The A_4-structure of a graph

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Abstract

We define the A_4-*structure* of a graph G to be the 4-uniform hypergraph on the vertex set of G whose edges are the vertex subsets inducing $2K_2$, C_4, or P_4. We show that perfection of a graph is determined by its A_4-structure. We relate the A_4-structure to the canonical decomposition of a graph as defined by Tyshkevich [Discrete Mathematics 220 (2000) 201-238]; for example, a graph is indecomposable if and only if its A_4-structure is connected. We also characterize the graphs having the same A_4-structure as a split graph.

1 Introduction

The P_4-*structure* of a simple graph G is the 4-uniform hypergraph with vertex set $V(G)$ whose edges are the vertex subsets inducing copies of P_4. Chvátal [4] introduced this concept in 1984 to study the complexity of recognizing perfect graphs. The P_4-structure is also used in refinements of modular decomposition (see [16, 20]) and in defining or characterizing several graph classes ([2] gives a hierarchy of graph classes defined in terms of P_4-structure).

For any set $\mathcal{F}$ of graphs, the $\mathcal{F}$-*structure* of a graph G as the hypergraph with vertex set $V(G)$ whose edges are the vertex subsets whose induced subgraphs lie in $\mathcal{F}$. Such structures have been considered in [11, 12, 13, 14, 15] when $\mathcal{F}$ is $\{P_3\}$, $\{C_5, \text{paw}, P_3+K_1\}$, $\{2K_2, C_4, C_5\}$, $\{P_3, K_2 + K_1\}$, or $\{K_3, 3K_1\}$, where $G + H$ is the disjoint union of graphs G and H, and $\overline{G}$ is the complement of G.

In this paper we consider the A_4-*structure* of a graph G, which we define as the 4-uniform hypergraph H with vertex set $V(G)$ having as edges the vertex subsets that induced a subgraph in $\{2K_2, C_4, P_4\}$. The name indicates that $2K_2, C_4$, and P_4 are the 4-vertex graphs exhibiting an *alternating 4-cycle*, which is a configuration on four vertices a, b, c, d such that ab and cd are edges while bc and ad are not. An alternating 4-cycle and the three corresponding subgraphs appear in Figure 1 (dashed segments in figures will always denote non-adjacencies).

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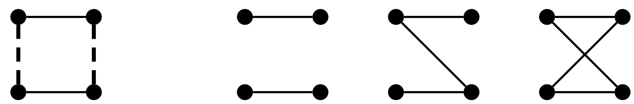

Figure 1: Alternating 4-cycle and A_4-graphs

Study of the A_4-structure has several motivations. First, alternating 4-cycles arise in connection with vertex degrees. A *2-switch* interchanges the edges and non-edges of an alternating 4-cycle. This may change the isomorphism class but does not change any vertex degree. A well-known result of Fulkerson, Hoffman, and McAndrew [8] states that two graphs have the same vertex degrees if and only if one can be changed into the other via 2-switches.

A second motivation is the relation of alternating 4-cycles to the *canonical decomposition* of a graph, defined by Tyshkevich in [23] (see also [24]). We explore this relation in Section 3, showing that a graph is indecomposable if and only if its A_4-structure is a connected hypergraph. In Section 4, we show that the A_4-structure is to the canonical decomposition as the P_4-structure of a graph is to a refinement of its modular decomposition.

Our third motivation is the role of alternating 4-cycles in characterizing threshold graphs, matroidal graphs, and matrogenic graphs. Threshold graphs were introduced in [5] in connection with set-packing problems and have been rediscovered several times (see [18] for the history). One of the many characterizations can be expressed in terms of the A_4-structure. Threshold graphs are precisely those graphs having no alternating 4-cycle [5], which is the statement that the A_4-structure has no edges.

A graph G is *matroidal* if the edge pairs appearing in alternating 4-cycles are the circuits of a matroid on $E(G)$. These were introduced and characterized in [19] as the graphs not containing an induced 5-cycle or the configuration $\mathcal{C}$ in Figure 2, where dashed segments join vertices required to be nonadjacent; $\mathcal{C}$ consists of a vertex triple S and vertices $x, y \notin S$ such that the neighborhoods of x and y partition S into two nonempty sets.

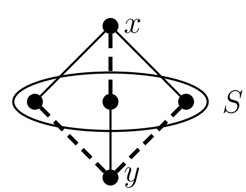

Figure 2: The configuration $\mathcal{C}$

A graph G is *matrogenic* if the vertex sets of alternating 4-cycles are the circuits of a matroid on $V(G)$. These graphs were introduced and characterized in [7] as the graphs forbidding $\mathcal{C}$ (but allowing induced 5-cycles). Among graphs on five vertices, C_5 is the only one having more than three edges in its A_4-structure, and the graphs in which $\mathcal{C}$ appears are those whose A_4-structures have two or three edges.

Observation 1.1. *A graph is matrogenic if and only if no five of its vertices induce exactly two or three edges in the A_4-structure. A graph is matroidal if and only if no five of its vertices induce more than one edge in the A_4-structure.*

A similar notion arises in the study of P_4-structures. The (q,t)*-graphs* were defined in [1] as the graphs in which any q vertices induce at most t copies of P_4; thus the P_4-free graphs are the $(4,0)$-graphs. The $(5,1)$-graphs are also known as the P_4*-sparse* graphs [10]. With $[q,t]$*-graphs* denoting the analogous notion for A_4-structure, the threshold graphs become the $[4,0]$-graphs, and the matroidal graphs become the $[5,1]$-graphs.

In Section 2 we show that perfect graphs are recognizable from their A_4-structures and that in a triangle-free graph the A_4-structure describes the structure of matchings. As mentioned above, Sections 3 and 4 discuss the relationship between A_4-structure and the Tyshkevich canonical decomposition. Finally, in Section 5 we discuss the problem of obtaining all graphs whose A_4-structure is a given hypergraph H; this leads us to characterize the A_4*-split graphs*, which are the graphs having the same A_4-structure as some split graph. A *split graph* is a graph whose vertex set can be partitioned into a clique (a set of pairwise adjacent vertices) and an independent set (a set of pairwise nonadjacent vertices).

