Acquisition-extremal Graphs

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Abstract

A *total acquisition move* in a weighted graph G moves all weight from a vertex u to a neighboring vertex v, provided that before this move the weight on v is at least the weight on u. The *total acquisition number*, $a_t(G)$, is the minimum number of vertices with positive weight that remain in G after a sequence of total acquisition moves, starting with a uniform weighting of the vertices of G. For $n \geq 2$, Lampert and Slater showed that $a_t(G) \leq \frac{n+1}{3}$ when G has n vertices, and this is sharp. We characterize the graphs achieving equality: $a_t(G) = \frac{|V(G)|+1}{3}$ if and only if $G \in \mathcal{T} \cup \{P_2, C_5\}$, where $\mathcal{T}$ is the family of trees that can be constructed from P_5 by iteratively growing paths with three edges from neighbors of leaves.

1 Introduction

Consider an army dispersed among many cities. We wish to consolidate the troops. Troops move only to neighboring occupied cities, and the number of troops in a move cannot exceed the number already at the destination. Can the troops all move to one city?

We model such situations using graphs with vertex weights. Initially, each vertex has weight 1. An *acquisition move* transfers some weight from a vertex u to a neighboring vertex v, provided that before this move the weight on v is at least the weight on u. The total weight is preserved. Lampert and Slater [1] introduced acquisition in graphs, using *total acquisition moves* that transfer all of the weight from a vertex to a neighbor. Other models (see [3, 4]) involve "partial" acquisition moves, in which only a portion of the weight on a vertex is transferred to a neighbor. In this paper we study only the total acquisition model and hence omit the adjective "total" when referring to acquisition moves.

We refer to a succession of acquisition moves as an *acquisition protocol*. The *residual set* left by an acquisition protocol is the set of vertices that remain with positive weight. The *total acquisition number* (or simply *acquisition number*), written $a_t(G)$, is the minimum

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possible size of a residual set left by an acquisition protocol (starting from the distribution with weight 1 on each vertex). An acquisition protocol on a graph G is *optimal* if it leaves a residual set of size $a_t(G)$. Lampert and Slater [1] proved that, for $n \geq 2$, the maximum of $a_t(G)$ over connected n-vertex graphs is $\lfloor \frac{n+1}{3} \rfloor$. In this paper we characterize the graphs G such that $a_t(G) = \frac{|V(G)|+1}{3}$. Our analysis of the acquisition number of trees yields another proof of the result of Lampert and Slater. For additional results about total acquisition and related acquisition parameters, see [2, 3, 4].

In our analysis we use three related families of trees. We develop characterizations of these families that may be of independent interest. The most important family, called $\mathcal{T}$, is the family of trees that can be obtained from P_5 by iteratively growing paths of length 3 from neighbors of leaves (here P_n denotes the path with n vertices). We will show that $\mathcal{T} \cup \{P_2, C_5\}$ is the complete family of graphs G such that $a_t(G) = \frac{|V(G)|+1}{3}$. The family $\mathcal{T}$ was introduced in [2], where the acquisition number of such graphs was computed. Thus our task is to show that $a_t(G) < \frac{|V(G)|+1}{3}$ when $G \notin \mathcal{T} \cup \{P_2, C_5\}$.

The related family $\mathcal{S}$ is the family of trees that can be obtained from P_4 by iteratively growing paths of length 2 from neighbors of leaves. Finally, the family $\mathcal{R}$ is the family of trees that can be obtained from P_3 or P_4 by iteratively growing paths of length at most 2 from neighbors of leaves.

In Section 2, we develop properties of trees in $\mathcal{T}$. These are used in Section 3 to bound acquisition numbers of trees in or "near" $\mathcal{T}$. Section 4 obtains properties of the related families $\mathcal{S}$ and $\mathcal{R}$. These and the properties of $\mathcal{T}$ are used in Section 5 to restrict the minimal counterexamples to the statement that $\mathcal{T} \cup \{P_2\}$ contains all the extremal trees, and then the characterization of the extremal graphs is completed in Section 6.

2 Properties of Trees in $\mathcal{T}$

Trees are critical for acquisition problems because deleting edges cannot reduce a_t, so the maximum of a_t over n-vertex graphs occurs on trees. A *leaf* in a tree is a vertex of degree 1; a 1-vertex tree has no leaf.

Definition 2.1. Starting with P_5, let $\mathcal{T}$ be the family of trees constructed by iteratively growing paths with three edges from neighbors of leaves. The operation of growing such a path from a neighbor of a leaf is *augmentation*.

Although P_5 can be obtained by augmenting P_2, we exclude P_2 from $\mathcal{T}$ for technical reasons (for example, its leaves are neighbors rather than being separated by distance at least 4). Define a *core* vertex in a tree to be a vertex having distance at least 2 from all leaves. We introduce seven properties that are easy to show are true of all trees in $\mathcal{T}$; in fact, Properties C–F characterize $\mathcal{T}$. The *order* of a graph is the size of its vertex set.

Definition 2.2. The following are properties that a tree T may satisfy.
Property A: The distance between any two leaves of T is at least 4.
Property B: The number of leaves in T is $\frac{|V(T)|+1}{3}$.
Property C: T has at least one core vertex.
Property D: If v is a core vertex in T, then $d_T(v) = 2$.
Property E: If v is a core vertex in T, then both components of $T - v$ lie in $\mathcal{T} \cup \{P_2\}$.
Property F: If w is a neighbor of a core vertex in T, then $T - w$ consists of an isolated vertex plus components whose orders are divisible by 3.
Property G: Every core vertex and every neighbor of a leaf lies on a path of length 4 joining two leaves.

