

Largest 2-regular subgraphs in 3-regular graphs

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Abstract

For a graph G , let $f_2(G)$ denote the largest number of vertices in a 2-regular subgraph of G . We determine the minimum of $f_2(G)$ over 3-regular n -vertex simple graphs G . To do this, we prove that every 3-regular multigraph with exactly c cut-edges has a 2-regular subgraph that omits at most $\max\{0, \lfloor (c-1)/2 \rfloor\}$ vertices. More generally, every n -vertex multigraph with maximum degree 3 and m edges has a 2-regular subgraph that omits at most $\max\{0, \lfloor (3n-2m+c-1)/2 \rfloor\}$ vertices. These bounds are sharp; we describe the extremal multigraphs.

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1 Introduction

For $\ell \in \mathbb{N}$, an ℓ -factor in a graph or multigraph is an ℓ -regular spanning subgraph. Let $f_i(G)$ denote the maximum number of vertices in an i -regular subgraph of G . A graph or

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multigraph is *cubic* if every vertex has degree 3.

A classical theorem by Petersen [11] says that every cubic multigraph with at most two cut-edges has a 2-factor and (equivalently) a 1-factor. Thus $f_1(G) = f_2(G) = |V(G)|$ when G is 3-regular and has at most two cut-edges. In this paper, we extend this result on $f_2(G)$ to the setting where there are more cut-edges and also to the setting of maximum degree 3.

For a $(2r + 1)$ -regular graph G with n vertices, Henning and Yeo [6] proved $f_1(G) \geq n - r \frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}$ (while studying matchings), and this is sharp. The formula reduces to $(8n - 2)/9$ for 3-regular graphs. O and West [9] gave a short proof of the Henning–Yeo result using the notion of a *balloon* in a graph, which they defined to be a maximal 2-edge-connected subgraph incident to exactly one cut-edge.

We use balloons to study the minimum of $f_2(G)$ when G is 3-regular with n vertices. For 3-regular graphs, the notion of balloon has a simpler equivalent description: a graph obtained from a 2-edge-connected 3-regular graph by subdividing one edge.

In order to solve the problem, we consider a more general question, determining a sharp lower bound on $f_2(G)$ in terms of the number of cut-edges in G . Our basic result is

Theorem 1.1. *If G is a cubic n -vertex multigraph with c cut-edges, then $f_2(G) \geq n - \max\{0, \lfloor \frac{c-1}{2} \rfloor\}$, and this bound is sharp.*

We will also describe all the multigraphs that achieve equality in the bound. Since O and West [9] showed that a cubic n -vertex graph has at most $(n - 7)/3$ cut-edges, Theorem 1.1 immediately yields a lower bound on $f_2(G)$ for a cubic graph G in terms of the number of vertices alone. It also yields a somewhat weaker guarantee for cubic loopless multigraphs.

Corollary 1.2. *If G is a cubic n -vertex graph, then $f_2(G) \geq \min\{n, \lceil \frac{5}{6}(n + 2) \rceil\}$. If G is a cubic n -vertex loopless multigraph, then $f_2(G) \geq \min\{n, \lceil \frac{3}{4}(n + 2) \rceil\}$. Both bounds are sharp.*

Theorem 1.1 is proved more simply by considering the broader class of *subcubic* multigraphs, which are those having maximum degree at most 3. Given an n -vertex multigraph G with maximum degree at most $2r + 1$, the r -deficit of G is the difference between $(2r + 1)n$ and the degree-sum of G , which can be computed as $(2r + 1)n - 2|E(G)|$.

Theorem 1.3. *If G is a subcubic n -vertex multigraph with c cut-edges and 1-deficit d , then $f_2(G) \geq n - \max\{0, \frac{d+c-1}{2}\}$, and this bound is sharp.*

Several constructions of sharpness examples together lead to a characterization of all sharpness examples.

Example 1.4. Trees. A subcubic n -vertex tree has $n - 1$ cut-edges. Its 1-deficit is $3n - 2(n - 1)$, so in this case $(d + c - 1)/2 = n$. Hence Theorem 1.3 guarantees nothing, and in fact a tree has no 2-regular subgraph.

Balloons. By definition, a balloon has no cut-edge and has 1-deficit 1. Theorem 1.3 guarantees a 2-factor, which achieves equality in the bound.