We use $V(G)$ and $E(G)$ to denote the vertex and edge sets of a graph G (no loops or multiedges). Let $N_G(v) = \{u\colon\ uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$. Let $d_G(v) = |N_G(v)|$. A vertex v in G is *isolated* if $d_G(v) = 0$, *pendant* if $d_G(v) = 1$, and *dominating* if $N_G[v] = V(G)$. For $S \subseteq V(G)$, the subgraph $G[S]$ *induced by* S is the graph with vertex set S whose edges are the edges of G with endpoints in S. Let $G - v = G[V(G) - \{v\}]$. Given a set $\mathcal{F}$ of graphs, a graph G is $\mathcal{F}$*-free* if no induced subgraph of G is isomorphic to an element of $\mathcal{F}$.

2 A_4-structure and cycles

Given a hypergraph H, any graph whose A_4-structure is (isomorphic to) H is a *realization* of H. Given G and G' realizing H and H', an A_4*-isomorphism* from G to G' is a bijection $\varphi\colon V(G) \to V(G')$ that is an isomorphism from H to H'. We say that G and G' are A_4*-isomorphic* or "have the same A_4-structure" if there is an A_4-isomorphism from G to G'.

From the definition, every graph has the same A_4-structure as its complement. In this section, we show that the A_4-structure of the cycle C_n is realized only by C_n and its complement $\overline{C}_n$ (except when $n \in \{3,4,6\}$). As a consequence, perfect graphs can be recognized from their A_4-structures. We conclude by showing how in triangle-free graphs the A_4-structure describes the vertex sets of matchings.

Chvátal [4] showed that when n is odd and at least 5, the P_4-structure of C_n is realized only by C_n and $\overline{C}_n$. Reed [21] proved Chv'atal's conjecture that two graphs with the same P_4-structure are either both perfect or both imperfect. Now known as the Semistrong Perfect Graph Theorem, this implies the Perfect Graph Theorem of Lovász [17] (a graph and its complement are both perfect or both imperfect), and it is in turn implied by the much later Strong Perfect Graph Theorem of Chudnovsky et al. [3] (a graph G is perfect if and only if no induced subgraph is an odd cycle of length at least 5 or its complement).

Motivated by the results of Chvátal and Reed, we show that for $n = 5$ and $n \geq 7$, the cycle C_n and its complement are the only realizations of their A_4-structure. By the Strong Perfect Graph Theorem, it then follows that graphs with the same A_4-structures are either both perfect or both imperfect.

Lemma 2.1. *Four vertices form an edge in the A_4-structure of a graph G*
(a) if and only if the subgraph they induce in G has no vertex of degree 0 or 3.
(b) if and only if the induced subgraph has no triangle or independent 3-set.

Proof. Recall that $2K_2$, P_4, and C_4 are the only 4-vertex graphs in which an alternating 4-cycle occurs. Checking the eleven 4-vertex graphs shows that these are also the only ones having neither a dominating nor an isolated vertex, and they are the only ones having no triangle or independent 3-set. □

In the next three lemmas, suppose that G is a graph A_4-isomorphic to C_n. Letting $u_1, \ldots, u_n$ be the vertices of C_n in order, name the vertices of G as $v_1, \ldots, v_n$ so that u_i is mapped to v_i by a given A_4-isomorphism φ. Note that C_n is A_4-isomorphic to both G and $\overline{G}$ under φ. We will show that if v_1v_2 is an edge in G, then φ is in fact a graph isomorphism. All addition and subtraction in the indices of vertices is performed modulo n.

Lemma 2.2. *No triangle or independent 3-set in G contains both v_i and v_{i+1} for some i.*

Proof. Let $S_{i,j} = \{v_i, v_{i+1}, v_j\}$. In the A_4-structure of C_n, all 4-tuples consisting of two disjoint pairs of consecutive vertices are edges, so some edge contains $\{u_i, u_{i+1}, u_j\}$. Hence some edge in the A_4-structure of G contains $S_{i,j}$. By Lemma 2.1, $G[S_{i,j}]$ has no triangle or independent 3-set. This applies for all j. □

Lemma 2.3. *For $n \geq 7$, there is no i such that v_i has exactly one neighbor in $\{v_{i-1}, v_{i+1}\}$.*

Proof. If the conclusion fails, then for some i exactly one of $v_{i-1}v_i$ and v_iv_{i+1} is an edge of G. We may use symmetry and the fact that G and $\overline{G}$ have the same A_4-structure to assume that $v_{i-1}v_i \in E(G)$ and $v_{i-1}v_{i+1}, v_iv_{i+1} \notin E(G)$.

Let H be the A_4-structure of G. Since $\{v_{i-2}, v_{i-1}, v_i, v_{i+1}\}, \{v_{i-1}, v_i, v_{i+1}, v_{i+2}\} \in E(H)$, Lemma 2.1(a) requires $v_{i-2}v_{i+1}, v_{i+2}v_{i+1} \in E(G)$. On the other hand, Lemmas 2.1(a) and (b) require $v_{i+1}v_{i-3}, v_{i-2}v_{i+2} \notin E(G)$, since $\{v_{i-3}, v_{i-2}, v_{i+1}, v_{i+2}\}, \{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\} \in E(H)$, Since $\{v_{i-3}, v_{i-2}, v_i, v_{i+1}\} \in E(H)$, Lemma 2.1(b) requires $v_{i-3}v_i \in E(G)$. The present status is shown in Figure 3.

Since $n \geq 7$, vertices v_{i+2} and v_{i-3} are not consecutive, so $\{v_i, v_{i-3}, v_{i+1}, v_{i+2}\} \notin E(H)$. Also $\{v_i, v_{i+1}, v_{i-2}, v_{i+2}\} \notin E(H)$. The former requires $v_iv_{i+2} \in E(G)$, while the latter requires $v_iv_{i+2} \notin E(G)$, a contradiction. □

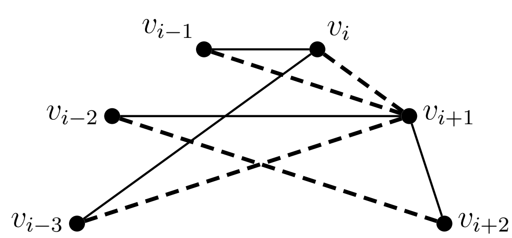

Figure 3: The subgraph of G from Lemma 2.3.

We specify an alternating 4-cycle by the notation $[a, b : c, d]$, indicating that $ab, cd \in E(G)$ and $bc, da \notin E(G)$.

Theorem 2.4. *If $n = 5$ or $n \geq 7$, then the only graphs A_4-isomorphic to C_n are C_n and $\overline{C}_n$.*

Proof. Since C_5 is the only graph with five vertices whose A_4-structure has more than three edges, we may assume that $n \geq 7$. Let G be A_4-isomorphic to C_n.