Lemma 2.3. *Each property in Definition 2.2 holds for every tree in $\mathcal{T}$.*

Proof. Each property holds for P_5 and is preserved by augmentation. Note that each augmentation preserves core vertices and introduces three new vertices: a new leaf, its neighbor, and a new core vertex. Properties D–G hold for the new core vertex and its new neighbor by construction. They continue to hold for old vertices because the augmentation just enlarges the tree; in particular, it augments one component of the relevant vertex-deleted subgraph (adding three vertices) or just adds a copy of P_3 as such a component (when w is the vertex where the augmentation occurs. □

Given a tree T, let T' be the tree obtained from T by deleting its leaves, and let $T'' = (T')'$.

Lemma 2.4. *If T is a tree satisfying Properties C–F of Definition 2.2, then every leaf x of T'' is a core vertex of T and belongs to a four-vertex path $\langle u, y, x, w \rangle$ such that u is a leaf of T, y and x have degree 2 in T, and w has a leaf neighbor in T.*

Proof. Let x be a leaf of T'' (so T'' is not just a single vertex), and let w be the neighbor of x in T''. Since x is a leaf of T'', it must have a neighbor in T' other than w, and all such vertices are leaves in T'. Similarly, at least one such neighbor y is not a leaf in T, but all neighbors of y in T other than x are leaves. Let u be a leaf of T adjacent to y.

Suppose that x is not a core vertex of T. This gives x a leaf neighbor u' in T (see Figure 1). Note that u and u' are leaves of T connected by the path induced by $\{u, y, x, u'\}$.

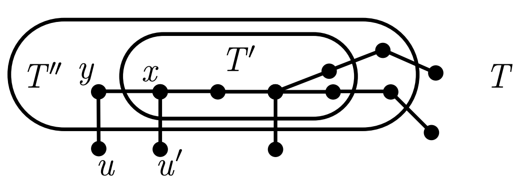

Figure 1: Ensuring that leaves of T'' are core vertices of T.

By Property C, T has some core vertex v. By the definition of core vertex, every core vertex of T remains in T''. Since x is not a core vertex, we have $v \neq x$, so $v \notin \{u, y, x, u'\}$. Hence these four vertices lie in the same component Q of $T - v$. By Property E, $Q \in \mathcal{T}$. However, the distance between u and u' is 3, so Q fails Property A, which violates Lemma 2.3. The contradiction implies that x is a core vertex of T.

By Property D, $d_T(x) = 2$; let y and w be the neighbors of x in T. By Property F, y and w each have exactly one leaf neighbor in T; call them u and u^*, respectively. Since x is a leaf in T'', exactly one of the two neighbors of x in T is a leaf in T'; we may let it be y (see Figure 2). Since y is a leaf of T', the component of $T - x$ containing y is a non-trivial star; by Properties E and A, it is P_2. Now, $\{u, y, x, w\}$ induces the desired path containing x, since u is a leaf and w has a leaf neighbor. □

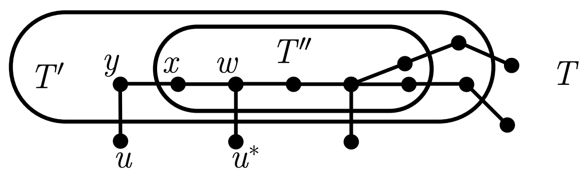

Figure 2: Finding the desired path through a leaf of T''.

The fact that $\mathcal{T}$ is characterized by Properties C–F will be used in Section 5.

Lemma 2.5. *The family of trees satisfying Properties C–F of Definition 2.2 is $\mathcal{T}$.*

Proof. Let $\mathcal{T}^*$ be the family of trees satisfying Properties C–F; we show that $\mathcal{T}^* \subseteq \mathcal{T}$. Note that P_5 is the unique smallest tree in $\mathcal{T}^*$. Consider a tree T in $\mathcal{T}^*$. It suffices to show that $T = P_5$ or T is obtained from a tree in $\mathcal{T}$ by an augmentation.

By Property C, T has a core vertex, so T'' has at least one vertex. If T'' has only one vertex, v, then every vertex of T is within distance 2 of v. By Property D, T consists of two stars sharing v as a leaf. By Property E or F, each of those stars is P_3, and $T = P_5$. Hence we may assume that $T \neq P_5$ and T'' has more than one vertex.

Now let v be a leaf of T'', and let $\langle u, y, v, w\rangle$ be the path guaranteed by Lemma 2.4, with u being a leaf of T and w having a leaf neighbor (see Figure 3). Let $\widehat{T} = T - \{u, y, v\}$; note that $\widehat{T}$ is a component of $T - v$. By Property D, $\widehat{T} \in \mathcal{T} \cup \{P_2\}$. Since $T \neq P_5$, we have $\widehat{T} \neq P_2$. Also, T arises from $\widehat{T}$ by an augmentation. Hence $T \in \mathcal{T}$, by the definition of $\mathcal{T}$.□

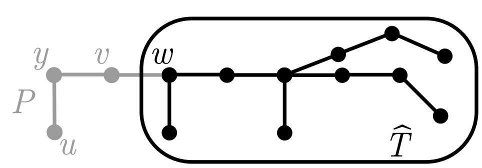

Figure 3: Expressing T as an augmentation of a tree in $\mathcal{T}$.

By definition, each tree in $\mathcal{T}$ is constructed via augmentations from *some* subgraph isomorphic to P_5. We show next that *any* four-edge path joining leaves of T may serve as the starting path in constructing T via augmentations. Thus by Property G, any non-leaf vertex of T may be taken as a vertex of the initial copy of P_5 from which T is constructed.

