The Number of Dependent Arcs in an Acyclic Orientation

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Let G be a graph with n nodes, e edges, chromatic number χ and girth g. In an acyclic orientation of G, an arc is *dependent* if its reversal creates a cycle. It is well known that if $\chi < g$, then G has an acyclic orientation without dependent arcs. Edelman showed that if G is connected, then every acyclic orientation has at most $e - n + 1$ dependent arcs. We show that if G is connected and $\chi < g$, then G has an acyclic orientation with exactly d dependent arcs for all $d \leq e - n + 1$. We also give bounds on the minimum number of dependent arcs in graphs with $\chi \geq g$.

Given an acyclic orientation of a graph, Edelman (as quoted in West [**6**]) defined an arc to be *dependent* if its reversal creates a cycle. We consider the number of dependent arcs in an acyclic orientation of a graph. Given a graph G, let $n(G)$ be the number of nodes, let $e(G)$ be the number of edges, and let $d_{\max}(G)$ be the maximum number of dependent arcs over all acyclic orientations of G. Edelman proved the following.

Theorem 1. *If G is a connected graph, then $d_{\max}(G) = e(G) - n(G) + 1$.*

For a connected graph G, an acyclic orientation with $e(G) - n(G) + 1$ dependent arcs can be obtained by orienting edges away from the root of a depth-first search tree. Applying Theorem 1 to each component of a graph answers the question for all graphs.

Corollary 2. *If G is a graph with k components, then $d_{\max}(G) = e(G) - n(G) + k$.*

Let $d_{\min}(G)$ be the the minimum number of dependent edges in an acyclic orientation of a graph G. A *cover graph* is the underlying graph of the Hasse diagram of a partially ordered set. It is easy to show that having an orientation with no dependent edges is equivalent to being a cover graph (see [**5**] for example). Unfortunately, recognition of cover graphs is NP-complete [3].

Theorem 3 gives a well-known sufficient condition for a graph to be a cover graph (see [**5**] for example). An *m-coloring* of a graph G is a labeling $c: V(G) \to \{0, 1, \ldots, m-1\}$ such that $c(x) \neq c(y)$ for each edge xy. The *chromatic number* $\chi(G)$ is the minimum m such that G has an m-coloring. The *girth* $g(G)$ is the minimum length of a cycle in a graph G.

Theorem 3. *If $\chi(G) < g(G)$, then $d_{\min}(G) = 0$.*

Proof. Let c be an m-coloring of G with $m < g(G)$. Orient each edge from the lower colored node to the higher. Clearly this orientation is acyclic. Suppose it has a dependent arc. Let C be the cycle formed by reversing that arc. Then with one exception, the colors must lie in increasing order on C. Thus C has at most m edges, which contradicts the hypothesis. □

The *interpolation question* is the question of whether G has an acyclic orientation with exactly d dependent arcs for every d with $d_{\min}(G) \leq d \leq d_{\max}(G)$. West [**6**] proved constructively that the answer is yes for complete bipartite graphs. Section 1 extends this result to every graph G with $\chi(G) < g(G)$. Section 2 presents results on $d_{\min}(G)$ for graphs G with $\chi(G) \geq g(G)$.