Since G and $\overline{G}$ have the same A_4-structure, we may assume $v_1v_2 \in E(G)$. By Lemma 2.3, G now has a spanning cycle with vertices $v_1, \ldots, v_n$ in order. By Lemma 2.2, $v_iv_{i+2} \notin E(G)$ for all i. Suppose that G has a chord v_jv_k with j and k at least 3 apart. By Lemma 2.2, $v_jv_{k-1}, v_jv_{k+1} \notin E(G)$. Now $[v_j, v_k : v_{k-2}, v_{k-1}]$ and $[v_j, v_k : v_{k+2}, v_{k+1}]$ are alternating 4-cycles in G. Since $n \geq 7$, at least one of $\{v_{k-2}, v_{k+2}\}$ is not consecutive to v_j, which contradicts the A_4-structure of C_n. Thus G has no chords and $G \cong C_n$. □

As noted earlier, Theorem 2.4 and the Strong Perfect Graph Theorem immediately yield the following:

Corollary 2.5. *Two graphs with the same A_4-structure are both perfect or both imperfect.*

The conclusion of Theorem 2.4 fails when $n = 6$; the graph C_6 is A_4-isomorphic to any graph obtained by deleting a matching (of size at most 3) from $K_{3,3}$. Note also that Theorem 2.4 applies also to even cycles, while Chvátal's analogous result for P_4-structure considers only odd cycles.

We next relate A_4-structure to matchings in triangle-free graphs.

Lemma 2.6. *Let G be a 6-vertex triangle-free graph. If there are three disjoint pairs of vertices in G such that the union of any two of them is an edge in the A_4-structure of G, then G has a perfect matching.*

Proof. Let A, B, and C be three such pairs, and let H be the A_4-structure of G. If each of A, B, and C induces an edge in G, then G has a perfect matching. Otherwise, we may assume by symmetry that $a_1a_2 \notin E(G)$, where $A = \{a_1, a_2\}$. Since $A \cup B \in E(H)$, there is a matching $\{a_1b_1, a_2b_2\}$ in $G[A \cup B]$. Similarly, there is a matching $\{a_1c_1, a_2c_2\}$ in $G[A \cup C]$. Since G is triangle-free, $b_1c_1, b_2c_2 \notin E(G)$. However, $B \cup C \in E(H)$, so $B \cup C$ induces two non-incident edges. Now G has a spanning cycle and hence a perfect matching. □

For graphs G and G', a bijection $\varphi \colon V(G) \to V(G')$ *preserves matchings* if for all $S \subseteq V(G)$ with $|S| \geq 4$, the graph $G[S]$ has a perfect matching if and only if $G'[\varphi(S)]$ has a perfect matching.

Theorem 2.7. *Given triangle-free graphs G and G', a bijection φ from $V(G)$ to $V(G')$ is an A_4-isomorphism if and only if it preserves matchings.*

Proof. Note: If S is a set of four vertices in a triangle-free graph F, then S is an edge in the A_4-structure H if and only if $F[S]$ has a perfect matching.

Suppose that φ preserves matchings. If $S \in E(H)$, then $G[S]$ has a perfect matching. Hence $G'[\varphi(S)]$ has a perfect matching. Since G' is triangle-free, $\varphi(S)$ is now an edge in the A_4-structure. Similarly, φ^{-1} also preserves edges of the A_4-structure.

Suppose that φ is an A_4-isomorphism. Let M be a matching of size at least 2 in G, with vertex set S. Partition the edges of M into pairs and triples of edges, yielding vertex sets $S_1, \ldots, S_k$. Using Lemma 2.6 and the Note above, each image $\varphi(S_i)$ is the vertex set of a matching in G'. The union of these matchings is a perfect matching in $G'[\varphi(S)]$. Thus φ preserves matchings, and the same argument applies to the A_4-isomorphism φ^{-1}. $\square$

3 Canonical decomposition and A_4-structure

Tyshkevich [23, 24] introduced "canonical decomposition" of graphs, which we now describe.

A *splitted graph* is a triple (G, A, B) such that G is a split graph in which A is an independent set, B is a clique, and $V(G) = A \cup B$. Splitted graphs (G, A, B) and (G', A', B') are *split-isomorphic* if some graph isomorphism maps A into A' and B into B'. Given a splitted graph (G, A, B) and a graph H, the *composition* of (G, A, B) and H is the graph $(G, A, B) \circ H$ formed by adding to $G + H$ all edges uv such that $u \in B$ and $v \in V(H)$. For example, when $H = K_3$ and $G = P_4$, with A the set of endpoints and B the set of midpoints of G, the composition $(G, A, B) \circ K_3$ is the graph on the left in Figure 4 (henceforth, heavy lines in figures indicate that all edges joining vertices from one set to the other are present). On the right we show $(G, A, B) \circ ((G, A, B) \circ K_3)$.

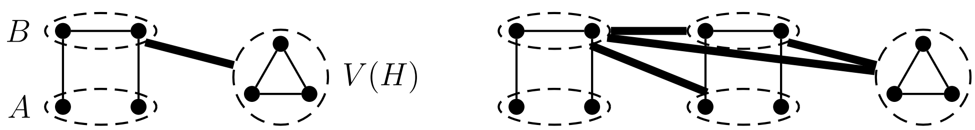

Figure 4: The compositions $(G, A, B) \circ H$ and $(G, A, B) \circ (G, A, B) \circ H$.

The operation $\circ$ is associative, so we omit parentheses when performing multiple compositions. In a composition $(G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$, each vertex in B_i is adjacent to all of $\bigcup_{j<i} V(G_j)$, each vertex in A_i is adjacent to none of $\bigcup_{j<i} V(G_j)$, and only the rightmost graph in the composition can fail to be a split graph.

A graph is *decomposable* if it arises as a composition $(G, A, B) \circ H$ with G and H both having at least one vertex. Otherwise, it is *indecomposable*. Tyshkevich showed the following:

Theorem 3.1 (Tyshkevich [24])**.** *Every graph G has a unique expression as a composition*

$$G = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0 \qquad (*)$$

of indecomposable components. Uniqueness is up to isomorphism in the components; that is, for decompositions of isomorphic graphs G and G', we have $k = k'$, $G_0 \cong G'_0$, and $(G_i, A_i, B_i) \cong (G'_i, A'_i, B'_i)$ under split-isomorphism for $1 \le i \le k$.

If G is indecomposable, then $k = 0$, and there are no splitted components in $(*)$. Due to the uniqueness of the decomposition, we call $(*)$ the *canonical decomposition* of G. We next characterize indecomposable graphs in terms of their A_4-structures.