Lemma 2.6. *If P is a copy of P_5 joining leaves of a tree T in $\mathcal{T}$, then T can be constructed from P via augmentations.*

Proof. We use induction on $|V(T)|$; the claim holds trivially for P_5. If $T \neq P_5$, then T'' has at least two leaves. By Lemma 2.4, T contains two paths of length 3 attached to the rest of T at neighbors of leaves of T. The path P must be edge-disjoint from one of these, which we call P'. Let v' be the vertex of P' adjacent to w', where w' is the end of P' that is not a leaf of T (see Figure 4). Let $\widehat{T}$ be the subtree of T obtained by deleting the vertices of P' other than w'; note that $\widehat{T}$ is a component of $T - v'$. Since v' is a core vertex of T, Property E implies $\widehat{T} \in \mathcal{T}$. Now P is a copy of P_5 joining leaves of $\widehat{T}$. By the induction hypothesis, $\widehat{T}$ can be constructed from P via augmentations, and then T is an augmentation of $\widehat{T}$. □

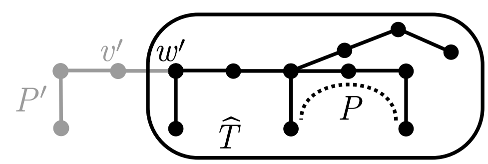

Figure 4: T is the result of augmenting $\widehat{T}$ by growing the path P' shown in gray.

3 Acquisition Numbers of Trees in or Near $\mathcal{T}$

The main step in characterizing the graphs G such that $a_t(G) = \frac{|V(G)|+1}{3}$ is to characterize such trees. When G is a tree, we show that G has this property if and only if $G \in \mathcal{T} \cup \{P_2\}$. Sufficiency was shown in [2], where the family was defined. For completeness of our characterization of the extremal graphs, we include the short proof of this.

Lemma 3.1. *Let x and y be vertices in a tree T. If the path joining x and y in T contains a vertex v of degree 2 not adjacent to x or y, then the initial weights from x and y cannot reach a common vertex via acquisition moves.*

Proof. For the weight from x and y to reach a common vertex, the initial weight from x or y must reach v by some acquisition move. The first acquisition move involving v transfers weight 1 to or from it, so this move can only transfer the original weight from a vertex in $N[v]$. After this move, some vertex in $N[v]$ has weight 0. Now a vertex of weight 0 separates the weights originally on x and y; this prevents them from combining. □

Lemma 3.2. *If $T \in \mathcal{T} \cup \{P_2\}$, then $a_t(T) = \frac{|V(T)|+1}{3}$.*

Proof. Clearly $a_t(P_2) = 1$. For the general upper bound, $a_t(P_5) = 2$, and the weight on the three new vertices in each augmentation can be acquired by the central vertex among them.

To prove the lower bound for $T \in \mathcal{T}$, it suffices to show that between any two leaves u and u' there is a core vertex adjacent to neither of them. Property B (guaranteeing $\frac{|V(T)|+1}{3}$ leaves) and Lemma 3.1 will then complete the proof.

We use induction on $|V(T)|$; the claim is immediate for P_5. By Properties C–E, T has a core vertex v such that $T - v$ has two components, both in $\mathcal{T} \cup \{P_2\}$. If u and u' are in the same component, then it is not a copy of P_2, since both u and u' are leaves of T. Hence the induction hypothesis applies in this case. If u and u' are in distinct components of $T - v$, then v is the desired vertex. □

Most of the rest of this paper is devoted to proving the converse of Lemma 3.2 in the universe of trees. In the remainder of this section, we study trees having a vertex whose deletion leaves components that are isolated vertices, isolated edges, or trees in $\mathcal{T}$. Before studying the general case of such trees, we need a preliminary lemma. Let a *near-optimal acquisition protocol* for T be an acquisition protocol leaving a residual set of size $a_t(T) + 1$.

Lemma 3.3. *Given $T \in \mathcal{T}$, recall that $a_t = \frac{|V(T)|+1}{3}$.*
(**a**) *If T^+ arises from T by adding a pendant edge at a core vertex v, then $a_t(T^+) \le \frac{|V(T^+)|-3}{3}$.*
(**b**) *If v is a core vertex in T, then some optimal acquisition protocol leaves weight 4 on v, and for $j \in \{1, 2, 3\}$ there is a near-optimal acquisition protocol leaving weight j on v.*
(**c**) *If w is a neighbor of a leaf in T, then some near-optimal acquisition protocol leaves weight 1 on v, and for $j \in \{2, 3, 4\}$ there is an optimal acquisition protocol leaving weight j on w.*

Proof. By inspection, such protocols exist when $T = P_5$. For $T \ne P_5$, Property G provides a four-edge path P joining leaves of T that contains the specified vertex (see Figure 5). By Lemma 2.6, T can be constructed from P via augmentations. Since $P \cong P_5$, we can apply the acquisition protocol for P_5 on P (or for P_5^+ on P^+) that gives the specified vertex the desired weight and leaves the desired size of residual set relative to P_5. Then the weight on each three-edge path added during an augmentation can be acquired by its central vertex, adding 1 to the size of the residual set and 3 to $|V(T)|$. □

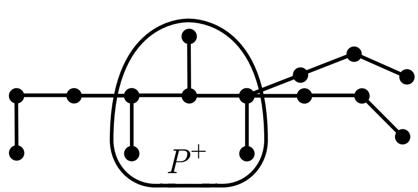

Figure 5: The tree T^+ can be constructed by augmenting P^+

Lemma 3.4. *Let T be a tree with a vertex x such that $T - x$ consists of p isolated vertices, q isolated edges, k components in $\mathcal{T}$ in which the neighbor of x is not a leaf, and possibly other components whose acquisition number is at most one-third of the number of vertices. If $p + q + k \geq 3$, then $a_t(T) \leq \frac{|V(T)|}{3}$.*

Proof. We may assume that $T - x$ has only the special $p + q + k$ components mentioned, since running protocols independently on the extra components will maintain the inequality. When $p \geq 1$, we begin an acquisition protocol by letting x acquire all weight from its leaf neighbors. Next, x can acquire all weight from the q isolated edges in $T - x$.