Bipartite multigraphs. Let H be a 2-connected cubic bipartite multigraph with parts X and Y ; note that $|X| = |Y|$. Let $G = H - \hat{y}$, where $\hat{y} \in Y$. If G is 2-connected, then G has

no cut-edge and has 1-deficit 3. Since the number of vertices in G is odd and all cycles in G are even, G has no 2-factor. Theorem 1.3 guarantees a 2-regular subgraph in G with $n - 1$ vertices, where $n = |V(G)|$. Hence G is a sharpness example.

The argument for bipartite multigraphs in Example 1.4 applies to confirm sharpness for a larger family. (Recall that a cubic multigraph is 2-connected if and only if it has no cut-edge, with the exception of the loopless multigraph with two vertices and three edges.)

Definition 1.5. Let \mathcal{G} be the family of multigraphs obtained in the following way:

- (1) Start with a 2-connected cubic bipartite multigraph H with parts X and Y .
- (2) Delete one vertex $\hat{y} \in Y$ such that $H - \hat{y}$ is 2-connected.
- (3) Explode (or not) each vertex y in $Y - \hat{y}$, where *exploding* y means taking the disjoint union of the current graph with a 2-connected cubic multigraph F and then replacing both y and a vertex z in F with three edges joining the neighborhoods of y and z so that all vertices have degree 3.

We will show that combining sharpness examples via cut-edges preserves sharpness. Trees are assembled in this way from single vertices, so we do not need them as fundamental building blocks for sharpness examples. With the characterization of sharpness, our main result (including all those mentioned previously and proved in Section 2) is then the following.

Theorem 1.6. *If G is a subcubic n -vertex multigraph with c cut-edges and 1-deficit d , then $f_2(G) \geq n - \max\{0, \frac{d+c-1}{2}\}$. When G is connected, equality holds if and only if each component after deleting all the cut-edges is a single vertex, a balloon, or a graph in \mathcal{G} .*

In Section 3, we offer additional enhancements. First, we generalize by restricting to graphs with girth at least g . Second, we show that one can restrict the initial bipartite multigraph H in the definition of \mathcal{G} by forbidding multi-edges. Third, one can alternatively restrict each multigraph F used to explode a vertex to be factor-critical, where *factor-critical* means having a matching that omits only any one vertex. However, one cannot ensure these latter two enhancements simultaneously.

Generalizing the problem, one would seek first a large 2-regular subgraph when G is $(2r + 1)$ -regular, and then more generally a large $2k$ -regular subgraph when G is $(2r + 1)$ -regular. It is reasonable to think that $f_2(G) \geq n - \max\{0, \lceil \frac{d+c-1}{2r} \rceil\}$ holds when G has maximum degree $2r + 1$ with c cut-edges and r -deficit d , because sharpness holds in two quite different classes. Equality holds for trees ($n - 1$ cut-edges, r -deficit $(2r - 1)n + 2$, and no 2-regular subgraph) and for $(2r + 1)$ -regular graphs with at most $2r$ cut-edges (Hanson, Loten, and Toft [5] showed that every such graph has a 2-factor).

For the general problem of minimizing $f_{2k}(G)$ when G is $(2r + 1)$ -regular, Kostochka et al. [7] generalized [5] by showing that if $k < (2r + 1)/3$ and G has at most $2r - 3(k - 1)$ cut-edges, then G has a $2k$ -factor. Therefore, we are interested in how large a $2k$ -regular subgraph is guaranteed when there are more cut-edges. In this paper, we settle the case $k = r = 1$.

2 The Main Result

To prove the desired bound on the number of vertices omitted by a largest 2-regular subgraph, in cases where the graph has no cut-edge we will need two earlier results.

First, a result of Edmonds [1] easily implies the following lemma.

Lemma 2.1 (O and West [10]). *Every edge-weighted 2-edge-connected 3-regular multigraph has a perfect matching containing at most $1/3$ of the total weight.*

The results of Edmonds [1] were used earlier in an essentially equivalent way by Naddef and Pulleyblank [8] to prove that every edge-weighted $(t - 1)$ -edge-connected t -regular multigraph of even order has a 1-factor with weight at least a fraction $1/t$ of the total weight. Here the *order* of a graph is its number of vertices.

We also use a special case for cubic graphs of a result of Plesník that strengthens the usual conclusion about 1-factors in regular graphs.

Lemma 2.2 (Plesník [12]). *Every $(t - 1)$ -edge-connected t -regular graph of even order has a 1-factor that avoids any $t - 1$ specified edges.*

We also use the special case of Tutte's 1-Factor Theorem [13] for 3-regular multigraphs G , stating that if G has no 1-factor, then $V(G)$ contains a *Tutte set* S such that $o(G - S) \geq |S| + 2$, where $o(H)$ is the number of components of H having odd order, called *odd components*.