Theorem 3.2. *A graph G is indecomposable if and only if its A_4-structure is connected. More generally, the vertex sets of the components of the canonical decomposition $(*)$ are the vertex sets of the components of the A_4-structure of G.*

The proof is lengthy; we first obtain several preliminary results. For an alternating 4-cycle C, let $V(C)$ denote the corresponding edge in the A_4-structure.

Observation 3.3. *A graph G with more than one vertex has an isolated or a dominating vertex if and only if $k \geq 1$ in the canonical decomposition $(*)$ and $G_k = K_1$. The vertex of G_k is then dominating in G if $A_k = \varnothing$ and isolated in G if $B_k = \varnothing$.*

Observation 3.4. *If $G = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$, then $\overline{G} = (\overline{G}_k, B_k, A_k) \circ \cdots \circ \overline{G}_1, B_1, A_1) \circ \overline{G}_0$.*

Lemma 3.5. *If G is an indecomposable graph with more than one vertex, then every vertex of G belongs to an alternating 4-cycle in G.*

Proof. We prove the contrapositive. Let v be a vertex of G in no alternating 4-cycle. Let $V_1 = N(v)$ and $V_2 = V(G) - N[v]$. If v is a dominating or isolated vertex, then G is decomposable by Observation 3.3, so we may assume $V_1, V_2 \neq \varnothing$.

Let $S = \{u \in V_1\colon\ N_G(u) \cap V_2 = \varnothing$, and let $B = V_1 - S$. We claim that if V_1 is not a clique, then $S \neq \varnothing$ and $G = (G', V_2, B) \circ (K_1, \varnothing, \{v\}) \circ G[S]$, where $G' = G[V_2 \cup B]$. If V_1 is not a clique, then there exist $u, w \in V_1$ with $uw \notin E(G)$. For $a \in V_2$, neither $[v, w : u, a]$ nor $[v, u : w, a]$ is an alternating 4-cycle (both contain v), so $u, w \notin N_G(a)$. Hence each vertex of B dominates V_1. Since we assumed that V_1 is not a clique, $S \neq \varnothing$. For $u \in S$, if a and b in V_2 are adjacent, then $[v, u : a, b]$ is an alternating 4-cycle containing v. Hence V_2 is independent, and the claimed decomposition is valid.

We may therefore assume that V_1 is a clique in G. Since G and $\overline{G}$ are A_4-isomorphic, v also belongs to no alternating 4-cycle in $\overline{G}$. Since $V_2 = N_{\overline{G}}(v) = V_2$, the same reasoning implies that V_2 is a clique in $\overline{G}$ and an independent set in G.

With V_1 being a clique and V_2 being an independent set, we have $G = (G', V_2, V_1) \circ G[\{v\}]$, where $G' = G[V_2 \cup V_1]$. Hence in all cases G is decomposable. $\square$

Henceforth let $H(G)$ denote the A_4-structure of a graph G. When A and B are edges in $H(G)$, we write $A \to B$ to mean that $G[A] \cong P_4$ and that each vertex of B is adjacent to the midpoints of $G[A]$ and not adjacent to the endpoints of $G[A]$.

Lemma 3.6. *If A and B are disjoint edges in $H(G)$ such that no edge of $H(G)$ intersects both A and B, then $A \to B$ or $B \to A$.*

Proof. Let $[a, b : c, d]$ and $[e, f : g, h]$ be alternating 4-cycles in $G[A]$ and $G[B]$, respectively. Since $\{a, b, e, f\} \notin E(H(G))$ and each of these vertices already has a neighbor among the other three, Lemma 2.1(a) implies that some vertex in $\{a, b, e, f\}$ dominates the other three; let a be such a vertex. Since neither $[a, f : g, h]$ nor $[a, e : h, g]$ is an alternating 4-cycle, $ag, ah \in E(G)$. Thus $B \subseteq N_G(a)$. It follows that d has no neighbor v in B, for otherwise $[a, u : v, d]$ would be an alternating 4-cycle, where u is a non-neighbor of v in B. The same argument starting with $\{c, d, g, h\}$ implies that $B \subseteq N_G(c)$ (since d has no neighbor in B) and that b has no neighbor in B.

Also, $bd \notin E(G)$ and $ac \in E(G)$, since otherwise $[b, d : e, f]$ or $[a, e : h, c]$ would be an alternating 4-cycle. We conclude that $G[A] \cong P_4$, so $A \to B$. The same conclusion holds by a symmetric argument if b dominates $\{a, e, f\}$. If instead e or f dominates the other three vertices of $\{a, b, e, f\}$, then we obtain $B \to A$. □

Lemma 3.6 implies that if two edges of $H(G)$ do not intersect a common edge of $H(G)$, then in G at least one of them induces P_4. Therefore, if each of x and y belongs to an induced copy of C_4 or $2K_2$ in G, then the distance between x and y in $H(G)$ is at most 3.

Corollary 3.7. *Let G be a graph. If A and B are edges in distinct components of $H(G)$, then $A \to B$ or $B \to A$.*

Lemma 3.8. *Let G be a graph. If A, B, and C are edges in $H(G)$ such that $A \to B$ and $A \cap C$ is nonempty, then $B \nrightarrow C$.*

Proof. If $B \to C$, then the midpoints of the path induced by B dominate C, and the endpoints have no neighbors in C. Hence no vertex in C can dominate or be independent of B. Since every vertex of A dominates or is independent of B, we conclude that $A \cap C = \varnothing$. □

Lemma 3.9. *Let G be a graph. Let Q_1 and Q_2 be distinct components of $H(G)$, and let A and B be edges in Q_1 and Q_2, respectively. If $A \to B$, then $C \to D$ whenever C is an edge in Q_1 and D is an edge in Q_2.*

Proof. Since $A, C \in E(Q_1)$, there are edges $R_0, \ldots, R_k \in E(Q_1)$ such that $A = R_0$, $C = R_k$, and $R_{i-1} \cap R_i \neq \varnothing$ for $1 \leq i \leq k$. By Corollary 3.7, $B \to R_i$ or $R_i \to B$ for each i. Inductively, from $R_i \to B$, Lemma 3.8 forbids $B \to R_{i+1}$, and hence $R_{i+1} \to B$. In particular, $C \to B$.