Case 1: $p \geq 1$ *and* $p + 2q \geq 3$. We have now given x weight at least 4. By definition, every non-leaf vertex of a tree is a core vertex or has a leaf neighbor. Hence in each of the k components of $T - x$ in $\mathcal{T}$, Lemma 3.3 guarantees an optimal acquisition protocol leaving weight 4 on the neighbor of x. After these protocols, positive weight remains on $\frac{|V(T)|-1-(p+2q)+k}{3} + 1$ vertices. Since x has weight at least 4, x can acquire the remaining weight on its neighbors. This reduces the size of the residual set by k, yielding $a_t(T) \leq \frac{|V(T)|-p-2q-2k+2}{3}$. Since $p + 2q \geq 3$, we obtain $a_t(T) \leq \frac{|V(T)|-1}{3}$.

Case 2: $p \geq 1$ *and* $p+2q \leq 2$. This case requires $q = 0$ and $k \geq 1$, since $p+q+k \geq 3$. We have given x weight at least 2. In one nontrivial component of $T-x$, Lemma 3.3 guarantees a near-optimal (or optimal) acquisition protocol leaving weight 2 on the neighbor of x, which can then be acquired to x. In the remaining $k-1$ nontrivial components of $T-x$, Lemma 3.3 guarantees optimal acquisition protocols leaving weight 4 on the neighbor of x. Computing as in Case 1, we have $a_t(T) \leq \frac{|V(T)|-1-p+(k+3)}{3} + 1 - k \leq \frac{|V(T)|-p+5-2k}{3}$.

If $k \geq 2$ or $p \geq 3$, then $a_t(T) \leq \frac{|V(T)|}{3}$. Otherwise, $p = 2$ and $k = 1$. In that case, after x acquired the weight from the neighboring leaves, Lemma 3.3 guarantees an optimal acquisition protocol in the one component of $T - x$ in $\mathcal{T}$, leaving weight 4 on the neighbor of x. Now the weight from x moves to that vertex, leaving a residual set of size $\frac{|V(T)|-2}{3}$.

Case 3: $p = 0$. Let A_1 be a smallest component of $T - x$, and let A_2 be a smallest component of $T - x$ after deleting A_1. By Lemma 3.3 (or by doing nothing when $A_1 = P_2$), there is a near-optimal protocol on A_1 that leaves weight 1 on the neighbor y of x in A_1; perform this and then acquire the weight from y to x. Now let z be the neighbor of x in A_2. Since $A_2 \in \mathcal{T} \cup \{P_2\}$, there is a protocol on A_2 leaving weight 2 on z that is optimal if z has a leaf neighbor in A_2 and is near-optimal if z is a core vertex of A_2. After this protocol, x acquires the weight from z.

In each of the remaining $q+k-2$ components of $T-x$, an optimal protocol brings weight 2 or 4 to the neighbor of x. Since x now has weight 4, it can acquire the weight from all those neighbors to complete the protocol. Since we used near-optimal protocols on at most two components of $T - x$ and later reduced the residual set by $q + k$, the size of the residual set is now at most $1 + \frac{|V(T)|-1+(q+k)}{3} + 2 - (q + k)$, with equality only when z is a core vertex of A_2. Otherwise, $a_t(T) \leq \frac{|V(T)|+8-2(q+k)}{3} - 1 \leq \frac{|V(T)|-1}{3}$, since $q + k \geq 3$ by hypothesis.

Thus if $a_t(T) > \frac{|V(T)|-1}{3}$, then we may assume that $q+k=3$ and that z is a core vertex of A_2. Let A_2^+ be the subtree of T induced by $V(A_2) \cup \{x\}$ (see Figure 6). By Lemma 3.3(a), $a_t(A_2^+) \le \frac{V(A_2^+)-3}{3}$. Since A_1 and the remaining component of $T-x$ lie in $\mathcal{T} \cup \{P_2\}$, Lemma 3.2 yields $a_t(T) \le a_t(A_1) + a_t(A_2^+) + a_t(A_3) \le \frac{|V(A_1)|+1}{3} + \frac{|V(A_2^+)|-3}{3} + \frac{|V(A_3)|+1}{3} = \frac{|V(T)|-1}{3}$. □

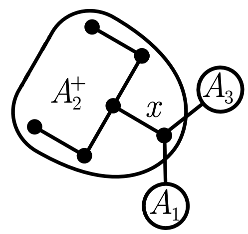

Figure 6: The last case of Lemma 3.3.

4 Two Related Families

To prove the converse of Theorem 3.2 we will also use two families defined similarly to $\mathcal{T}$. Starting with P_4, let $\mathcal{S}$ be the family of trees constructed by iteratively growing paths of length 2 from neighbors of leaves. Let $\mathcal{R}$ be the family of trees constructed from P_3 or P_4 by iteratively growing paths of length at most 2 from neighbors of leaves. The *corona* of a graph H, denoted $H \circ K_1$ is the graph obtained from H by adding one pendant edge to each vertex of H.