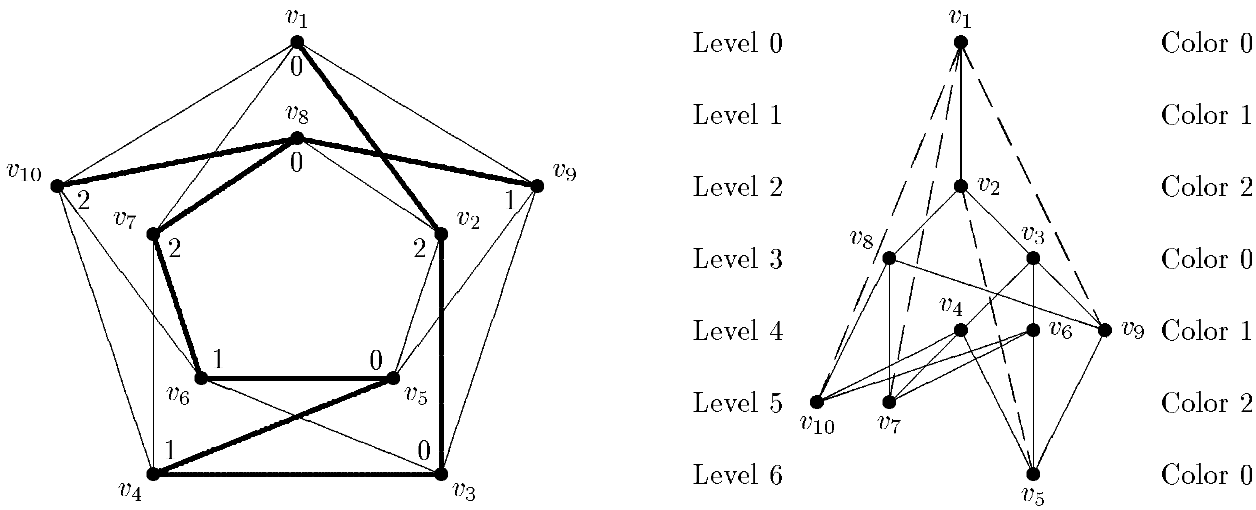

Figure 1. The construction from Theorem 4 giving an acyclic orientation with precisely four dependent arcs for a graph with girth 4 and chromatic number 3.

1 The Interpolation Problem

Let c be an m-coloring of a connected graph G, where $m = \chi(G)$. A *color-first* tree T of G with respect to c is constructed as follows: Choose some node v with $c(v) = 0$. This node is at level 0. We construct level k recursively by adding all unreached nodes x with $c(x) \equiv k \pmod{m}$ that are adjacent to nodes already reached, together with one edge from each such x to a node at a previous level. Note that a level may be empty. This procedure generates a spanning tree (if $\chi(G) = 2$, it generates the familiar breadth-first search tree). As in a breadth-first tree, there are no edges of $G - T$ between nodes at the same level, since each level contains only nodes of a single color.

Theorem 4. *Let G be a connected graph. If $\chi(G) < g(G)$ and $0 \le d \le e(G) - n(G) + 1$, then G has an acyclic orientation with exactly d dependent arcs.*

Proof. Suppose $0 \le d \le e(G) - n(G) + 1$; we construct an orientation with exactly d dependent arcs.

Step 1. Index the nodes as $v_1, v_2, \ldots, v_n$ in the order of a depth-first search. Let T be the tree that results from the depth-first search.

Step 2. Find an m-coloring c with $m < g(G)$. Relabel the colors so $c(v_1) = 0$.

Step 3. Label each edge $v_i v_j$ of $G - T$ by the pair (i, j) where $i < j$. Lexicographically order the $e(G) - n(G) + 1$ edges of $G - T$ by their labels. Let G' be the subgraph obtained from G by removing the first d edges in this ordering.

Step 4. Construct a color-first tree T' of G' starting with v_1.

Step 5. Orient all edges of G from lower level to higher level of T'. Since all paths in the orientation pass through nodes of increasing level, the orientation is acyclic.

This construction when $d = 4$ is illustrated in Figure 1 for a graph with chromatic number 3 and girth 4. The right side shows Steps 1 and 2. The nodes receive depth-first labeling $v_1, v_2, \ldots, v_{10}$ and a 3-coloring in which v_1 has color 0. The depth-first tree T appears as the thick edges. Step 3 sorts the edges not in T lexicographically: v_1v_7, v_1v_9, v_1v_{10}, v_2v_5, v_2v_8, ... Removing the first four of these, the left side shows the level sets of the color-first tree constructed in Step 4. After replacing the removed edges (dashed lines), the desired acyclic orientation is found by orienting all edges downwards.