Similarly, since $B, D \in E(Q_2)$, there are edges $S_0, \ldots, S_\ell \in E(Q_2)$ such that $B = S_0$, $D = S_\ell$, and $S_{i-1} \cap S_i \neq \varnothing$ for $1 \leq i \leq \ell$. Corollary 3.7 implies that $S_i \to C$ or $C \to S_i$ for each i. If $S_1 \to C$, then Lemma 3.8 yields $C \nrightarrow B$, which is false; hence $C \to S_1$. Repeating this argument iteratively yields $C \to S_i$ for each i. Thus $C \to D$. □

A *tournament* is an orientation of a complete graph.

Lemma 3.10. *Let T be a directed graph whose vertices are the components of $H(G)$, defined by putting $Q_1Q_2 \in E(T)$ if Q_1 has an edge A and Q_2 has an edge B such that $A \to B$. If G is indecomposable, then T is a transitive tournament.*

Proof. We write $Q_1 \to Q_2$ if $Q_1Q_2 \in E(T)$. Since G is indecomposable, Lemma 3.5 implies that each component of $H(G)$ has an edge. By Corollary 3.7, any two vertices of T are adjacent. By Lemma 3.9, $Q_1 \to Q_2$ implies $Q_2 \nrightarrow Q_1$. Hence T is a tournament.

If $Q_1 \to Q_2 \to Q_3$, then by the definition of "$\to$" there exist $A_i \in E(Q_i)$ for $i \in \{1, 2, 3\}$ such that $A_1 \to A_2 \to A_3$. We claim $A_1 \to A_3$, which again by definition yields $Q_1 \to Q_3$.

Since $A_1 \to A_2 \to A_3$, we have $G[A_1] \cong G[A_2] \cong P_4$. For $i \in \{1, 2\}$, let a_i be a leaf in $G[A_i]$, and let b_i be its neighbor in $G[A_i]$. Since Q_1, Q_2, Q_3 are distinct components, when $c \in A_3$ neither $[b_1, a_1 : b_2, c]$ nor $[b_2, a_2 : c, a_1]$ is an alternating 4-cycle. Hence $cb_1 \in E(G)$ and $ca_1 \notin E(G)$. Since a_1 may be either leaf in $G[A_1]$, we conclude that $A_1 \to A_3$. □

Lemma 3.11. *For any graph G, no edge of $H(G)$ contains vertices from distinct components of the canonical decomposition of G.*

Proof. Let $(G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$ be the canonical decomposition of G. Suppose that some alternating 4-cycle $[u, v : w, x]$ has vertices in more than one G_i. Let j be the largest index such that G_j contains some vertex of the alternating 4-cycle. If such a vertex is in B_j, then its nonneighor on the alternating 4-cycle must lie in A_j, since B_j dominates $V(G_i)$ for $i < j$. If such a vertex is in A_j, then its neighor on the alternating 4-cycle must lie in B_j, since vertices of A_j are independent of $V(G_i)$ for $i < j$. Hence following along the alternating 4-cycle implies that its vertices all lie in $V(G_j)$. □

We are now ready to prove our main result:

Proof of Theorem 3.2. Given a graph G, suppose first that $H(G)$ is connected. For $u, v \in V(G)$, there exist edges $E_0, \ldots, E_k$ of $H(G)$ such that $u \in E_0$ and $v \in E_k$, and $E_i \cap E_{i-1} \neq \varnothing$ for $1 \leq i \leq k$. Applying Lemma 3.11 to vertices in the sets $E_0, \ldots, E_k$ in turn yields that u and v belong to the same component in the canonical decomposition of G. Thus G is indecomposable.

If G is indecomposable, then the digraph T of Lemma 3.10 is a transitive tournament. If $H(G)$ is disconnected, then T is nontrivial; let Q be the source vertex. By Lemma 3.5, every component of $H(G)$ contains an edge. Every edge of Q in $H(G)$ corresponds to an induced P_4 in G whose midpoints dominate every vertex outside $V(Q)$ and whose endpoints have neighbors (in G) only in $V(Q)$.

Therefore, no vertex in $V(Q)$ is both a midpoint of some induced P_4 and an endpoint of another, and $V(Q)$ splits into sets A and B consisting of the endpoints and the midpoints of the induced copies of P_4 in $G[V(Q)]$, respectively. Two adjacent vertices of A would create an alternating 4-cycle with any two adjacent vertices of $G - V(Q)$; similarly, two nonadjacent vertices of B would create an alternating 4-cycle with any two nonadjacent vertices of $G - V(Q)$. We conclude that B is a clique and A is an independent set in G. Hence $G = (G', A, B) \circ G[V(G) - V(Q)]$, where $G' = G[A \cup B]$, and G is decomposable.

Having shown that G is indecomposable if and only if $H(G)$ is connected, it follows that the components of H partition the set $V(G)$ into exactly the same subsets that the components in the canonical decomposition do. □

Using Theorem 3.2, we relate the A_4-structure of a graph to its degree sequence. This is not surprising, since 2-switches along alternating 4-cycles convert one realization of a degree sequence into another. Tyshkevich [23, 24] gave a characterization of indecomposable graphs in terms of their degree sequences that has a quick explanation using these ideas.

Proposition 3.12. *Applying a 2-switch to a graph G does not change the partition of $V(G)$ given by the components of the canonical decomposition of G.*

Proof. By Lemma 3.11, every alternating 4-cycle C lies in a single component G_i of the canonical decomposition. Since B_i is a clique and A_i is independent, C alternates between B_i and A_i. Hence a 2-switch along C leaves B_i and A_i as a clique and an independent set, thereby preserving the components of the canonical decomposition other than G_i. Furthermore, G_i cannot change into a decomposable graph, because the vertices of C would then form an alternating 4-cycle intersecting more than one component of the decomposition. □

Corollary 3.13 (Tyshkevich [23, 24])**.** *For every graph G, the degree sequence of G uniquely determines the number of indecomposable components in the canonical decomposition of G and the degree sequences of those components.*

Corollary 3.14. *If G and G' are graphs with the same vertex degrees, then $H(G)$ and $H(G')$ have the same number of components and the same vertex partitions into components.*

4 A_4-structure and modules

Expanding on the previous section, we show that A_4-structure is to canonical decomposition as P_4-structure is to other graph decompositions. In particular, we develop an analogue of the "primeval decomposition" of Jamison and Olariu [16] (see Theorem 4.7), which itself refines the well known "modular decomposition" of Gallai [9] (see Theorem 4.6).

The key is an analogue of the notion of "module". A *module* in a graph G is a nonempty set $S \subseteq V(G)$ such that every vertex outside S is adjacent to all of S or to none of S. A module S is *trivial* if $|S| = 1$ or $S = V(G)$.