Lemma 4.1. *For a tree S with at least four vertices, the following are equivalent:*
(**a**) $S \in \mathcal{S}$,
(**b**) $S = S' \circ K_1$ *for some subtree* S' *of* S, *and*
(**c**) *each non-leaf vertex is adjacent to exactly one leaf.*

Proof. Each tree in $\mathcal{S}$ is iteratively constructed from P_4, and $P_4 = P_2 \circ K_1$. For a tree, the property of being a corona is preserved by appending a path of two edges to a neighbor of a leaf, so (a) implies (b). By the definition of corona, (b) implies (c).

To prove that (c) implies (a), we use induction on $|V(S)|$; note that P_4 is the smallest tree with at least four vertices satisfying (c), and $P_4 \in \mathcal{S}$. For a larger tree S satisfying (c), let S' be the tree obtained by deleting the leaves, and let w be a leaf in S'. Let u be the unique leaf of S adjacent to w and let $\widehat{S} = S - \{w, u\}$, as in Figure 7. Note that $\widehat{S}$ satisfies (c). By the induction hypothesis, $\widehat{S} \in \mathcal{S}$. Let v be the neighbor of w in S'. The tree S arises from $\widehat{S}$ by growing a two-edge path from v; hence $S \in \mathcal{S}$. □

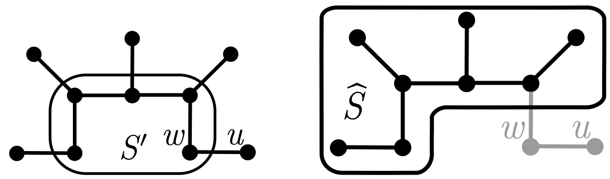

Figure 7: S' and $\widehat{S}$ are subtrees of S.

Lemma 4.2. *A tree R with at least three vertices is in $\mathcal{R}$ if and only if each vertex that is not a leaf is adjacent to at least one leaf.*

Proof. Both P_3 and P_4 have the specified property, and the property is preserved by growing a path of length at most 2 from a neighbor of a leaf. Hence the condition is necessary.

For sufficiency, we use induction on the number of leaves. The trees with two leaves satisfying the condition are P_3 and P_4, which lie in $\mathcal{R}$. Let R be a tree with more leaves satisfying the condition. If some non-leaf vertex w has more than one leaf neighbor, then R arises from a tree in $\mathcal{R}$ by growing a path of length 1 from w. If each non-leaf vertex has exactly one leaf neighbor, then $R \in \mathcal{S}$ by Lemma 4.1, and $\mathcal{S} \subseteq \mathcal{R}$ by definition. □

Core vertices play an important role in the characterization of trees in $\mathcal{T}$ given in Lemma 2.5. By Lemma 4.2, there are no such vertices in a tree in $\mathcal{R}$. In the proof of the converse of Lemma 3.2 in Sections 5 and 6, we will need to exclude counterexamples in $\mathcal{R}$. The two lemmas below will be useful.

Lemma 4.3. *If $S \in \mathcal{S}$, and S' is the result of deleting all leaves from S, then $a_t(S) \leq a_t(S')$.*

Proof. Begin an acquisition protocol by having each non-leaf acquire the weight from the unique leaf to which it is adjacent. Now, the vertices with positive weight are exactly the vertices in S', and they all have weight 2. Any optimal acquisition protocol for S' now leaves a residual set of size $a_t(S')$. □

Lemma 4.4 below gives a property of trees in $\mathcal{R}$ analogous to Property E of Definition 2.2.

Lemma 4.4. *If R is a tree in $\mathcal{R}$ and w is a vertex that is not a leaf, then $R - w$ consists of isolated vertices, isolated edges, and components in $\mathcal{R}$.*

Proof. The conclusion holds for P_3 and P_4, and is preserved by growing paths of length at most 2 from neighbors of leaves. □

5 Minimal Counterexamples

In the next section, we complete the proof that $a_t(G) \le |V(G)|/3$ when $|V(G)| \ge 3$ and $G \notin \mathcal{T}$. The proof that this holds for trees is inductive, showing that there is no minimal counterexample. In this section, we develop properties of a purported such counterexample. Throughout this section we assume that T is a smallest tree among those for which $a_t(T) \ge \frac{|V(T)|+1}{3}$ and $T \notin \mathcal{T}$ and $|V(T)| \ge 3$. Any smaller tree T_0 with at least three vertices satisfies $a_t(T_0) \le \frac{|V(T_0)|+1}{3}$, with equality only if $T_0 \in \mathcal{T}$.

Lemma 5.1. *Every minimal counterexample T has a core vertex (and hence is not in $\mathcal{R}$).*

Proof. The characterization of $\mathcal{R}$ in Lemma 4.2 implies that $\mathcal{R}$ is precisely the set of trees having at least three vertices but no core vertices. Hence it suffices to show $T \notin \mathcal{R}$.

For $T \in \mathcal{S}$, obtain T' from T by deleting all leaves. Lemma 4.3 and minimality yield $a_t(T) \le a_t(T') \le \frac{|V(T')|+1}{3} = \frac{|V(T)|+2}{6} < \frac{|V(T)|}{3}$.