We claim that the orientation constructed in Step 5 has exactly d dependent arcs. Let edge xy be in G' where x is at level i, and y is at level j with $i < j$. Since y is adjacent to a node at level i in Step 5, it was placed into a level set when its color was next considered. Since there are only m colors, $j - i < m$. Suppose xy is dependent. Then there is a path $x = z_1, z_2, \ldots, z_k = y$ in the orientation. Since the level increases by at least one for each arc in the path, $k - 1 \le j - i$. Thus $k \le m < g(G)$, which is not possible since the length of the cycle $z_1, z_2, \ldots, z_k$ is k. Hence no edge of G' is a dependent arc in this orientation.

Now let v_iv_j be an edge in $G - G'$ with $i < j$. In the depth-first search, v_i is an ancestor of v_j, and the unique v_i, v_j-path in T passes through v_i. Since every edge in $G - T$ with an endpoint that is an ancestor in T of v_i was removed to form G', there is a unique v_1, v_i-path in G'. Thus every v_1, v_j-path in G' passes through v_i, and v_i is an ancestor of v_j in T'. All edges of the unique v_i, v_j-path in T' are oriented away from v_i, as is the edge v_iv_j. Reversing the orientation of the edge v_iv_j creates a cycle in G, so v_iv_j is a dependent arc. Hence every edge of $G - G'$ is a dependent arc in this orientation.

Therefore, the orientation has precisely d dependent arcs. □

Applying Theorem 4 to each component of a graph gives a generalization that holds for all graphs G with $\chi(G) < g(G)$.

Corollary 5. *Suppose G is a graph with k components. If $\chi(G) < g(G)$ and $0 \le d \le e(G) - n(G) + k$, then G has an acyclic orientation with d dependent arcs.*

2 On the Minimum Number of Dependent Arcs

Theorem 3 shows that if the chromatic number of a graph is less than its girth, then the graph has an orientation with no dependent arcs. Theorem 6 gives a partial result for graphs with chromatic number at least as large as girth.

Theorem 6. *If G is a graph G with chromatic number m, then $d_{\min}(G) \le \frac{m-2}{m} e(G)$.*

Proof. Let c be an m-coloring of G. For $0 \le j, k < m$, let e_{jk} be the number of edges between nodes colored j and nodes colored k. Note that $e_{jk} = e_{kj}$ and $e_{jj} = 0$. Given a permutation π of $\{0, 1, \ldots, m-1\}$, let $e(\pi) = e_{\pi(0)\pi(1)} + e_{\pi(1)\pi(2)} + \ldots + e_{\pi(m-2)\pi(m-1)}$. Summing $e(\pi)$ over all permutations counts each edge $2(m-1)!$ times, since there are $m-1$ pairs of adjacent positions in π in which its color classes can appear, $(m-2)!$ ways to permute the remaining classes, and two ways to order the colors on the edge. Since $\sum_\pi e(\pi) = 2(m-1)! e(G)$, there must be some permutation π with $e(\pi) \ge \frac{2}{m} e(G)$. Orient the edges of G toward the node

