than half of $V(H)$ and must intersect. Now the vertex of degree at least 3 in $C$ guarantees a vertex of $H'$ outside $Q$. \qed

**Lemma 4.10.** When $D$ is ambiguous and $n = 2\ell + 1 \geq 7$, in any non-cyclic reconstruction $H$ no $\hat{k}$-vine has more than $n - \ell$ vertices.
Proof. By Lemma 4.9, every reconstruction has \(d + \ell \) \(k\)-central edges, where \(d\) is the maximum of \(d_C\) over short cards \(C\). By Lemma 4.7, \(H\) has a unique cycle, \(Q\).

Let \(H'\) be the component of \(H\) containing \(Q\). Every card has \(\ell + 1\) vertices and hence intersects \(Q\), so \(H'\) contains a short card \(C\). Since \(H\) is unicyclic, \(C \cap Q\) is connected. Since \(C\) has diameter \(2k + 2\), we conclude that \(C\) shares at most \(2k + 3\) vertices with \(Q\).

Now \(|V(H')| \geq (n - \ell + 1) + (n - \ell) - (2k + 3) = n - (2k + 1)\). Thus at most \(2k + 1\) vertices lie outside \(H'\). If some \(k\)-center lies outside \(H'\), then there is only one, and its component is a path with \(2k + 1\) vertices.

Each vertex on \(Q\) is a \(k\)-center, and each edge on \(Q\) is a \(k\)-central edge. For every \(k\)-center \(v\) in \(H' - V(Q)\), the edge leaving \(v\) on the path to \(Q\) is a \(k\)-central edge (and the endpoint farther from \(Q\) in any \(k\)-central edge of \(H' - V(Q)\) is a \(k\)-center). Thus the difference between the numbers of \(k\)-centers and \(k\)-central edges is the same in \(H'\) as in \(Q\), where it is 0.

We proved in Lemma 4.9 that \(F\) and \(H\) have the same number of \(k\)-central edges and that \(F\) has more \(k\)-centers than \(k\)-central edges. We also know from Corollary 2.6 that \(F\) and \(H\) have the same number of \(k\)-centers. Hence \(H\) also has more \(k\)-centers than \(k\)-central edges. This requires \(H = H' + P_{2k+1}\), with \(C\) and \(Q\) sharing \(2k + 3\) vertices in \(H'\).

Now consider in \(H\) a \(\hat{k}\)-vine \(B\) with center \(v\). Since \(B\) is connected with at least \(2\hat{k} + 1\) vertices, it must lie in \(H'\) and omit the \(2k + 1\) vertices of the outside path. Since \(H'\) is unicyclic, \(B \cap Q\) is connected. Being a tree with diameter \(2\hat{k}\), the tree \(B\) contains at most \(2\hat{k} + 1\) vertices among the \(\ell + 2\) vertices of \(Q\). Therefore, \(B\) omits at least \(\ell + 2 - (2k + 3)\) vertices of \(H'\) and \(2k + 1\) vertices outside \(H'\), which means \(B\) has at most \(n - \ell\) vertices. \(\square\)

**Theorem 4.11.** For \(n \geq 2\ell + 1 \geq 7\), the family of \(n\)-vertex acyclic graphs is \(\ell\)-recognizable.

Proof. By Theorem 3.6, we may assume \(n = 2\ell + 1\). By Lemmas 4.9 and 4.10, in every reconstruction no \(\hat{k}\)-vine has more than \(n - \ell\) vertices. Thus all \(\hat{k}\)-vines are seen in the deck. The deck then provides the number of maximal \(\hat{k}\)-vines with \(n - \ell\) vertices, and none are larger. Also any reconstruction has girth at least \(n - \ell + 1\), which by Lemma 4.5 is at least \(2\hat{k} + 2\). Hence by Corollary 2.6 \(\mathcal{D}\) determines the number of \(\hat{k}\)-centers in any reconstruction.

Any short card \(C\) has radius \(\hat{k}\) and diameter \(2\hat{k}\), so it is a \(\hat{k}\)-vine with a unique center \(z\). We now modify the marking argument of Definition 3.1 and Lemma 3.2 so that in \(F\) each \(\hat{k}\)-center \(x\) other than \(z\) marks a vertex at distance \(\hat{k}\) from \(x\) along an extension of the \(z, x\)-path in \(F\). Since \(C\) has radius \(\hat{k}\), the marked vertices are outside \(C\), and a vertex can only be marked by one \(\hat{k}\)-center. Hence the number of \(\hat{k}\)-centers in \(F\) is at most \(1 + \ell\).

However, every vertex on a cycle in \(H\) is a \(\hat{k}\)-center, since \(2\hat{k} \leq n - \ell\), so the number of \(\hat{k}\)-centers in \(H\) is at least \(n - \ell + 1\), which equals \(\ell + 2\). This contradicts that \(F\) and \(H\) have the same number of \(\hat{k}\)-centers and completes the proof. \(\square\)
References