If $T \in \mathcal{R} - \mathcal{S}$, then choose a vertex w neighboring at least two leaves. By Lemma 4.4, the components of $T - w$ are vertices, edges, and members of $\mathcal{R}$. Form $\widehat{T}$ by deleting from T the vertices of the components of $T - w$ in $\mathcal{R}$. All weight in $\widehat{T}$ can be acquired by w, and $a_t(\widehat{T}) = 1 \le |V(\widehat{T})|/3$ since w has at least two leaf neighbors. By minimality, $a_t(R) \le |V(R)|/3$ for each remaining component of $T - w$. Together, $a_t(T) \le |V(T)|/3$, a contradiction. □

Lemma 5.2. *If e is a non-pendant edge in a minimal counterexample T, then $T - e$ has a component with order congruent to 2 modulo 3, and any such component belongs to $\mathcal{T} \cup \{P_2\}$.*

Proof. Let A_1 and A_2 be the components of $T - e$. Minimality of T yields $a_t(A_i) \le \frac{|V(A_i)|+1}{3}$. If $|V(A_i)| \not\equiv 2 \mod 3$, then integrality of a_t permits removal of $+1$ from the numerator, yielding $a_t(T) \le a_t(A_1) + a_t(A_2) \le \frac{|V(A_1)|}{3} + \frac{|V(A_2)|}{3} = \frac{|V(T)|}{3}$, a contradiction.

Now let A be a component of $T - e$ with $|V(A)| \equiv 2 \pmod 3$. If $A \notin \mathcal{T} \cup \{P_2\}$, then minimality of T yields $a_t(A) \le \frac{|V(A)|-2}{3}$ and $a_t(T - V(A)) \le \frac{|V(T)|-|V(A)|+1}{3}$. Now $a_t(T) \le a_t(A) + a_t(T - V(A)) \le \frac{|V(T)|-1}{3}$. We conclude $A \in \mathcal{T} \cup \{P_2\}$. □

Lemma 5.3. *If T is a minimal counterexample, then $|V(T)| \equiv 2 \pmod 3$. Furthermore, if $e \in E(T)$ and A is a component of $T - e$ with $|V(A)| \equiv 2 \pmod 3$ and $A \ne P_2$, then the endpoint of e in A is not a leaf of A.*

Proof. First suppose $|V(T)| \equiv 0 \pmod 3$. The only trees with every edge incident to a leaf are stars, which belong to $\mathcal{R}$ and are not counterexamples (Lemma 5.1). Hence T has an edge e' incident to no leaf. By Lemma 5.2, the orders of the components A_1 and A_2 of $T - e'$ are not both divisible by 3, so we may assume $|V(A_i)| \equiv i \pmod 3$. Minimality of T now yields $a_t(T) \le a_t(A_1) + a_t(A_2) \le \frac{|V(A_1)|-1}{3} + \frac{|V(A_2)|+1}{3} = \frac{|V(T)|}{3}$. Hence $|V(T)| \not\equiv 0 \pmod 3$.

Now consider $e \in E(T)$, and let A be a component of $T - e$ with $|V(A)| \equiv 2 \pmod 3$. Since $|V(T)| \not\equiv 0 \pmod 3$, the endpoint of e outside A is not a leaf. By Lemma 5.2, $A \in \mathcal{T} \cup \{P_2\}$. If e is incident to a leaf u of A, then let w be the neighbor of u in A. Let $\widehat{A}$ be the component of $T - uw$ other than $A - u$ (see Figure 8). Since $|V(A-u)| \equiv 1 \pmod 3$, Lemma 5.2 requires $|V(\widehat{A})| \equiv 2 \pmod 3$, which contradicts $|V(T)| \not\equiv 0 \pmod 3$.

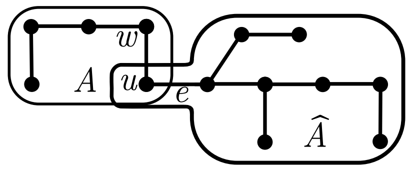

Figure 8: The edge e is adjacent to a leaf in A.

Finally, suppose $|V(T)| \equiv 1 \pmod 3$. By Lemma 5.1, we may choose a core vertex v of T. By Lemma 5.2, the orders of the components left by deleting any non-pendant edge are both congruent to 2 modulo 3. Applying this to the edges incident to v shows that every component of $T - v$ has order congruent to 2 modulo 3. By the preceding paragraph, $T - v$ consists of isolated edges or components in $\mathcal{T}$ whose leaves are not adjacent to v. The number of such components is divisible by 3, since $|V(T)| \equiv 1 \pmod 3$. Now Lemma 3.4 applies and yields $a_t(T) \leq \frac{|V(T)|}{3}$. Hence T is not a counterexample. $\square$

Lemma 5.4. *Let v be a core vertex in a minimal counterexample T. If $d_T(v) = 2$, then both components of $T - v$ have order congruent to 2 modulo 3.*

Proof. Since $|V(T)| \equiv 2 \pmod 3$, failure requires components A_0 and A_1 that satisfy $|V(A_i)| \equiv i \pmod 3$. Now deleting the edge joining v to A_1 as in Figure 9 contradicts Lemma 5.2. $\square$

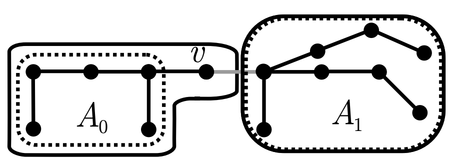

Figure 9: Both components of $T - v$ have order congruent to 2 modulo 3.

Lemma 5.5. *Let v be a core vertex in a minimal counterexample T. If $d_T(v) = 2$, then each neighbor of v is a neighbor of a leaf in T.*

Proof. Let A be a component of $T - v$, with w the neighbor of v in A. By Lemmas 5.4 and 5.2, $A \in \mathcal{T} \cup \{P_2\}$. If $A = P_2$, then w is a neighbor of a leaf, so we may assume $A \neq P_2$.