We can now prove the main result, which we restate for ease of reference.

Theorem 2.3. *If G is a subcubic n -vertex multigraph with c cut-edges and 1-deficit d , then $f_2(G) \geq n - \max\{0, \frac{d+c-1}{2}\}$. When G is connected, equality holds if and only if each component after deleting all the cut-edges is a single vertex, a balloon, or a graph in \mathcal{G} .*

Proof. By Petersen's Theorem [11], a cubic graph G with at most two cut-edges has a 2-factor. Hence we may assume $d > 0$ or $c > 2$. Indeed, we may assume this in each component. Hence we may also assume that G is connected and must prove $f_2(G) \geq n - (d + c - 1)/2$.

The difficult case is when $c = 0$ and $d > 0$. We postpone this basis step for a proof by induction on the number of cut-edges, considering first the induction step.

Deleting a cut-edge e from G leaves its endpoints with degree less than 3. Letting $G - e$ be the disjoint union of G_1 and G_2 , each containing an endpoint of e . For $i \in \{1, 2\}$, let c_i be the number of cut-edges and d_i be the 1-deficit of G_i . Since neither G_1 nor G_2 is 3-regular, and both are subcubic with fewer cut-edges than G , the induction hypothesis applies to each.

That is, G_i has a 2-regular subgraph H_i omitting at most $(d_i + c_i - 1)/2$ vertices, and the disjoint union $H_1 + H_2$ is a 2-regular subgraph of G omitting at most $(d_1 + d_2 + c_1 + c_2 - 2)/2$ vertices. Since $d = d_1 + d_2 - 2$ and $c = c_1 + c_2 + 1$, the graph $H_1 + H_2$ omits at most $(d + c - 1)/2$ vertices of G . Equality holds if and only if it holds in both G_1 and G_2 , which implies inductively that equality holds in G if and only if G has the claimed description.

Now consider the basis step: G has no cut-edge, but $d > 0$. If G has only one vertex, then the formula holds with equality whether the vertex has a loop or not. Hence we are reduced to a connected subcubic multigraph with more than one vertex.

Since G has no cut-edge, G now has minimum degree 2. We may also assume that G has maximum degree 3, since $f_2(G) = n$ when G is 2-regular. We use Lemma 2.1. A *thread* in a graph is a maximal path whose internal vertices have degree 2 (it may have just one edge); the endpoints of each thread in G have degree 3. Let a j -*vertex* be a vertex of degree j .

Suppress each 2-vertex of G by turning each thread through 2-vertices into one weighted edge whose weight equals the length of the thread. The total weight of the resulting graph G' is the number of edges in G . Deleting from G' the matching guaranteed by Lemma 2.1 leaves a 2-factor of G' whose total weight is at least $2/3$ of the total weight of G' . This 2-factor expands back into a 2-regular subgraph of G that has at least $2/3$ of the edges of G .

Hence G has a 2-regular subgraph H with at least $2m/3$ vertices, where $m = |E(G)|$. Let $t = |V(H)|$. Since $d = 3n - 2m$, we have $n - t \leq n - (2m/3) = d/3$. If $d > 3$, then $d/3 < (d - 1)/2 = (d + c - 1)/2$, so here the bound holds and cannot hold with equality.

If $d \in \{1, 2\}$, then the formula requires a 2-factor. Suppressing the 2-vertex or the two 2-vertices leaves a 3-regular graph G' with no cut-edge. By Lemma 2.2, the graph G' has a 1-factor that omits the edge(s) formed by suppressing 2-vertices. Deleting this 1-factor leaves a 2-factor in G' that uses those edge(s), and it expands to a 2-factor in G . When $d = 2$, equality cannot hold in the formula, since the formula is not an integer. When $d = 1$, equality holds, and G is a balloon, as claimed.

Finally, assume $d = 3$. At each of the three 2-vertices of G , add a cut-edge and a balloon to form a 3-regular graph G' . If G' has a 1-factor, then deleting its edges (and the added vertices) leaves a 2-factor of G . Otherwise, G' has a Tutte set S such that $o(G' - S) \geq |S| + 2$. By parity of the degree-sum, an odd number of edges join S to any odd component of $G' - S$.

Let m be the number of edges joining S to $V(G' - S)$; note that $m \leq 3|S|$. Since G has no cut-edge, each odd component of $G' - S$ other than an added balloon receives at least three edges from S . Therefore, $m \geq 3 + 3(|S| - 1)$, and equality must hold. Since G' is connected, also $G' - S$ has no even components, the components of $G' - S$ are the added balloons and others receiving exactly three edges, and S is an independent set.