Lemma 4.1 (Seinsche [22])**.** *The following hold for every graph G.*

(i) *The vertex set of an induced P_4 in G and a module in G can intersect only in zero, one, or four vertices.*

(ii) *G is P_4-free if and only if every induced subgraph with at least three vertices has a nontrivial module.*

We introduce a restricted notion of module that plays for A_4-structure the role that modules play for P_4-structure. An *alternating path* is a list of vertices whose consecutive pairs alternate being adjacent and nonadjacent; the first may equal the last, but otherwise the vertices are distinct. We write $\langle v_0, \dots, v_p \rangle$ for an alternating path of *length* p with *endpoints* v_0 and v_p. An alternating path is *S-terminal* if it has length at least 2 and its only vertices in S are its endpoints. Modules are characterized by forbidding S-terminal alternating paths of length 2. A *strict module* is a set S in $V(G)$ such that G has no S-terminal alternating path of any length. In fact, it suffices to forbid only the short alternating paths.

Proposition 4.2. *A vertex subset S of a graph G is a strict module if and only if G has no S-terminal alternating paths of length 2 or 3.*

Proof. If S is a strict module, then by definition G has no short S-terminal alternating paths. If S is not a strict module, then G has an S-terminal alternating path; let $\langle v_0, \dots, v_p \rangle$ be a shortest one. If $p \geq 4$, then consider v_2. Whether v_2 is adjacent to v_0 or not, we can start from v_0 and continue from v_2 to v_1 or v_3. That is, $\langle v_0, v_2, v_1, v_0 \rangle$ or $\langle v_0, v_2, v_3, \dots, v_p \rangle$ is a shorter S-terminal alternating path. Thus $p \leq 3$. □

The full vertex set is a *trivial* strict module. Single vertices form modules, but they need not form strict modules. Proposition 4.4 below is analogous to Lemma 4.1. Recall that the threshold graphs are the graphs having no alternating 4-cycles.

Theorem 4.3 (Chvátal–Hammer [5])**.** *A graph G is a threshold graph if and only if G arises from a single vertex by iteratively adding an isolated vertex or a dominating vertex.*

Proposition 4.4. *The following hold for every graph G.*

(i) *Every alternating 4-cycle and strict module intersect in zero or four vertices.*

(ii) *G has no alternating 4-cycles if and only if every induced subgraph with at least two vertices has a nontrivial strict module.*

Proof. (i) If an alternating 4-cycle C intersects both a vertex set S and its complement, then there is an S-terminal alternating path along C, so S is not a strict module.

(ii) If G has an alternating cycle, then by (i) it has a 4-vertex induced subgraph with no nontrivial strict module. If G has no alternating cycle, then every induced subgraph G' is a threshold graph. By Theorem 4.3, G' has a dominating or isolated vertex u, and $V(G')-\{u\}$ is a strict module in G'. □

Existence of strict modules corresponds to decomposability.

Proposition 4.5. *A graph G is decomposable with respect to canonical decomposition if and only if it has a nontrivial strict module.*

Proof. If S is a nontrivial strict module in a graph G (so $S \neq V(G)$), then let $A = \{u \in V(G) - S\colon\ N_G(u) \cap S = \varnothing$ and $B = \{u \in V(G) - S\colon\ S \subseteq N_G(u)$. Now A is an independent set and B is a clique, since adjacent vertices in A or non-adjacent vertices in B would be the midpoints of an S-terminal alternating path of length 3, preventing S from being a strict module. Hence $G = (G', A, B) \circ G[S]$, where $G' = G[A \cup B]$.

Conversely, if G is decomposable, then $G = (G', A, B) \circ G_0$ for some nonempty subgraphs G' and G_0, where A and B are an independent set and a clique partitioning $V(G')$. Let $S = V(G_0)$. By the definition of decomposition, every vertex in S is adjacent to all of B and to none of A. An alterating path beginning in S must alternate thereafter between A and B and never is at the right parity to return to S. Thus S is a nontrivial strict module. □

Proposition 4.5 shows that for strict modules, indecomposable graphs play a role like that of prime graphs for modules. A graph is *prime* if it has no nontrivial modules. A module S is *proper* if $S \neq V(G)$; single-vertex modules are proper. Gallai [9] showed that if G and $\overline{G}$ are both connected, then every vertex in G belongs to a unique maximal proper module. This leads recursively to a "modular decomposition tree"

Theorem 4.6 (Gallai [9])**.** *For a graph G with at least two vertices, exactly one of the following conditions holds.*

(i) *G is disconnected.*

(ii) *$\overline{G}$ is disconnected.*

(iii) *The maximal proper modules partitition $V(G)$, and the subgraph induced by a set consisting of one vertex from each maximal proper module is a maximal prime subgraph.*

Jamison and Olariu [16] developed a refinement called *primeval decomposition* using the P_4-structure. A graph G is *p-connected* if for every partition of its vertex set into two nonempty disjoint sets, some edge in the P_4-structure intersects both sets. A maximal p-connected induced subgraph of G is a *p-component*. A p-connected graph G is *separable* if its vertex set splits into two nonempty disjoint sets such that each P_4 not contained within one of the sets has its endpoints in one set and its midpoints in the other. The primeval decomposition of a graph partitions its vertex set into modules via the following theorem.

Theorem 4.7 (Jamison–Olariu [16])**.** *For a graph G, exactly one of the following holds.*

(i) *G is disconnected.*

(ii) *$\overline{G}$ is disconnected.*

(iii) *G is p-connected.*

(iv) *There is a unique proper separable p-component Q of G with a partition Q_1, Q_2 of $V(Q)$ such that every vertex not in $V(Q)$ is adjacent to all of Q_1 and none of Q_2.*

We present in Theorem 4.8 an analogue of Theorem 4.7 for A_4-structure and canonical decomposition. An *A_4-component* of a graph G is a component of the canonical decomposition of G, which by Theorem 3.2 is the subgraph induced by the vertices of a component of the A_4-structure of G. A *separating partition* of a graph G is a partition (U, W) of $V(G)$ such that every induced subgraph having an alternating 4-cycle has an alternating 4-cycle that alternates between U and W. A graph is *A_4-separable* if has a separating partition, as illustrated in Figure 5.

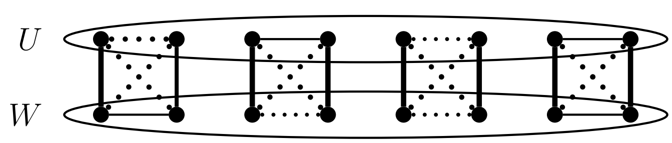

Figure 5: Alternating 4-cycles in an A_4-separable graph.