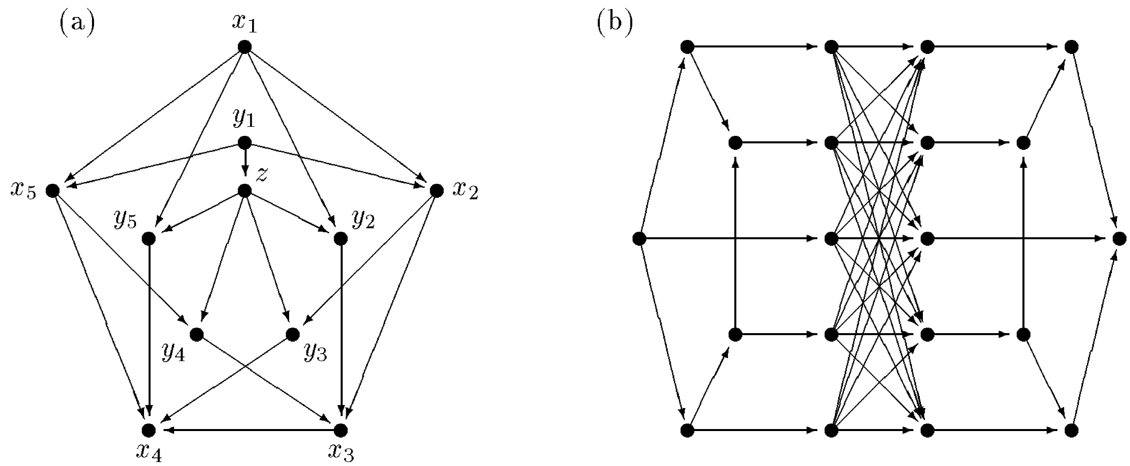

Figure 2. Each of these graphs has chromatic number 4 and girth 4. Graph (a) is Grötzsch's graph, which by Theorem 7 is not a cover graph. The orientation shown has only one dependent arc (the arc from x_5 to x_4). Graph (b) is a cover graph. The orientation shown has no dependent arcs.

whose color appears later in π. This orientation is clearly acyclic. Furthermore, reversing an arc xy whose endpoints have colors consecutive in π does not form a cycle. Hence this orientation has at least $\frac{2}{m}e(G)$ arcs that are not dependent. □

All acyclic orientations of a complete graph on k nodes are isomorphic and have $\binom{k-1}{2}$ dependent edges. Hence Theorem 6 is best possible for graphs of girth 3.

The first case where the best upper bound for $d_{\min}(G)$ in terms of chromatic number and girth is not known is $\chi(G) = g(G) = 4$. The smallest triangle-free graph with chromatic number 4 is Grötzsch's graph (see Figure 2(a)). Theorem 7 shows that Grötzsch's graph has no acyclic orientation without dependent arcs. We give a short combinatorial proof of this well-known result (see [**4**] and [**5**] for other approaches).

Theorem 7. *Grötzsch's graph is not a cover graph.*

Proof. Grötzsch's graph has ten 4-cycles. In its depiction in Figure 2, these 4-cycles appear in the plane without crossings, so the orientation of an edge in (with respect to) a 4-cycle containing it can be specified as clockwise or counterclockwise. Each edge lies in two 4-cycles. In every orientation, each of the five outer arcs has the same orientation in both of its 4-cycles, while each of the other fifteen arcs is clockwise in one of its 4-cycles and counterclockwise in the other.

Suppose Grötzsch's graph has an orientation with no dependent arcs. In such an orientation, two arcs of each 4-cycle have a clockwise orientation. This yields a count of twenty clockwise orientations. Each of the five outer arcs contributes an even number (either 0 or 2) of clockwise orientations, while each of the other fifteen arcs contributes one clockwise orientation. Hence the number of clockwise orientations is always odd and cannot equal twenty. □

The orientation in Figure 2(a) has only one dependent arc; thus $d_{\min}(G) = 1$ for Grötzsch's graph. Some graphs with chromatic number 4 and girth 4 can be oriented so that there are no dependent arcs; Figure 2(b) is such an example.

It is interesting to consider the minimum percentage of dependent arcs in an orientation of a graph. Given m and k, let $r_{m,k}$ be the supremum of $d_{\min}(G)/e(G)$ over all graphs G with $\chi(G) = m$ and $g(G) = k$. Theorem 3 shows that $r_{m,k} = 0$ if $m < k$. Theorem 6 shows that $r_{m,k} \leq \frac{m-2}{m}$ for all m and k, and equality holds when $k = 3$. Theorems 6 and 7 show that $\frac{1}{20} \leq r_{4,4} \leq \frac{1}{2}$.

Determining $r_{m,k}$ when $m \geq k \geq 4$ requires studying graphs with $\chi(G) \geq g(G) > 4$ that are not cover graphs. In general, deciding whether a graph is a cover graph is NP-complete [**4**]. Probabilistic arguments show that most graphs of each girth are not cover graphs [**1**,**4**], but few specific examples of girth greater than 4 are known. Erdős [**2**] proved the existence of graphs with $\chi(G) = m$ and $g(G) = k$ for all m and k. Constructions of such graphs were obtained in [**3**] and in [**4**], but these examples are quite large even for $m = k = 5$.

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