The components of $G' - S$ other than the added balloons are the set T of components of $G - S$, each having odd order. The edges in G joining S to T form a bipartite multigraph F with parts S and T obtainable by deleting one vertex of a 3-regular bipartite graph (which produces the three 2-vertices in G). To obtain G from F , each vertex of T is left alone or is exploded. Thus every extremal graph with $d = 3$ has the form described.

Also every such graph is extremal. To prove this, it remains only to show that every graph $G \in \mathcal{G}$ has no 2-factor. The construction of G according to Definition 1.5 begins with a bipartite graph H having parts X and Y . A vertex $y \in Y - \{\hat{y}\}$ may be exploded using a 2-connected multigraph F , but the vertices of $F - z$ that are made adjacent to the neighborhood of y in H lie in the same component of $G - X$.

Suppose that G has a 2-factor and orient each cycle consistently. Each vertex of X is followed on its cycle by a vertex that corresponds to a particular vertex y in Y , and the cycle can only leave that component of $G - X$ via a vertex corresponding to the same vertex y . Also, since H is 3-regular, that vertex y cannot serve in this way for any other vertex $x \in X$. Since $|X| > |Y - \{\hat{y}\}|$, there cannot be disjoint cycles covering all the vertices of X . \square

3 Enhancements

In this section, we consider several refinements of the main result.

As noted in Corollary 1.2, Theorem 1.6 specializes for cubic graphs ($d = 0$) to say for $n > 4$ that every cubic n -vertex graph has a 2-regular subgraph with at least $5(n + 2)/6$ vertices; this uses that such a graph has at most $(n - 7)/3$ cut-edges [9].

Example 3.1. Equality holds in Corollary 1.2 for every graph G obtained by starting with a tree whose internal vertices all have degree 3 and attaching a 5-vertex balloon at each leaf. When all internal vertices have degree 3, the number of leaves in the tree exceeds the number of internal vertices by 2. The internal vertices lie in no cycle and hence in no 2-regular subgraph, while the balloons have 2-factors. With t internal vertices and n vertices altogether, we have $n = t + 5(t + 2)$ and $f_2(G) = 5(t + 2)$, so $f_2(G) = 5(n + 2)/6$.

Furthermore, equality holds in $f_2(G) \geq 3(n + 2)/4$ for cubic multigraphs by using the balloon obtained by subdividing one edge of a triple-edge instead of the 5-vertex simple balloon. In both cases, this describes all examples achieving equality (see [9]).

We can generalize Corollary 1.2 and Example 3.1 in terms of girth by considering the minimum number of vertices in a balloon with girth g . When $g \geq 2$, a smallest balloon with girth g arises from a smallest 3-regular (multi)graph with girth g by subdividing one edge. We can pick an edge to subdivide that does not increase the girth as long as there is an edge that does not belong to every shortest cycle. Such an edge exists because the vertex degrees are not 2.

A smallest k -regular graph with girth g is called a (k, g) -cage (for $g = 2$ it consists of two vertices joined by k edges). Determining the minimum number $h(k, g)$ of vertices in a (k, g) -cage is a well-known and very difficult problem. The smallest balloon with girth g will have $h(3, g) + 1$ vertices. For $g \in \{2, \dots, 12\}$, the number of vertices is 3, 5, 7, 11, 15, 25, 31, 59, 71, 113, 127, respectively (see [3], for example).

Corollary 3.2. *If G is a cubic n -vertex multigraph with girth g , then*

$$f_2(G) \geq \min \left\{ n, \frac{h'}{h' + 1}(n + 2) \right\},$$

where $h' = h(3, g) + 1$. *The bound is sharp. All examples achieving equality arise by attaching a smallest balloon with girth g at each leaf of a tree whose internal vertices have degree 3.*

Proof. (Sketch) The argument for the upper bound of O and West [9] on the number of cut-edges depends on the smallest order of balloons. The number of cut-edges is maximized by attaching smallest subcubic balloons of girth g to the leaves of a tree with internal vertices of degree 3, which yields the given formula.

The proof is inductive. Achieving equality for a larger graph requires achieving equality in both graphs obtained by deleting a cut-edge. This leads to the structure described. \square

Next we refine Theorem 1.6 by showing that \mathcal{G} can be produced in a more restricted way.