Split graphs with at least two vertices are A_4-separable. Each vertex of an alternating 4-cycle C has a neighbor and a nonneighbor along C, so no clique or independent set can have three vertices of C. Also, C cannot have consecutive vertices in a clique and the other two in an independent set, because adjacency in C is the same for both pairs. Hence every alternating cycle in a split graph alternates between the clique and the independent set.

Theorem 4.8. *For a graph G with more than one vertex, exactly one of the following holds:*

(i) *G has an isolated vertex.*

(ii) *$\overline{G}$ has an isolated vertex.*

(iii) *The A_4-structure of G is connected.*

(iv) *There is a unique proper A_4-separable A_4-component Q of G with a partition Q_1, Q_2 of $V(Q)$ such that every vertex not in $V(Q)$ is adjacent to none of Q_1 and to all of Q_2.*

Proof. Here "proper" means $V(Q) \neq V(G)$. Let $G_k, \ldots, G_0$ be the components of the canonical decomposition $(*)$. Since $G_k, \ldots, G_0$ also are the A_4-components of G, the only candidate for Q in (iv) is G_k; for any other G_i, the vertices of G_k ensure that no nontrivial partition of G_i can satisfy the adjacency condition.

Indecomposable graphs have no isolated or dominating vertex, so no two of (i), (ii), (iii) can simultaneously hold. If (iii) holds, then there is no proper A_4-component. If (i) or (ii) holds, then G_k is not A_4-separable. Hence at most one of the conditions holds.

If (iii) fails, then $k \geq 1$. If (i) and (ii) fail, then A_k and B_k are both nonempty. Since split graphs are A_4-separable, G_k is an A_4-component of G having the properties in (iv). We have noted that G_k is the only candidate, so at least one of the conditions holds. □

5 A_4-split graphs

In this section we characterize the *A_4-split graphs*, those having the same A_4-structure as some split graph. As motivation, we show that this problem arises in the problem of constructing all graphs having a given A_4-structure.

Example 5.1. *Graphs with the same A_4-structures.* By Lemma 3.11, every alternating 4-cycle in a graph G lies in one component of the canonical decomposition. Thus permuting the components of the canonical decomposition does not change the A_4-structure $H(G)$.

By Theorem 3.2 and Lemma 3.11, each component of $H(G)$ is determined by the corresponding component G_i of the canonical decomposition. Replacing G_i with another subgraph having the same A_4-structure yields a graph with the same A_4-structure as G.

To illustrate these operations, let G_2 and G_1 be copies of K_1 with vertices u and v, and let $G_0 = K_2 + P_3$. Given $G = (G_2, \varnothing, \{u\}) \circ (G_1, \{v\}, \varnothing) \circ G_0$, let G' be the graph formed by transposing the first two components; that is, $G' = (G_1, \{v\}, \varnothing) \circ (G_2, \varnothing, \{u\}) \circ G_0$. Let G'_0 be the 5-vertex graph with degree sequence $(3, 2, 1, 1, 1)$; note that G_0 and G'_0 have the same A_4-structure. Let $G'' = (G_2, \varnothing, \{u\}) \circ (G_1, \{v\}, \varnothing) \circ G'_0$. The graphs G, G', and G'' appear in Figure 6; they are pairwise nonisomorphic but have the same A_4-structure.

When $G = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$, the components $G_k, \ldots, G_1$ are all split graphs. Permuting them yields a graph having the same A_4-structure as G. If G_0 is not a split graph, then we cannot move the vertices of G_0 to a different position in the canonical decomposition unless we first replace G_0 by a split graph G'_0 having the same A_4-structure as G_0. To determine whether this is possible, we characterize the graphs having the same A_4-structure as a split graph; these are the *A_4-split graphs.* □

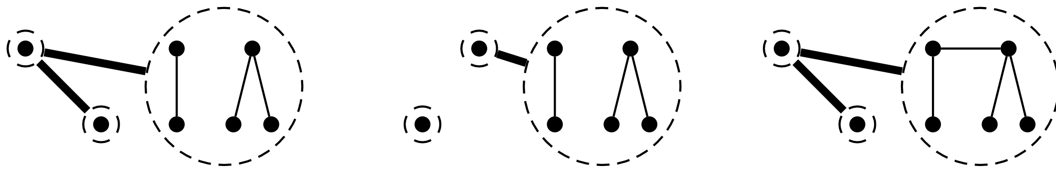

Figure 6: Different graphs with the same A_4-structure.

To characterize A_4-split graphs, we need several more concepts. A *split partition* of a graph G is a partition (Q, S) of $V(G)$ such that Q is a clique and S is an independent set. A *balancing partition* is a partition (V_1, V_2) of $V(G)$ such that every alternating 4-cycle in G has two vertices in each set; a graph with such a partition is *A_4-balanced* (we also say its A_4-structure is A_4-balanced). By definition, every A_4-separable graph (Section 4) is A_4-balanced. The converse fails, since bipartite graphs are A_4-balanced, but C_6 is not A_4-separable.

For an A_4-balanced A_4-structure H with balancing partition (V_1, V_2) and a vertex $v \in V_i$, the *v-restriction* of H is the graph on V_{3-i} in which two vertices are adjacent if and only if they both lie in an edge of H containing v. An A_4-balanced A_4-structure H has the *bipartite restriction property* if there is a balancing partition of H such that for all $v \in V(H)$ the v-restriction of H is bipartite.

The *k-pan* is the graph obtained by attaching a pendant vertex to a vertex of a k-cycle; the *co-k-pan* is its complement. The 4-pan and co-4-pan are shown in Figure 7.

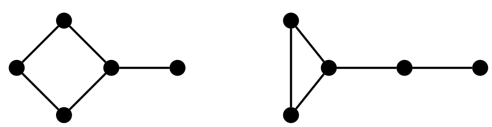

Figure 7: The 4-pan and the co-4-pan.