By Lemma 5.3, w is not a leaf of A, so all leaves of A are leaves of T. If w has no leaf neighbor in T, then w is a core vertex of A. Let A^+ be the tree induced by $V(A) \cup \{v\}$, and let $B = T - V(A^+)$ (see Figure 10). By Lemma 3.3, $a_t(A^+) \leq \frac{|V(A^+)|-3}{3}$; by the minimality of T, $a_t(B) \leq \frac{|V(B)|+1}{3}$. Hence $a_t(T) \leq a_t(A^+) + a_t(B) \leq \frac{|V(T)|-2}{3}$. □

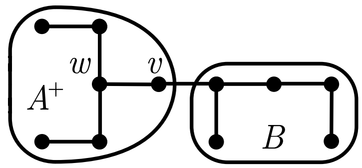

Figure 10: A core vertex of degree 2 cannot be adjacent to core vertex.

Lemma 5.6. *If some core vertex in a minimal counterexample T has degree 2, then all core vertices have degree 2.*

Proof. Let v and v' be core vertices with $d_T(v) = 2$ and $d_T(v') \geq 3$. Let the components of $T - v$ be A_1 and A_2. By Lemmas 5.4 and 5.2, $A_1, A_2 \in \mathcal{T} \cup \{P_2\}$. By symmetry, we may assume $v' \in V(A_1)$. By Lemma 5.3, the neighbor of v in A_1 is not a leaf of A_1, so all leaves of A_1 are leaves of T. Now v' is a core vertex of A_1. By Lemma 5.5, v is not adjacent to v'. Hence v' is a core vertex of degree at least 3 in a member of $\mathcal{T}$, contradicting Property D.□

6 Characterization of acquisition-extremal graphs

The main effort remaining is to characterize the extremal trees.

Theorem 6.1. *If $a_t(T) \geq \frac{|V(T)|+1}{3}$ for a tree T with $|V(T)| \geq 2$, then $T \in \mathcal{T} \cup \{P_2\}$.*

Proof. If the implication fails, then let T be a smallest counterexample. By Lemma 5.3, $|V(T)| \equiv 2 \pmod 3$. By Lemma 5.1, T has a core vertex. By Lemma 5.6, either all core vertices have degree 2 or all have degree at least 3. Let v be a core vertex.

Case 1: $d_T(v) = 2$. Let the components of $T - v$ be A_1 and A_2. Note that T satisfies Property D of Definition 2.2 for this vertex v. By Lemmas 5.4 and 5.2, $A_1, A_2 \in \mathcal{T} \cup \{P_2\}$, so Property E holds at this vertex. By Lemma 2.5, it remains only to show that T satisfies Property F.

By symmetry, it suffices to show that if w is the neighbor of v in A_1, then $T - w$ consists of an isolated vertex and components whose orders are divisible by 3. Since $|V(A_2)| \equiv 2 \pmod 3$, the component containing v causes no trouble. If $A_1 = P_2$, then there is only one other component of $T - w$, an isolated vertex.

We may therefore assume $A_1 \neq P_2$. By Lemma 5.5, w has a leaf neighbor in A_1. The non-leaf neighbors of w in A_1 are core vertices of A_1, since otherwise A_1 has leaves at distance 3, contradicting Property A. Now Property F for A_1 implies that $A_1 - w$ consists of an isolated vertex and components with order divisible by 3 (Figure 11); hence T satisfies Property F.

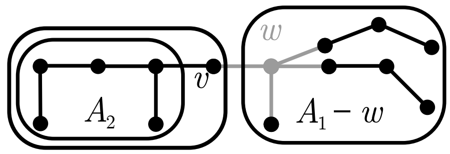

Figure 11: Verifying Property F at w.

Case 2: $d_T(v) \geq 3$. Since $|V(T)| \equiv 2 \pmod 3$ and v has no leaf neighbor, Lemma 5.2 implies that no component of $T - v$ has order congruent to 1 modulo 3. Let the components of $T - v$ be $A_1, \ldots, A_k$ and $B_1, \ldots, B_\ell$, where $|V(A_i)| \equiv 2 \pmod 3$ and $|V(B_j)| \equiv 0 \pmod 3$. Since $|V(T)| \equiv 2 \pmod 3$, we have $k \equiv 2 \pmod 3$ (see Figure 12).

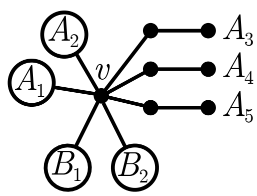

Figure 12: If $k \geq 3$, then $a_t(T) \leq |V(T)|/3$.

By Lemma 5.2, $A_i \in \mathcal{T} \cup \{P_2\}$ for each $i \in [k]$. Let x_i be the neighbor of v in A_i. If x_i is a leaf of A_i, then $A_i = P_2$, by Lemma 5.3. Minimality of T yields $a_t(B_j) \leq |V(B_j)|/3$ for each j. If $k \geq 3$, then Lemma 3.4 applies and yields $a_t(T) \leq |V(T)|/3$. Hence $k = 2$ and $\ell \geq 1$, since $d_T(v) \geq 3$.

Let w be the neighbor of v in $V(B_1)$. By Lemma 5.2, $T - w$ consists of isolated vertices $u_1, \ldots, u_r$ and components $C_1 \ldots, C_s$ and $D_1, \ldots, D_t$ such that $C_i \in \mathcal{T} \cup \{P_2\}$ (since $|V(C_i)| \equiv 2 \mod 3$) and $|V(D_j)| \equiv 0 \pmod 3$. Again by Lemma 5.3 the neighbor of w in C_i is not a leaf of C_i unless $C_i = P_2$.