Proposition 3.3. *In Definition 1.5 for the family \mathcal{G} , the initial bipartite multigraph H generating any member of \mathcal{G} can be taken to be simple, without changing the resulting family.*

Proof. For every multigraph in \mathcal{G} , the 1-deficit is 3, there is no cut-edge, and every largest 2-regular subgraph omits exactly one vertex, as proved in Theorem 2.3.

Hence in Theorem 2.3 the graphs in \mathcal{G} arise only in the case $d = 3$ when the augmented graph G' has no 1-factor. As described there, for any Tutte set S in G' , the edges joining S and T form a bipartite multigraph that can serve as the multigraph H in the construction of G as a graph in \mathcal{G} .

To ensure that H can be chosen to be simple, we let S be a minimal Tutte set in G' . It is an elementary exercise that for any minimal Tutte set S in a cubic multigraph G' , each vertex in S has all its neighbors in distinct components of $G' - S$. (If the neighbors of any $x \in S$ are confined to fewer than three odd components of $G' - S$, then deleting x from S reduces $|S|$ by as much as it reduces the number of resulting odd components, thereby yielding a smaller Tutte set.)

Since the neighbors of each $x \in S$ are in distinct components of $G' - S$, the neighbors of x in the resulting bipartite multigraph H are distinct vertices of $Y - \hat{y}$. \square

Finally, the famous Gallai–Edmonds Structure Theorem [2, 4] that describes all largest matchings in a multigraph leads to another refinement of the structure of members of \mathcal{G} .

Definition 3.4. In a multigraph G , let B be the set of vertices that are covered by every maximum matching in G . Let A be the set of vertices in B having at least one neighbor outside B , let $C = B - A$, and let $D = V(G) - B$. The *Gallai–Edmonds Decomposition* of G is the partition of $V(G)$ into the three sets A, C, D . The *deficiency* $\text{def}(G)$ of a graph G is $\max_{S \subseteq V(G)} \{o(G - S) - |S|\}$.

Theorem 3.5 (Gallai–Edmonds Structure Theorem). *Let A, C, D be the Gallai–Edmonds Decomposition of a multigraph G . Let G_1, \dots, G_q be the components of $G - A - C$. If M is a maximum matching in G , then the following properties hold.*

- a) M covers C and matches A into distinct components of $G - A - C$.
- b) Each G_i is factor-critical.
- c) $o(G - A) - |A| = \text{def}(G) = q - |A|$.

Proposition 3.6. *In the construction of any graph $G \in \mathcal{G}$, the bipartite multigraph H with parts X and Y can be chosen so that each component of $G - X$ is a factor-critical graph.*

Proof. In the Gallai–Edmonds Decomposition (A, C, D) of a graph not having a 1-factor, the set A is a Tutte set. For the auxiliary 3-regular multigraph G' in the proof of Theorem 2.3 when the 1-deficit is 3, take the Tutte set S to be A in the Gallai–Edmonds Decomposition. As argued in the proof of Theorem 2.3, we have C empty and A independent. By the Gallai–Edmonds Structure Theorem, with this choice of the Tutte set and the resulting bipartite graph H , the set A becomes X , and the components of $G - X$ are factor-critical. \square

Example 3.7. The refinements in Propositions 3.3 and 3.6 cannot be guaranteed simultaneously (that is, using one initial bipartite graph H). An example showing this appears in Figure 1, shown in solid edges. This is a bipartite multigraph G obtained from the complete bipartite graph $K_{2,3}$ by replacing one edge with a thread of length 3 and then duplicating the middle edge xy of that thread to reach degree 3 at its endpoints.

The augmented graph G' (including the dashed edges) grows a cut-edge from each 2-vertex and adds a balloon at the other end of each cut-edge. Every maximum matching in G' covers all the vertices of X . Using $H - \hat{y}$ as the full multigraph G , with no vertices exploded, the components of $G - X$ are factor-critical, but this H is not simple.

The Tutte set X has size 4. Also $X - \{x\}$ is a Tutte set. This Tutte set constructs G by starting with $H - \hat{y} = K_{2,3}$ and exploding one vertex of Y by using the 2-connected 3-regular multigraph consisting of a 4-cycle with two opposite edges duplicated.

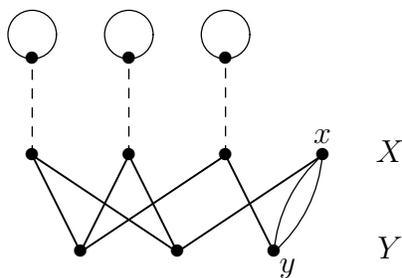


Figure 1: Graph for Example 3.7.

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