Theorem 5.2 (Földes–Hammer [6])**.** *G is a split graph if and only if G is $\{2K_2, C_4, C_5\}$-free.*

Corollary 5.3. *For any split partition (Q, S) of a split graph, every induced 4-vertex path has its midpoints in Q and its endpoints in S, and (Q, S) is a balancing partition.*

Proof. The placement of 4-vertex paths is immediate. By Theorem 5.2, the vertices of every alternating 4-cycle induce P_4. □

Theorem 5.4. *Let G_0 be the last graph in the canonical decomposition of a graph G, and let H be the A_4-structure of G. The following statements are equivalent.*

(a) *G is A_4-split.*

(b) *H is A_4-balanced and has the bipartite restriction property.*

(c) *G and $\overline{G}$ are both $\{C_5, P_5, K_2 + K_3,$ co-4-pan$, K_2 + P_4, K_2 + C_4, 2K_2 \vee 2K_1\}$-free.*

(d) *G is a split graph, or one of $\{G_0, \overline{G}_0\}$ is a disjoint union of stars.*

(e) *G is A_4-separable.*

Proof. We show that each condition implies the next and that (e) implies (a).

(a) ⇒ (b). Let G' be a split graph with the same A_4-structure H as G, and let (Q, S) be a split partition of G'. By Corollary 5.3, (Q, S) is a balancing partition, so H is A_4-balanced. For any induced P_4 in G' with vertices $a_1, a_2 \in Q$ and $b_1, b_2 \in S$, each a_i has one neighbor in $\{b_1, b_2\}$ and each b_i has one neighbor in $\{a_1, a_2\}$. Hence if $v \in V(G')$ and B is the v-restriction of H, then $N_{G'}(v)$ has exactly one vertex in each edge of B. Giving the neighbors and nonneighbors of v in $V(B)$ opposite colors shows that B is bipartite. Hence H has the bipartite restriction property.

(b) ⇒ (c). Since G and $\overline{G}$ have the same A_4-structure, existence of balancing partitions and the bipartite restriction property are preserved by complementation. They are also preserved by taking induced subgraphs. It thus suffices to show that the A_4-structure of each graph in (c) is not A_4-balanced or does not have the bipartite restriction property. For C_5, each 4-set contains an alternating 4-cycle, but no bipartition of $V(C_5)$ splits each 4-set equally. The co-4-pan, P_5, and $K_2 + K_3$ have the same A_4-structure H^* with three edges. In H^*, the only balancing partition has two vertices in one set and three vertices in the other. The v-restriction of H^* for a vertex v in the 2-set is K_3, so H^* does not have the bipartite restriction property. The A_4-structures of $K_2 + P_4$, $K_2 + C_4$, and $2K_2 \vee 2K_1$ also each have a unique balancing partition and a vertex v such that the v-restriction is K_3.

(c) ⇒ (d). Suppose that G and $\overline{G}$ have no graph in (c) as an induced subgraph. If G is not split, then G_0 is not split. By hypothesis G_0 is C_5-free, so G_0 induces $2K_2$ or C_4 (by Theorem 5.2). By complementation, we may assume that G_0 induces $2K_2$, with edges ab and cd; let $U = \{a, b, c, d\}$. Since G is $\{K_2 + K_3, P_5, \text{co-4-pan}\}$-free, every vertex of G_0 outside U has 0, 1, or 4 neighbors in U. Partition $V(G_0) - U$ into sets X, Y, A, B, C, and D, by those whose neighborhoods in U are U, $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$, respectively (see Figure 8).

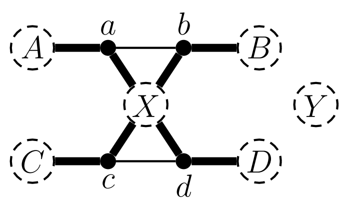

Figure 8: The graph G_0 from Theorem 5.4.

Since G_0 is $(2K_2 \vee 2K_1)$-free, X is a clique. If $X \neq \varnothing$, choose $x \in X$. Since G_0 is co-4-pan-free, $A = B = C = D = \varnothing$. Let Y'' be the set of isolated vertices in $G_0[Y]$. Let $Y' = Y - Y''$. Adjacent vertices $y_1, y_2 \in Y'$ are both adjacent to x; otherwise, $\{y_1, y_2, x, a, b\}$ induces $K_2 + K_3$ or the co-4-pan. Thus all of X is adjacent to all of Y'. Now $G_0 = (G_0[X \cup Y''], Y'', X) \circ G_0[\{a, b, c, d\} \cup Y']$, contradicting the indecomposability of G_0.

Hence $X = \varnothing$. Since G_0 is $(K_2 + P_4)$-free, A or B is empty, as is C or D. By symmetry, we may assume $B = D = \varnothing$. Since G_0 is $\{K_2 + K_3, P_5\}$-free, $A \cup C$ is independent. Since G_0 is $(K_2 + P_4)$-free, no vertex of Y has a neighbor in $A \cup C$. Thus $G_0[A \cup \{a, b\}]$ and $G_0[C \cup \{c, d\}]$ are components of G_0 that are stars. Since G_0 is $\{K_2 + K_3, K_2 + P_4, K_2 + C_4\}$-free, $G_0[Y]$ is $\{K_3, P_4, C_4\}$-free. The $\{K_3, P_4, C_4\}$-free graphs are forests with diameter at most 2 and hence also are disjoint unions of stars.

(d) ⇒(e). If G is a split graph, then G is A_4-separable. Hence we may assume by complementation that G_0 is a disjoint union of stars. Let A' be a largest independent set in G_0, and let $B' = V(G_0) - A'$. Any 4-vertex induced subgraph of G_0 having an alternating 4-cycle is isomorphic to $2K_2$ and has two nonadjacent vertices in each of A' and B'; thus (A', B') is an A_4-separating partition of G_0. Since the other components $G_k, \ldots, G_1$ of the canonical decomposition are split graphs and every alternating 4-cycle lies entirely within a single component (by Lemma 3.11), the partition of $V(G)$ into sets $A_k \cup \cdots \cup A_1 \cup A'$ and $B_k \cup \cdots \cup B_1 \cup B'$ is A_4-separating.

(e) ⇒(a). Let (V_1, V_2) be an A_4-separating partition of $V(G)$. Obtain a split graph G' from G by deleting all edges of $G[V_1]$ and adding all edges missing from $G[V_2]$, so (V_2, V_1) is a split partition of G'. Let H' be the A_4-structure of G'; we show that $H' = H$.

For each edge of H, there is an alternating 4-cycle in G that alternates between V_1 and V_2, and hence the cut between V_1 and V_2 on these four vertices is a matching. In G', these vertices induce P_4, so $E(H) \subseteq E(H')$. Conversely, in G' every alternating 4-cycle alternates between V_1 and V_2. Those edges and non-edges are the same as in G, so $E(H') \subseteq E(H)$. Thus equality holds, and G has the same A_4-structure as the split graph G'. □

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