Since $k = 2$, the component of $T - w$ induced by $V(A_1) \cup V(A_2) \cup \{v\}$ has order congruent to 2 modulo 3; thus $s \geq 1$, and we may call this component C_1 (see Figure 13). Minimality of T yields $a_t(D_j) \leq \frac{|V(D_j)|}{3}$ for each j. When $r + s \geq 3$, Lemma 3.4 applies and yields $a_t(T) \leq |V(T)|/3$.

Hence $r + s \leq 2$. Note that $|V(T)| \equiv r + 2s + 1 \pmod 3$ and $r \leq 2 - s \leq 1$. If $r = 1$, then also $s = 1$, which contradicts $|V(T)| \equiv 2 \pmod 3$. Hence we may assume $r = 0$ and $s = 2$ (see Figure 14). Now w has distance at least 2 from each leaf of T, so $d_T(w) \geq 3$. Since $d_T(w) = s + t$, there is a component D_1 of $T - w$ whose order is divisible by 3. Let

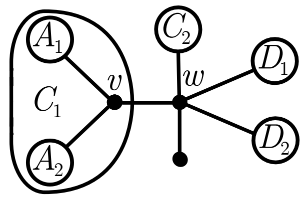

Figure 13: If $r + s \geq 3$, then $a_t(T) \leq |V(T)|/3$.

$T^* = T - V(D_1)$. Deleting the edge from D_1 to w leaves components D_1 and T^*. Since $|V(T^*)| \equiv 2 \pmod 3$, Lemma 5.2 requires $T^* \in \mathcal{T} \cup \{P_2\}$. Now v is a core vertex of T^* with $d_{T^*}(v) = 3$. This contradicts Property D for T^* and completes the proof. □

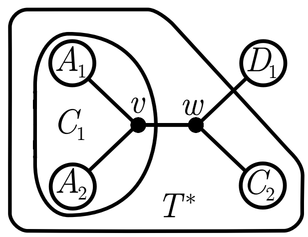

Figure 14: Both v and w are at distance at least 2 from all leaves of T^*.

The immediate corollary was first proved by Lampert and Slater [1].

Corollary 6.2. *If G is a connected graph with at least two vertices, then $a_t(G) \leq \frac{|V(G)|+1}{3}$.*

Proof. Let T be a spanning tree of G. If $T \in \mathcal{T} \cup \{P_2\}$, then by Theorem 3.2, $a_t(G) \leq a_t(T) = \frac{|V(G)|+1}{3}$. If $T \notin \mathcal{T} \cup \{P_2\}$, then by Theorem 6.1, $a_t(G) \leq a_t(T) < \frac{|V(T)|+1}{3}$. □

We can now characterize the extremal connected graphs. This is our main result.

Theorem 6.3. *If G is a connected graph with at least two vertices, then $a_t(G) \leq \frac{|V(G)|+1}{3}$, with equality if and only if $G \in \mathcal{T} \cup \{P_2, C_5\}$.*

Proof. Corollary 6.2 gives the upper bound. Among trees, the characterization of equality follows immediately from Theorem 6.1. If G is not a tree, then equality requires that every spanning tree of G is in $\mathcal{T} \cup \{P_2\}$. For $|V(G)| \leq 5$, every spanning tree must be P_2 or P_5, so G has five vertices and maximum degree 2 and must be a 5-cycle.

For larger graphs, it suffices to show that adding an edge e to a tree T in $\mathcal{T}$ creates a graph G having a spanning tree not in $\mathcal{T}$. Recall that each non-leaf vertex of T is a core vertex or a neighbor of a leaf. Furthermore, the set of core vertices is independent (by Property F), as is the set of neighbors of leaves (by Property A). In each case for the possible endpoints of e, we name an edge e' of the path P in T joining the endpoints of e and show that the tree T^* formed from T by adding e and deleting e' is not in $\mathcal{T}$.

Suppose $e = vz$, where v is a core vertex of T; note that $d_T(v) = 2$ (Property D). Let e' be the edge of P incident to z; since P has length at least 2, e' is not incident to v. Let w be the neighbor of v in $V(T) - V(P)$. Suppose $T^* \in \mathcal{T}$. Since $d_{T^*}(v) = 3$, Property D for T^* implies that v is not a core vertex in T^* and has a leaf neighbor. Since w has the same leaf neighbor as in T, now T^* has adjacent vertices with leaf neighbors and cannot be in $\mathcal{T}$.

Suppose $e = ww'$, where w and w' are neighbors of leaves. Let e' be any edge of P. Now T^* has adjacent vertices with leaf neighbors; again $T^* \notin \mathcal{T}$.

Suppose $e = uu'$, where u and u' are leaves of T. Let w and w' be the neighbors of u and u', respectively. The distance between leaves in T is at least 4 (Property A), so the length ℓ of P is at least 4. When $\ell > 4$, choose e' to be an edge of P incident to none of $\{u, u', w, w'\}$. When $\ell = 4$, since $T \in \mathcal{T} - \{P_5\}$, one of $\{w, w'\}$ has degree at least 3; let e' be the non-pendant edge of P incident to that vertex. Now u and u' are adjacent core vertices in T^*; again $T^* \notin \mathcal{T}$.

The remaining possibility is $e = wu$, where w is a neighbor of a leaf and u is a leaf of T. Let e' be the edge of P incident to u. Again $T^* \notin \mathcal{T}$, since w has two leaf neighbors in T^*.□

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